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ANALYTIC FORMULAE FOR DETERMINANT SYSTEMS FOR A CERTAIN CLASS OF FREDHOLM OPERATORS IN BANACH SPACES

1. Introduction

R. Sikorski [6] has constructed a determinant system for any linear and continuous Fredholm operator $I+T$ in a Banach space, where T is a quasi-nuclear operator.

A. Buraczewski [3] has obtained analytic formulae for a determinant system for any linear continuous operator $I+T$ in a Banach space, where T^k is a quasi-nuclear operator for some positive integer k.

The purpose of this paper is to show how to construct effectively a determinant system for any linear and continuous operator $A = S+T$ in a Banach space, where S is a fixed Fredholm operator, U is a quasi-inverse of S, T is a such operator, that if index $d(S) = d \geq 0$, then $(UT)^k$ is a quasi-nuclear operator for some positive integer k. Similarly, if $d(S) = d < 0$, then $(TU)^k$ is a quasi-nuclear operator. The obtained result is a generalization of the determinant theory of operators of the form $I+T$, where T^k is a quasi-nuclear operator for some positive integer k.

The possibility of the generalization was suggested by Prof. A. Buraczewski to whom the author is very much indebted.

2. Preliminaries

Let X, Ξ be fixed Banach spaces over the same real or complex field K. The norms in X and Ξ are denoted by $\|\|_X$ and $\|\|_\Xi$, respectively.

A pair (Ξ, X) is said to be a *pair of conjugate Banach spaces*, if there exists a continuous bilinear functional $I : \Xi \times X \to K$ whose value at a point $(\xi, x) \in \Xi \times X$ is denoted by ξx (i.e. $I(\xi, x) = \xi x$) and which satisfies the following conditions:

(a) if $\xi x = 0$ for every $\xi \in \Xi$, then $x = 0$;
(a′) if $\xi x = 0$ for every $x \in X$, then $\xi = 0$.

The bilinear functional I is called the *scalar product on* $\Xi \times X$.

It follows from conditions (a), (a′) and from continuity of I, that every element $\xi \in \Xi$ can be interpreted as a linear continuous functional on X and, analogously, every element $x \in X$ can be interpreted as a linear continuous functional on Ξ. In symbols

$$X \subset \Xi^*, \quad \Xi \subset X^*. \tag{1}$$

Hence, to each element $\xi \in \Xi$ we can assign two norms: $\|\xi\|_{\Xi}$ and $\|\xi\|_{X^*}$, where

$$\|\xi\|_{X^*} = \sup\{|\xi x| : \|x\|_X \le 1\}. \tag{2}$$

Similarly, to each element $x \in X$ we can assign two norms: $\|x\|_X$ and $\|x\|_{\Xi^*}$, where

$$\|x\|_{\Xi^*} = \sup\{|\xi x| : \|\xi\|_{\Xi} \le 1\}. \tag{2′}$$

If $\|\ \|_{\Xi}$, $\|\ \|_{X^*}$ are equivalent norms in Ξ and $\|\|_X$, $\|\|_{\Xi^*}$ are equivalent norms in X, then a pair (Ξ, X) is said to be a *pair of isomorphically conjugate Banach spaces*. A pair (Ξ, X) is a pair of isomorphically conjugate Banach spaces if and only if, in interpretation (1), Ξ is a closed subspace of X^*, and X is a closed subspace of Ξ^*.

Let $op(\Xi, X)$ be the set of all continuous bilinear functionals $A : \Xi \times X \to K$ whose value at a point $(\xi, x) \in \Xi \times X$ is denoted by ξAx (i.e. $A(\xi, x) = \xi Ax$) and satisfying the following conditions:

(b) for every $\xi \in \Xi$ there exists $\eta \in \Xi$ such that $\eta x = \xi Ax$ for every $x \in X$;

(b′) for every $x \in X$ there exists $y \in X$ such that $\xi y = \xi Ax$ for every $\xi \in \Xi$.

Note that such η and y have to be unique.

Every functional $A \in op(\Xi, X)$ can be interpreted as a linear continuous mapping $A : \Xi \to \Xi$ defined by the formula $\xi A = \eta$, where η is the element satisfying the condition (b) and also as a linear continuous mapping $A : X \to X$ defined by the formula $Ax = y$, where y is the element satisfying the condition (b′). Elements of $op(\Xi, X)$ are called *operators*.

For any $A \in op(\Xi, X)$ let us introduce the following notation:

$$R(A) = \{Ax : x \in X\}, \quad N(A) = \{x \in X : Ax = 0\},$$
$$\mathcal{R}(A) = \{\xi A : \xi \in \Xi\}, \quad \mathcal{N}(A) = \{\xi \in \Xi : \xi A = 0\}.$$

The set $op(\Xi, X)$ is a linear space with respect to natural definitions of algebraic operations. It is a Banach space equipped with the norm defined as follows

$$\|A\| = \sup\{|\xi Ax| : \|\xi\|_{\Xi} \leq 1, \|x\|_X \leq 1\}$$
$$= \sup\{\|Ax\|_X : \|x\|_X \leq 1\} = \sup\{\|\xi A\|_{\Xi} : \|\xi\|_{\Xi} \leq 1\}$$

for every $A \in op(\Xi, X)$.

It is also a Banach algebra with the unity, where the multiplication is defined by the formula

$$\xi(A_1A_2)x = (\xi A_1)(A_2x) \quad \text{for } A_1, A_2 \in op(\Xi, X), (\xi, x) \in \Xi \times X. \tag{4}$$

The unity of this algebra is the scalar product I defined by the formula $\xi Ix = \xi x$ for $(\xi, x) \in \Xi \times X$.

An operator $B \in op(\Xi, X)$, such that

$$ABA = A, \quad BAB = B. \tag{5}$$

is said to be a *quasi-inverse of an operator* $A \in op(\Xi, X)$.

For fixed elements $x_0 \in X, \xi_0 \in \Xi$ let $x_0 \cdot \xi_0$ stands for the operator defined as follows

$$\xi(x_0 \cdot \xi_0)x = (\xi x_0)(\xi_0 x) \quad \text{for } (\xi, x) \in \Xi \times X. \tag{6}$$

The operator $x_0 \cdot \xi_0 \in op(\Xi, X)$ is called a *one-dimensional operator*.

Any finite sum of one-dimensional operators is called a *finitely dimensional operator*. Thus

$$\sum_{i=1}^{n} x_i \cdot \xi_i, \tag{7}$$

is an n-dimensional operator, where $x_i \in X$, $\xi_i \in \Xi$, $i = 1, \ldots, n$ are fixed.

Let (Ξ, X) be a pair of isomorphically conjugate Banach spaces. Let $cn(\Xi, X)$ denote the space of all linear continuous functionals $\mathcal{F}$ on $op(\Xi, X)$, which determine functionals $T_{\mathcal{F}} \in op(\Xi, X)$ defined by the formula

$$\xi T_{\mathcal{F}} x = \mathcal{F}(x \cdot \xi) \quad \text{for } (\xi, x) \in \Xi \times X. \tag{8}$$

Elements of the Banach space $cn(\Xi, X)$ are called *quasi-nuclei*. If for an operator $T \in op(\Xi, X)$ there exists a quasi-nucleus $\mathcal{F} \in cn(\Xi, X)$ such that $T = T_{\mathcal{F}}$, then T is said to be *quasi-nuclear* and $\mathcal{F}$ is said to be a *quasi-nucleus of* T.

The operator (7) is quasi-nuclear and the functional $\sum_{i=1}^{n} \xi_i \otimes x_i$ defined by the formula

$$\Big(\sum_{i=1}^{n} \xi_i \otimes x_i\Big)(A) = \sum_{i=1}^{n} \xi_i A x_i, \quad \text{for every } A \in op(\Xi, X) \tag{9}$$

is its quasi-nucleus. The quasi-nucleus $\sum_{i=1}^{n} \xi_i \otimes x_i$ is said to be *finitely dimensional*.

For a fixed operator $C \in op(\Xi, X)$ let us define quasi-nuclei $C\mathcal{F}$ and $\mathcal{F}C$ on $op(\Xi, X)$ by the formulae

(10) $(C\mathcal{F})(A) = \mathcal{F}(AC), \quad (\mathcal{F}C)(A) = \mathcal{F}(CA) \quad$ for every $A \in op(\Xi, X)$.

Let D be an $(n+m)$-linear functional on $\Xi^n \times X^m$. The value of D at a point $(\xi_1, \ldots, \xi_n, x_1, \ldots, x_m) \in \Xi^n \times X^m$ is denoted by

$$D\begin{pmatrix} \xi_1, \ldots, \xi_n \\ x_1, \ldots, x_m \end{pmatrix}. \tag{11}$$

For fixed $\mathcal{F} \in cn(\Xi, X)$, if D can be interpreted as a function of the variables ξ_1, x_1, let us define the $((n-1)+(m-1))$-linear functional $\mathcal{F} \square D$ on $\Xi^{n-1} \times X^{m-1}$ (see [4]) by the formula

$$(\mathcal{F} \square D)\begin{pmatrix} \xi_2, & \ldots, & \xi_n \\ x_2, & \ldots, & x_m \end{pmatrix} = \mathcal{F}(A), \tag{12}$$

where

$$\xi_1 A x_1 = D\begin{pmatrix} \xi_1, & \ldots, & \xi_n \\ x_1, & \ldots, & x_m \end{pmatrix} \quad \text{for } \xi_i \in \Xi,\ x_j \in X,\ i = 1, \ldots, n,\ j = 1, \ldots, m. \tag{12'}$$

If $n > 1$ and $m > 1$, then interpreting $\mathcal{F} \square D$ as a function of the variables ξ_2, x_2, for fixed $\mathcal{G} \in cn(\Xi, X)$, we can define the $((n-2)+(m-2))$-linear functional $\mathcal{G} \square \mathcal{F} \square D$ on $\Xi^{n-2} \times X^{m-2}$ by the formula

$$(\mathcal{G} \square \mathcal{F} \square D)\begin{pmatrix} \xi_3, & \ldots, & \xi_n \\ x_3, & \ldots, & x_m \end{pmatrix} = \mathcal{G}(B), \tag{13}$$

where

$$\xi_2 B x_2 = (\mathcal{F} \square D)\begin{pmatrix} \xi_2, & \ldots, & \xi_n \\ x_2, & \ldots, & x_m \end{pmatrix} \quad \text{for } \xi_i \in \Xi,\ x_j \in X,\ i = 2, \ldots, n,\ j = 2, \ldots, m. \tag{13'}$$

Let $k = \min(n, m)$, $\mathcal{F}_i \in cn(\Xi, X)$, $i = 1, \ldots, k$. Interpreting $\mathcal{F}_{k-1} \square \ldots \square \mathcal{F}_1 \square D$ as a function of variables ξ_k, x_k, we can define the $((n-k)+(m-k))$-linear functional $\mathcal{F}_k \square \mathcal{F}_{k-1} \square \ldots \square \mathcal{F}_1 \square D$ on $\Xi^{n-k} \times X^{m-k}$ as follows

$$(\mathcal{F}_k \square \mathcal{F}_{k-1} \square \ldots \square \mathcal{F}_1 \square D)\begin{pmatrix} \xi_{k+1}, & \ldots, & \xi_n \\ x_{k+1}, & \ldots, & x_m \end{pmatrix} = \mathcal{F}_k(A_k), \tag{14}$$

where

$$(14') \qquad \xi_k A_k x_k = (\mathcal{F}_{k-1} \square \ldots \square \mathcal{F}_1 \square D)\begin{pmatrix} \xi_k, & \ldots, & \xi_n \\ x_k, & \ldots, & x_m \end{pmatrix}$$

for $\xi_i \in \Xi$, $x_j \in X$, $i = k, \ldots, n$, $j = k, \ldots, m$.

If $\mathcal{F}$ is a fixed quasi-nucleus in $cn(\Xi, X)$, then $\mathcal{F}\square$ will denote the mapping which assigns $\mathcal{F} \square D$ to every $(n+m)$-linear functional D on $\Xi^n \times X^m$. Clearly,

$$(15) \qquad \mathcal{F}_k \square \mathcal{F}_{k-1} \square \ldots \square \mathcal{F}_1 \square$$

is the composition of mappings $\mathcal{F}_1\square, \ldots, \mathcal{F}_k\square$ determined by quasi-nuclei $\mathcal{F}_1, \ldots, \mathcal{F}_k$ in $cn(\Xi, X)$. We will denote by $\mathcal{F}^k$ the modified k-th power of a quasi-nucleus $\mathcal{F}$ in $cn(\Xi, X)$, that is,

$$(16) \qquad \mathcal{F}^k = \frac{1}{k!} \underbrace{\mathcal{F} \square \mathcal{F} \square \ldots \mathcal{F} \square}_{k-\text{times}}.$$

In particular, $\mathcal{F}^1 = \mathcal{F}\square$.

3. Structure of analytic formulae for the determinant system

Let (Ξ, X) be a pair of isomorphically conjugate Banach spaces. Let $S \in op(\Xi, X)$ be a fixed Fredholm operator of order $r(S) = 0$ and of index $d(S) = d \geq 0$. Let $s_1, \ldots, s_d$ be a complete system of solutions of the equation $Sx = 0$, and $U \in op(\Xi, X)$ be a quasi-inverse of S. Let $T \in op(\Xi, X)$ denote such an operator that $(UT)^k$ is quasi-nuclear for some positive integer k. It will be given an effective analytic formulae for the determinant system of the linear continuous mapping $A = S + T$. The operator A can be represented in the form $A = S(I + UT)$. We will show that operator A has a determinant system. It follows from the main theorem of [3], that for every integer $l \geq k$ there exists an integer $s \geq l$ such that

$$(17) \quad I + UT = (A_0)^{-1}[I + (-1)^{s+1}(UT)^s] = [I + (-1)^{s+1}(UT)^s](A_0)^{-1},$$

where

$$(17') \qquad A_0 = \prod_{i=1}^{s-1}(I + \alpha_i UT)$$

for all roots $\alpha_1, \ldots, \alpha_{s-1}$ of the equation $\alpha^s = 1$, different from 1.

Let $\mathcal{F} \in cn(\Xi, X)$ be a quasi-nucleus of the operator $(UT)^k$, i.e.

$$(18) \qquad \xi(UT)^k x = \mathcal{F}(x \cdot \xi) \quad \text{for } (\xi, x) \in \Xi \times X.$$

It follows from formulae (10) that the quasi-nuclei

$$(19) \qquad [(-1)^{s+1}(UT)^{s-k}]\mathcal{F} \quad \text{and} \quad \mathcal{F}[(-1)^{s+1}(UT)^{s-k}]$$

determine the same quasi-nuclear operator $(-1)^{s+1}(UT)^s$. Since the operator A_0 defined by (17′) is invertible and $(-1)^{s+1}(UT)^s$ is a quasi-nuclear operator, then in view of (17) we conclude that $I + UT$ is a Fredholm operator and, consequently, A is Fredholm as the composition of Fredholm operators S and $I + UT$. It follows from the main theorem of determinant theory (see [5]) that A has a determinant system. Our purpose is to construct this determinant system.

Let (Θ_n) be a determinant system for I, i.e.

$$\Theta_0 = 1, \tag{20}$$

$$\Theta_n\begin{pmatrix}\xi_1, & \dots, & \xi_n\\ x_1, & \dots, & x_n\end{pmatrix} = \begin{vmatrix}\xi_1 x_1 & \dots & \xi_1 x_n\\ \dots & & \dots\\ \xi_n x_1 & \dots & \xi_n x_n\end{vmatrix}$$

for $n = 1, 2, \dots$, $\xi_i \in \Xi$, $x_i \in X$, $i = 1, \dots, n$. Let us recall (see [4]), that

$$\Theta_{n,m}(\mathcal{F}) = \mathcal{F}^m \Theta_{n+m} \quad \text{for } n, m \in N, \tag{21}$$

and the sequence $(\Theta_n(\mathcal{F}))$ defined by the formula

$$\Theta_n(\mathcal{F}) = \sum_{m=0}^{\infty} \Theta_{n,m}(\mathcal{F}) \tag{22}$$

is a determinant system for the operator $I + (UT)^k$.

In view of (19), (21), (22) the sequences

$$(\Theta_n([(-1)^{s+1}(UT)^{s-k}]\mathcal{F})) \quad \text{and} \quad (\Theta_n(\mathcal{F}[(-1)^{s+1}(UT)^{s-k}])) \tag{23}$$

are determinant systems for $(-1)^{s+1}(UT)^s$. Consequently, we obtain

$$\Theta_n([(-1)^{s+1}(UT)^{s-k}]\mathcal{F}) = \sum_{m=0}^{\infty} \Theta_{n,m}([(-1)^{s+1}(UT)^{s-k}]\mathcal{F}), \tag{24}$$

where

$$\Theta_{n,m}([(-1)^{s+1}(UT)^{s-k}]\mathcal{F})\begin{pmatrix}\xi_1, & \dots, & \xi_n\\ x_1, & \dots, & x_n\end{pmatrix} = \tag{24'}$$

$$= ([(-1)^{s+1}(UT)^{s-k}]\mathcal{F})^m \Theta_{n+m}\begin{pmatrix}\eta_1, & \dots, & \eta_m, & \xi_1, & \dots, & \xi_n\\ y_1, & \dots, & y_m, & x_1, & \dots, & x_n\end{pmatrix} =$$

$$= (-1)^{m(s+1)} \mathcal{F}^m \Theta_{n+m}\begin{pmatrix}\eta_1, & \dots, & \eta_m, & \xi_1, & \dots, & \xi_n\\ (UT)^{s-k}y_1, & \dots, & (UT)^{s-k}y_m, & x_1, & \dots, & x_n\end{pmatrix}$$

for $\xi_i, \eta_j \in \Xi$, $x_i, y_j \in X$, $i = 1, \dots, n$, $j = 1, \dots, m$ and similarly

$$\Theta_n(\mathcal{F}[(-1)^{s+1}(UT)^{s-k}]) = \sum_{m=0}^{\infty} \Theta_{n,m}(\mathcal{F}[(-1)^{s+1}(UT)^{s-k}]), \tag{25}$$

where

$$\Theta_{n,m}(\mathcal{F}[(-1)^{s+1}(UT)^{s-k}])\begin{pmatrix}\xi_1, \dots, \xi_n\\ x_1, \dots, x_n\end{pmatrix} = \tag{25'}$$

$$= (\mathcal{F}[(-1)^{s+1}(UT)^{s-k}])^m \Theta_{n+m}\begin{pmatrix} \eta_1, & \dots, & \eta_m, & \xi_1, & \dots, & \xi_n \\ y_1, & \dots, & y_m, & x_1, & \dots, & x_n \end{pmatrix} =$$

$$= (-1)^{m(s+1)}\mathcal{F}^m \Theta_{n+m}\begin{pmatrix} \eta_1(UT)^{s-k}, & \dots, & \eta_m(UT)^{s-k}, & \xi_1, & \dots, & \xi_n \\ y_1, & \dots, & y_m, & x_1, & \dots, & x_n \end{pmatrix}$$

for $\xi_i, \eta_j \in \Xi$, $x_i, y_j \in X$, $i = 1, \dots, n$, $j = 1, \dots, m$.

It follows from (20) that sequences (23) coincide.

Let us denote

$$\overline{D}_n = \Theta_n([(-1)^{s+1}(UT)^{s-k}]\mathcal{F}) = \Theta_n(\mathcal{F}[(-1)^{s+1}(UT)^{s-k}]) \quad \text{for } n \in N. \tag{26}$$

We shall recall the following property of determinant systems (see [5]).

If (D_n) is a determinant system for $A \in op(\Xi, X)$ and $B \in op(\Xi, X)$ is invertible, then

$$D_n\begin{pmatrix} \xi_1, & \dots, & \xi_n \\ B^{-1}x_1, & \dots, & B^{-1}x_n \end{pmatrix} \quad \text{for } n \in N,\ \xi_i \in \Xi,\ x_i \in X,\ i = 1, \dots, n \tag{27}$$

is a determinant system for BA, and

$$D_n\begin{pmatrix} \xi_1 B^{-1}, & \dots, & \xi_n B^{-1} \\ x_1, & \dots, & x_n \end{pmatrix} \quad \text{for } n \in N,\ \xi_i \in \Xi,\ x_i \in X,\ i = 1, \dots, n \tag{27'}$$

is a determinant system for AB. Then, in view of (27), (27′), bearing in mind (24), (26) and (17), the determinant system (D_n) for $I + UT$ is of the form

$$D_n = \sum_{m=0}^{\infty} D_{n,m} \quad \text{for } n, m \in N, \tag{28}$$

where

$$D_{n,m}\begin{pmatrix} \xi_1, \dots, \xi_n \\ x_1, \dots, x_n \end{pmatrix} = \overline{D}_{n,m}\begin{pmatrix} \xi_1, & \dots, & \xi_n \\ A_0x_1, & \dots, & A_0x_n \end{pmatrix} = \tag{28'}$$

$$= (-1)^{m(s+1)}\mathcal{F}^m \Theta_{n+m}\begin{pmatrix} \eta_1, & \dots, & \eta_m, & \xi_1, & \dots, & \xi_n \\ (UT)^{s-k}y_1, & \dots, & (UT)^{s-k}y_m, & A_0x_1, & \dots, & A_0x_n \end{pmatrix}$$

or

$$D_{n,m}\begin{pmatrix} \xi_1, \dots, \xi_n \\ x_1, \dots, x_n \end{pmatrix} = \overline{D}_{n,m}\begin{pmatrix} \xi_1 A_0, & \dots, & \xi_n A_0 \\ x_1, & \dots, & x_n \end{pmatrix} =$$

$$= (-1)^{m(s+1)}\mathcal{F}^m \Theta_{n+m}\begin{pmatrix} \eta_1(UT)^{s-k}, & \dots, & \eta_m(UT)^{s-k}, & \xi_1 A_0, & \dots, & \xi_n A_0 \\ y_1, & \dots, & y_m, & x_1, & \dots, & x_n \end{pmatrix}$$

for $\xi_i, \eta_j \in \Xi$, $x_i, y_j \in X$, $i = 1, \dots, n$, $j = 1, \dots, m$.

Since $S+T=S(I+UT)$, $r(S)=0$, $d(S)=d\geq 0, U$ is a quasi-inverse of S, $s_1,\dots,s_d$ is a basis of $N(S)$, then, applying Theorem (ix) in [1] and formulae (28′), (28″), we obtain

THEOREM 1. *The sequence* $(\mathcal{D}_n)$ *defined by*

$$\mathcal{D}_n=\sum_{m=0}^{\infty}\mathcal{D}_{n,m} \quad \textit{for } n,m\in N, \tag{29}$$

where

$$\mathcal{D}_{n,m}\begin{pmatrix}\xi_1,\dots,\xi_{n+d}\\ x_1,\dots,\ x_n\end{pmatrix}= \tag{29′}$$

$$=D_{n+d,m}\begin{pmatrix}\xi_1, & \dots, & & \xi_{n+d}\\ Ux_1,\dots,Ux_n, & s_1, & \dots, & s_d\end{pmatrix}=$$

$$=(-1)^{m(s+1)}\mathcal{F}^m\Theta_{n+d+m}$$

$$\begin{pmatrix}\eta_1, & \dots, & \eta_m, & \xi_1, & \dots, & \xi_{n+d}\\ (UT)^{s-k}y_1, & \dots, & (UT)^{s-k}y_m, & A_0Ux_1,\dots, & A_0Ux_n, & A_0s_1,\dots,A_0s_d\end{pmatrix}$$

or

$$\mathcal{D}_{n,m}\begin{pmatrix}\xi_1,\dots,\xi_{n+d}\\ x_1,\dots,\ x_n\end{pmatrix}=D_{n+d,m}\begin{pmatrix}\xi_1, & \dots, & & \xi_{n+d}\\ Ux_1, & \dots, Ux_n, & s_1, & \dots, & s_d\end{pmatrix}= \tag{29″}$$

$$=(-1)^{m(s+1)}\times$$

$$\mathcal{F}^m\Theta_{n+d+m}\begin{pmatrix}\eta_1(UT)^{s-k}, & \dots, & \eta_m(UT)^{s-k}, & \xi_1A_0, & \dots, & \xi_{n+d}A_0 & \\ y_1, & \dots, & y_m, & Ux_1, & \dots, & Ux_n, & s_1,\dots,s_d\end{pmatrix}$$

for $\xi_i,\eta_j\in\Xi$, $x_l,y_j\in X$, $i=1,\dots,n+d$, $j=1,\dots,m$, $l=1,\dots,n$ *is a determinant system for* $A=S+T$, *where* S *is a Fredholm operator of order* $r(S)=0$, *and of index* $d(S)=d\geq 0$, $(UT)^k$ *is a quasi-nuclear operator for some positive integer* k *and a quasi-inverse* U *of* S.

Now if $S=I$, then $U=I$ and we obtain Buraczewski's formulae for the determinant system for operator $I+T$, where T^k is quasi-nuclear for some positive integer k, given in [3]. If $k=1$, i.e. UT is quasi-nuclear, then T is also quasi-nuclear and $A_0=I$, $s=k=1$. Hence, we obtain formulae given in [1] for the determinant system for $S+T$, where S is a Fredholm operator and T is quasi-nuclear. Moreover, the obtained analytic formulae are independent of the choice of a quasi-inverse U of S.

Assume that $V\in op(\Xi,X)$ is another quasi-inverse of the operator $S(r(S)=0,\ d(S)=d\geq 0)$. It follows from [2], that

$$V = U + \sum_{i=1}^{d} s_i \cdot \zeta_i \tag{30}$$

for some $\zeta_i \in \mathcal{R}(U) = N(U)^{\perp} = \Xi$ for $i = 1, \ldots, d$.

It can be verified by the induction argument, the validity of the lemma

LEMMA 1. *The following identity holds*

$$(VT)^n = (UT)^n + \sum_{l=2}^{n} \sum_{i_1+\ldots+i_l=n-l+1} \sum_{j_1,\ldots,j_{l-1}=1}^{d} \Big[\Big(\prod_{m=1}^{l-2} \zeta_{j_m} T(UT)^{i_m} s_{j_{m+1}}\Big)(UT)^{i_{l-1}} s_{j_1} \cdot \zeta_{j_{l-1}} T(UT)^{i_l}\Big] + \sum_{j_1,\ldots,j_n=1}^{d} \Big[\Big(\prod_{m=1}^{n-1} \zeta_{j_m} T s_{j_{m+1}}\Big) s_{j_1} \cdot \zeta_{j_m} T\Big] \tag{31}$$

for all positive integers n.

Since $(UT)^k$ is a quasi-nuclear operator for some positive integer k, determined by a quasi-nucleus $\mathcal{F}$, then $(VT)^k$, in view of (31) is also a quasi-nuclear operator, determined by a quasi-nucleus $\overline{\mathcal{F}}$, defined as follows

$$\overline{\mathcal{F}} = \mathcal{F} + \sum_{l=2}^{k} \sum_{i_1+\ldots+i_l=k-l+1} \sum_{j_1,\ldots,j_{l-1}=1}^{d} \Big[\Big(\prod_{m=1}^{l-2} \zeta_{j_m} T(UT)^{i_m} s_{j_{m+1}}\Big)\zeta_{j_{l-1}} T(UT)^{i_l} \otimes (UT)^{i_{l-1}} s_{j_1}\Big] + \sum_{j_1,\ldots,j_k=1}^{d} \Big[\Big(\prod_{m=1}^{k-1} \zeta_{j_m} T s_{j_{m+1}}\Big)\zeta_{j_k} T \otimes s_{j_1}\Big]. \tag{32}$$

Similarly as $(\mathcal{D}_n)$ let us define a determinant system $(\mathcal{D}_n^*)$ for $A = S+T$, replacing $(UT)^k$ by $(VT)^k$. It follows from (29), (29′) and (29″), that

$$\mathcal{D}_n^* = \sum_{m=0}^{\infty} \mathcal{D}_{n,m}^* \quad \text{for } n, m \in N, \tag{33}$$

where

$$\mathcal{D}_{n,m}^* \begin{pmatrix} \xi_1, & \ldots, & \xi_{n+d} \\ x_1, & \ldots, & x_n \end{pmatrix} = \tag{33′}$$

$$= (-1)^{m(s+1)}\overline{\mathcal{F}}^m \Theta_{n+d+m}$$
$$\begin{pmatrix} \eta_1, & \dots, & \eta_m, & \xi_1, & \dots, & & & \xi_{n+d} \\ (VT)^{s-k}y_1, & \dots, & (VT)^{s-k}y_m, & \overline{A}_0 V x_1, & \dots, & \overline{A}_0 V x_n, & \overline{A}_0 s_1, \dots, & \overline{A}_0 s_d \end{pmatrix}$$

or

$$(33'') \qquad \mathcal{D}^*_{n,m}\begin{pmatrix} \xi_1, & \dots, & \xi_{n+d} \\ x_1, & \dots, & x_n \end{pmatrix} =$$
$$= (-1)^{m(s+1)} \times$$
$$\overline{\mathcal{F}}^m \Theta_{n+d+m}\begin{pmatrix} \eta_1(VT)^{s-k}, & \dots, & \eta_m(VT)^{s-k}, & \xi_1\overline{A}_0, \dots, & & \xi_{n+d}\overline{A}_0 \\ y_1, & \dots, & y_m, & Vx_1, \dots, Vx_n, & s_1, \dots, & s_d \end{pmatrix}$$

for $\xi_i, \eta_j \in \Xi$, $x_l, y_j \in X$, $i = 1, \dots, n+d$, $j = 1, \dots, m$, $l = 1, \dots, n$ and $\overline{A}_0 = \prod_{i=1}^{s-1}(I + \alpha_i VT)$, where $\alpha_1, \dots, \alpha_{s-1}$ are roots of the equation $\alpha^s = 1$, different from 1.

Let us consider the class of quasi-inverses of S, which can be represented in the form (30), where $\zeta_i \in \mathcal{N}(T)$ for $i = 1, \dots, d$, i.e.

$$(34) \qquad \left\{U + \sum_{i=1}^{d} s_i \cdot \zeta_i : \zeta_i \in \mathcal{N}(T)\right\}.$$

Let V belong to the class (34). Then, in view of (31) and (32), it is easily seen, that

$$(35) \qquad (VT)^n = (UT)^n \quad \text{for all } n \in N, \text{ and } \overline{\mathcal{F}} = \mathcal{F}.$$

Moreover, $\overline{A}_0 = A_0$. Hence, bearing in mind (33′), we obtain

$$(36') \qquad \mathcal{D}^*_{n,m}\begin{pmatrix} \xi_1, & \dots, & \xi_{n+d} \\ x_1, & \dots, & x_n \end{pmatrix} =$$
$$= (-1)^{m(s+1)}\mathcal{F}^m \Theta_{n+d+m}\left(\begin{matrix} \eta_1, & \dots, & \eta_m, \\ (UT)^{s-k}y_1, & \dots, & (UT)^{s-k}y_m, \end{matrix}\right.$$
$$\left.\begin{matrix} \xi_1, & \dots, & & \xi_{n+d} \\ A_0Ux_1 + \sum_{i=1}^{d}(\zeta_i x_1)A_0 s_i, & \dots, & A_0Ux_n + \sum_{i=1}^{d}(\zeta_i x_n)A_0 s_i, & A_0 s_1, \dots, A_0 s_d \end{matrix}\right)$$
$$= (-1)^{m(s+1)}\mathcal{F}^m \Theta_{n+d+m}$$
$$\begin{pmatrix} \eta_1, & \dots, & \eta_m, & \xi_1, & \dots, & & & \xi_{n+d} \\ (UT)^{s-k}y_1, & \dots, & (UT)^{s-k}y_m, & A_0Ux_1, & \dots, & A_0Ux_n, & A_0 s_1, \dots, & A_0 s_d \end{pmatrix}.$$

Similarly, using (33″), we have

$$\mathcal{D}^*_{n,m}\begin{pmatrix}\xi_1, & \ldots, & \xi_{n+d}\\ x_1, & \ldots, & x_n\end{pmatrix} = \tag{36″}$$

$$= (-1)^{m(s+1)}\mathcal{F}^m\Theta_{n+d+m}$$

$$\begin{pmatrix}\eta_1(UT)^{s-k}, \ldots, \eta_m(UT)^{s-k}, & \xi_1 A_0, & \ldots, & \xi_{n+d}A_0\\ y_1, \quad \ldots, \quad y_m, & Ux_1 + \sum_{i=1}^{d}(\zeta_i x_1)s_i, & \ldots, & Ux_n + \sum_{i=1}^{d}(\zeta_i x_n)s_i, s_1, \ldots, s_d\end{pmatrix}$$

$$= (-1)^{m(s+1)}\times$$

$$\mathcal{F}^m\Theta_{n+d+m}\begin{pmatrix}\eta_1(UT)^{s-k}, & \ldots, & \eta_m(UT)^{s-k}, & \xi_1 A_0, \ldots, & & \xi_{n+d}A_0 & \\ y_1, & \ldots, & y_m, & Ux_1, \ldots, Ux_n, & s_1, & \ldots, & s_d\end{pmatrix}.$$

In view od (36′) and (36″) the determinant systems $(\mathcal{D}_n)$, $(\mathcal{D}^*_n)$ defined by formulae (29) and (33), respectively, coincide. In other words, a determinant system for $S+T$, where S is a Fredholm operator of order zero and of non-negative index, and $(UT)^k$ is quasi-nuclear, does not depend on the choice of a quasi-inverse of S from the class (34). It is not yet known whether a determinant system for the considered operator $S+T$ is independent of the choice of an arbitrary quasi-inverse of S.

Let us consider $A = S+T$, where $S \in op(\Xi, X)$ is a Fredholm operator of order $r(S) = 0$ and of index $d(S) = d < 0$, $T \in op(\Xi, X)$ is a such operator that $(TU)^k$ is quasi-nuclear for some positive integer k and for a quasi-inverse U of S. Let $\sigma_1, \ldots, \sigma_{-d}$ denote all linearly independent solutions of $\xi S = 0$. Similarly as in the previous case, the operator A can be represented in the form $A = (I+TU)S$. It can be shown on the basis of the main theorem of [3], that for every integer $l \geq k$ there exists an integer $s \geq l$ such that

$$I + TU = (B_0)^{-1}[I + (-1)^{s+1}(TU)^s] = [I + (-1)^{s+1}(TU)^s](B_0)^{-1}, \tag{37}$$

where

$$B_0 = \prod_{i=1}^{s-1}(I + \alpha_i TU) \tag{37′}$$

for all roots $\alpha_1, \ldots, \alpha_{s-1}$ of the equation $\alpha^s = 1$, different from 1.

If $\mathcal{F} \in cn(\Xi, X)$ is a quasi-nucleus of $(TU)^k$, then $(-1)^{s+1}(TU)^s$ is a quasi-nuclear operator determined by quasi-nucleus $[(-1)^{s+1}(TU)^{s-k}]\mathcal{F}$ and also by $\mathcal{F}[(-1)^{s+1}(TU)^{s-k}]$.

Hence, in view of (37), $I+TU$ is Fredholm. Consequently, A is a Fredholm operator as a composition of Fredholm operators $I+TU$ and S.

Since $(\Theta_n(\mathcal{F}))$ is a determinant system for $I+(TU)^k$, the sequence $(\overline{D}_n)$, defined by

$$\begin{aligned}\overline{D}_n &= \Theta_n([(-1)^{s+1}(TU)^{s-k}]\mathcal{F}) \\ &= \Theta_n(\mathcal{F}[(-1)^{s+1}(TU)^{s-k}]) \quad \text{for } n \in N,\end{aligned} \tag{38}$$

is a determinant system for $(-1)^{s+1}(TU)^s$. Thus we obtain

$$\overline{D}_n = \sum_{m=0}^{\infty} \overline{D}_{n,m} \quad \text{for } n, m \in N, \tag{39}$$

where

$$\overline{D}_{n,m}\begin{pmatrix}\xi_1, & \ldots, & \xi_n \\ x_1, & \ldots, & x_n\end{pmatrix} = \tag{39'}$$

$$= (-1)^{m(s+1)}\mathcal{F}^m\Theta_{n+m}\begin{pmatrix}\eta_1, & \ldots, & \eta_m, & \xi_1, \ldots, \xi_n \\ (TU)^{s-k}y_1, & \ldots, & (TU)^{s-k}y_m, & x_1, \ldots, x_n\end{pmatrix},$$

and also

$$\overline{D}_{n,m}\begin{pmatrix}\xi_1, & \ldots, & \xi_n \\ x_1, & \ldots, & x_n\end{pmatrix} = \tag{39''}$$

$$= (-1)^{m(s+1)}\mathcal{F}^m\Theta_{n+m}\begin{pmatrix}\eta_1(TU)^{s-k}, & \ldots, & \eta_m(TU)^{s-k}, & \xi_1, \ldots, \xi_n \\ y_1, & \ldots, & y_m, & x_1, \ldots, x_n\end{pmatrix},$$

for $\xi_i, \eta_j \in \Xi$, $x_i, y_j \in X$, $i = 1, \ldots, n$, $j = 1, \ldots, m$.

It follows from properties (27), (27′) and from the formula (37) that a determinant system (D_n) for $I + TU$ is of the form

$$D_n = \sum_{m=0}^{\infty} D_{n,m} \quad \text{for } n, m \in N, \tag{40}$$

where

$$D_{n,m}\begin{pmatrix}\xi_1, & \ldots, & \xi_n \\ x_1, & \ldots, & x_n\end{pmatrix} = \tag{40'}$$

$$= (-1)^{m(s+1)}\mathcal{F}^m\Theta_{n+m}\begin{pmatrix}\eta_1, & \ldots, & \eta_m, & \xi_1, & \ldots, & \xi_n \\ (TU)^{s-k}y_1, & \ldots, & (TU)^{s-k}y_m, & B_0x_1, & \ldots, & B_0x_n\end{pmatrix}$$

or

$$D_{n,m}\begin{pmatrix}\xi_1, & \ldots, & \xi_n \\ x_1, & \ldots, & x_n\end{pmatrix} = \tag{40''}$$

$$= (-1)^{m(s+1)} \mathcal{F}^m \Theta_{n+m} \begin{pmatrix} \eta_1 (TU)^{s-k}, & \ldots, & \eta_m (TU)^{s-k}, & \xi_1 B_0, & \ldots, & \xi_n B_0 \\ y_1, & \ldots, & y_m, & x_1, & \ldots, & x_n \end{pmatrix}$$

for $\xi_i, \eta_j \in \Xi$, $x_i, y_j \in X$, $i = 1, \ldots, n$, $j = 1, \ldots, m$.

Similarly as in Theorem 1, applying the formulae (40), (40′), (40″), we obtain

THEOREM 2. *The sequence* $(\mathcal{D}_n)$ *defined by*

$$\mathcal{D}_n = \sum_{m=0}^{\infty} \mathcal{D}_{n,m} \quad \text{for } n, m \in N, \tag{41}$$

where

$$\mathcal{D}_{n,m} \begin{pmatrix} \xi_1, & \ldots, & \xi_n \\ x_1, & \ldots, & x_{n-d} \end{pmatrix} \tag{41'}$$

$$= D_{n-d,m} \begin{pmatrix} \xi_1 U, & \ldots, & \xi_n U, & \sigma_1, & \ldots, & \sigma_{-d} \\ x_1, & \ldots, & x_{n-d} & & & \end{pmatrix} =$$

$$= (-1)^{m(s+1)} \times$$

$$\mathcal{F}^m \Theta_{n-d+m} \begin{pmatrix} \eta_1, & \ldots, & \eta_m, & \xi_1 U, & \ldots, & \xi_n U, \sigma_1, \ldots, \sigma_{-d} \\ (TU)^{s-k} y_1, & \ldots, & (TU)^{s-k} y_m, & B_0 x_1, & \ldots, & B_0 x_{n-d} \end{pmatrix}$$

or

$$\mathcal{D}_{n,m} \begin{pmatrix} \xi_1, & \ldots, & \xi_n \\ x_1, & \ldots, & x_{n-d} \end{pmatrix} = \tag{41''}$$

$$D_{n-d,m} \begin{pmatrix} \xi_1 U, & \ldots, & \xi_n U, & \sigma_1, & \ldots, & \sigma_{-d} \\ x_1, & \ldots, & & & x_{n-d} & \end{pmatrix} =$$

$$= (-1)^{m(s+1)} \mathcal{F}^m \Theta_{n-d+m}$$

$$\begin{pmatrix} \eta_1 (TU)^{s-k}, & \ldots, & \eta_m (TU)^{s-k}, & \xi_1 U B_0, & \ldots, & \xi_n U B_0, \sigma_1 B_0, & \ldots, & \sigma_{-d} B_0 \\ y_1, & \ldots, & y_m, & x_1, & \ldots, & & x_{n-d} & \end{pmatrix}$$

for $\xi_i, \eta_j \in \Xi$, $x_l, y_j \in X$, $i = 1, \ldots, n$, $j = 1, \ldots, m$, $l = 1, \ldots, n-d$, *is a determinant system for* $A = S + T$, *where* S *is a Fredholm operator of order zero and of index* $d < 0$, $(TU)^k$ *is quasi-nuclear for some positive integer* k *and for a quasi-inverse* U *of* S.

Suppose that $V \in op(\Xi, X)$ is another quasi-inverse of the operator S with $r(S) = 0$ and $d(S) = d < 0$. It follows from [2], that

$$V = U + \sum_{i=1}^{-d} z_i \cdot \sigma_i \tag{42}$$

for some $z_i \in R(U) = \mathcal{N}(U)^{\perp} = X$ for $i = 1, \ldots, -d$.

Analogously, as in the case of a non-negative index (formula (31)), we can verify by the induction on n

LEMMA 2. *The following identity holds*

$$(43)\quad (TV)^n = (TU)^n + \sum_{l=2}^{n} \sum_{i_1+\ldots+i_l=n-l+1} \sum_{j_1,\ldots,j_{l-1}=1}^{-d} \Big[\Big(\prod_{m=1}^{l-2} \sigma_{j_m}(TU)^{i_m} Tz_{j_{m+1}}\Big)(TU)^{i_l-1}Tz_{j_1}\cdot\sigma_{j_{l-1}}(TU)^{i_l}\Big]+ + \sum_{j_1,\ldots,j_n=1}^{-d} \Big[\Big(\prod_{m=1}^{n-1}\sigma_{j_m}Tz_{j_{m+1}}\Big)Tz_{j_{m+1}}\cdot\sigma_{j_n}\Big]$$

for all $n = 1, 2, \ldots$.

Similarly, as in (32), if $(TU)^k$ is a quasi-nuclear operator for some positive integer k, determined by some quasi-nucleus $\mathcal{F}$, then $(TV)^k$, in view of (43), is also a quasi-nuclear operator, determined by a quasi-nucleus $\overline{\mathcal{F}}$, defined as follows

$$(44)\quad \overline{\mathcal{F}} = \mathcal{F} + \sum_{l=2}^{k} \sum_{i_1+\ldots+i_l=k=l+1} \sum_{j_1,\ldots,j_{l-1}=1} \Big[\Big(\prod_{m=1}^{l-2} \sigma_{j_m}(TU)^{i_m}Tz_{j_{m+1}}\Big)\sigma_{j_{l-1}}(TU)^{i_l}\otimes(TU)^{i_l-1}z_{j_1}\Big]+ + \sum_{j_1,\ldots,j_k=1}^{-d}\Big[\Big(\prod_{m=1}^{k-1}\sigma_{j_m}Tz_{j_{m+1}}\Big)\sigma_{j_k}\otimes Tz_{j_1}\Big].$$

Replacing U by V and $\mathcal{F}$ by $\overline{\mathcal{F}}$ in (37′), (41′) and (41″), we obtain a determinant system for $A = S + T$, where $(TV)^k$ is quasi-nuclear.

Let us consider the class of quasi-inverses of S, which can be represented in the form (42), where $z_i \in N(T)$ for $i = 1, \ldots, -d$, i.e.

$$(46)\qquad \{U + \sum_{i=1}^{-d} z_i\cdot\sigma_i : z_i \in N(T)\}.$$

In a similar way as in (36′), (36″), applying elementary properties of determinants, it is easy to verify, that a determinant system for $S + T$, where S is a Fredholm operator of order zero and of negative index, and $(TU)^k$ is quasi-nuclear, does not depend on the choice of a quasi-inverse of S from the class (46).

Now let us consider the most general case. Let $S \in op(\Xi, X)$ be any Fredholm operator, and let $U \in op(\Xi, X)$ be a quasi-inverse of S. Let $T \in op(\Xi, X)$ be such, that if the index of S is non-negative, then $(UT)^k$ is quasi-nuclear, and similarly if the index of S is negative, then $(TU)^k$ is quasi-nuclear for some positive integer k. Let $s_1, \ldots, s_m$ and $\sigma_1, \ldots, \sigma_\mu$ denote bases of $N(S)$ and $\mathcal{N}(S)$, respectively. It follows from the basic properties of Fredholm operators that there exist elements $u_1, \ldots, u_\mu \in X$ and $\tau_1, \ldots, \tau_m \in \Xi$ such that

$$(47) \qquad SU = I - \sum_{i=1}^{\mu} u_i \cdot \sigma_i \quad \text{where } \sigma_i u_j = \delta_{ij} \quad \text{for } i, j = 1, \ldots, \mu,$$

$$US = I - \sum_{i=1}^{m} s_i \cdot \tau_i \quad \text{where } \tau_i s_j = \delta_{ij} \text{ for } i, j = 1, \ldots, m.$$

Let us take $r = \min(\mu, m)$ and let $R = \sum_{i=1}^{r} s_i \cdot \sigma_i$ and $L = \sum_{i=1}^{r} u_i \cdot \tau_i$. Then the following identities hold:

$$(48)(S+L)(U+R) = I \text{ and } (U+R)(S+L) = I - \sum_{i=1}^{m-r} s_{r+i}\tau_{r+i} \text{ if } d(S) \geq 0;$$

$$(U+R)(S+L) = I \text{ and } (S+L)(U+R) = I - \sum_{i=1}^{\mu-r} u_{r+i}\sigma_{r+i} \text{ if } d(S) < 0.$$

By (48), $S_0 = S + L$ is a Fredholm operator of order $r(S_0) = 0$ and of index $d(S_0) = d(S)$, and that $U_0 = U + R$ is a quasi-inverse of S_0. Hence, the operator $S + T$ can be expressed in the form

$$(49) \qquad S + T = S_0 + (T - L).$$

Suppose that $d(S) \geq 0$. Since $(UT)^k$ is quasi-nuclear, R and L are finitely dimensional, then, evidently, $[U_0(T - L)]^k$ is a quasi-nuclear operator. Let us denote by $\mathcal{F}_0$ its quasi-nucleus. In view of (49), bearing in mind that $s_{r+1}, \ldots, s_m$ is a basis of $N(S_0)$, in order to obtain effective formulae for determinant systems for the operator $A = S + T$ it is sufficient to substitute S_0 for $S, T - L$ for T, U_0 for U and $\mathcal{F}_0$ for $\mathcal{F}$ in (29′) and (29″).

Similarly, suppose that $d(S) < 0$. Since $(TU)^k$ is quasi-nuclear, then it is easy to verify that $[(T - L)U_0]^k$ is also quasi-nuclear. Denoting by $\mathcal{F}_0$ its quasi-nucleus, bearing in mind that $\sigma_{r+1}, \ldots, \sigma_\mu$ is the basis of $\mathcal{N}(S_0)$ and substituting S_0 for $S, T - L$ for T, U_0 for U and $\mathcal{F}_0$ for $\mathcal{F}$ in (41′) and (41″), we obtain analytic formulae for determinant systems for the operator $A = S + T$.

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