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ON A SYSTEM OF NONLINEAR SINGULAR INTEGRAL EQUATIONS IN AN EUCLIDEAN SPACE R^n

1. Introduction

Consider in an Euclidean space R^n, $(n \geq 3)$, a system of $p+1$ closed $(n-1)$-dimentional Lapunov surfaces $S_0, S_1, \dots, S_p$, $(p \geq 0)$, having no common points. The surface S_0 is the boundary of a bounded region Ω_0 containing the surfaces $S_1, S_2, \dots, S_p$. Let Ω denote the set of all those points of the region Ω_0 which do not lie on the surfaces $S_1, S_2, \dots, S_p$. If $p = 0$, then $\Omega = \Omega_0$. So $\Omega = \sum_{i=1}^{p+1} \Omega_i$, where Ω_i are separable, simply-connected or multi-connected regions. Let $f(y)$ be a complex function in any one of the regions $\Omega_1, \Omega_2, \dots, \Omega_{p+1}$ and $N(x)$ – a complex function defined at each point $x \neq (0,0,\dots,0)$ by the formula

$$N(x) = F(x')|x|^{-n} \tag{1}$$

where x' denotes the central projection of the point x on the unitary sphere ω, with centre $(0,0,\dots,0)$. We then have $x = |x|x'$. Assume that $F(x')$ satisfies on the sphere ω the condition

$$|F(x') - F(y')| \leq K_\omega |x' - y'|^{h_\omega}, \quad 0 < h_\omega \leq 1, \quad k_\omega > 0, \tag{2}$$

and, moreover, the condition

$$\int_\omega F(x')dx' = 0. \tag{3}$$

After Zygmund ([6], p. 468–505) and Pogorzelski ([4], p. 8), we define the singular integral

$$\int_\Omega N(x-y)f(y)dy = \lim_{\epsilon \to 0} \int_{\Omega_\epsilon} N(x-y)f(y)dy, \tag{4}$$

where Ω_ϵ denotes the set of all points y of the set Ω satisfying the condition $|x - y| > \epsilon$.

Let us now recall the definition of a certain class of the functions introduced by Pogorzelski ([4], p.6) in connection with investigation of properties of multidimensional singular integral (4).

We denote by $\mathcal{H}_\alpha^h$ the class of all complex functions $f(x)$ defined for $x \in \Omega$ and satisfying the conditions

$$|x - x_s|^\alpha |f(x)| \le \mu_f, \tag{5}$$

$$|x - x_s|^{\alpha+h} |f(x) - f(y)| \le K_f |x - y|, \tag{6}$$

where $|x-y|$ denotes the Euclidean distance of two arbitrary points x and y situated within any of the regions $\Omega_1, \Omega_2, \ldots, \Omega_{p+1}$; x_s is the point of one of the surfaces $S_0, S_1, \ldots, S_p$ for which the distance $|x - x_s|$ reaches, for a fixed $x \in \Omega$, a lower limit. We assume that $|x - x_s| \le |y - y_s|$ and parameters α and h are fixed for a given class and satisfy the conditions

$$0 \le \alpha < 1, \quad 0 < h < 1, \quad \alpha + h < 1; \tag{7}$$

μ_f and K_f are positive constants which may depend on f.

Denote by $\mathcal{H}_\alpha^h(\mu_f, K_f)$ the subclass of $\mathcal{H}_\alpha^h$ which is obtained by fixing the values of μ_f and K_f independently of f.

In the sequel we make use of properties ([4], p. 17) of n-dimensional singular integral

$$\Phi(x, u) = \int_\Omega N(x - y, u) f(y) dy, \tag{8}$$

where $f(y) \in \mathcal{H}_\alpha^h$.

The aim of this paper is to give a proof for the existence of a solution of a system of nonlinear n-dimensional singular integral equations, that system being of arbitrary power.

If the system is finite (see [4], p. 33) or countable (see [5]), we can apply one of the hitherto used methods, consisting in considering the Cartesian product $\prod_{k=1}^m \Lambda_k$ or $\prod_{k=1}^\infty \Lambda_k$, respectively, of the metric spaces $\Lambda = \Lambda_1 = \Lambda_2 = \ldots$ with a natural metric induced by the metric in Λ.

However, if the system is uncountable, this method fails, since an uncountable Cartesian product of metric spaces is not, in general, metric space. In this situation we shall use a topological method based on more general theorem of Schauder-Tikhonov ([1], p.227). As far as author knows, uncountable system of singular integral equations in R^n has not yet been examined.

2. Statement of the problem

Consider in the set Ω a system of nonlinear integral equations

$$(9)\qquad \varphi_\nu(x) = f_\nu(x) + \int_\Omega N_\nu(x-y,x)\Phi_\nu[x,y,\{\varphi_\gamma(y)\}_{\gamma\in T}]dy, \qquad \nu\in T,$$

with unknown functions $\{\varphi_\gamma(x)\}_{\gamma\in T}$, where T is an arbitrary set of indices.

We make the following assumptions:

I. Ω is defined as in Section 1.

II. $N_\nu(x,u)$ is a complex function defined as follows in the domain $\{(x,u) : x\in R^n - \{0\},\ u\in\Omega\}$

$$(10)\qquad N_\nu(x,u) = \frac{F_\nu(x',u)}{|x|^n}, \qquad (x = x'|x|), \qquad \nu\in T,$$

where $F_\nu(x',u)$ is defined in the domain $\{(x',u) : x'\in\omega,\ u\in\Omega\}$ and satisfies for $\nu\in T$ the conditions

$$(11)\qquad |F_\nu(x',u) - F_\nu(\widetilde{x}',\widetilde{u})| \le K_N[|x'-\widetilde{x}'|^{h_N} + |u-\widetilde{u}|^{h_1}],$$
$$u,\widetilde{u}\in\Omega_i,\ i = 1,2,\ldots,p+1,$$

$$(12)\qquad \int_\omega F_\nu(x',\cdot)dx' = 0.$$

III. Complex functions $f_\nu(x), x\in\Omega, \nu\in T$, satisfy the conditions

$$(13)\qquad \begin{cases} |f_\nu(x)| \le \dfrac{M_f}{|x-x_s|^\alpha}, \\ |f_\nu(x) - f(\widetilde{x})| \le \dfrac{K_f|x-\widetilde{x}|^h}{|x-x_s|^{\alpha+h}}, \end{cases}$$

where $|x-x_s| \le |\widetilde{x}-\widetilde{x}_s|$.

IV. Complex functions $\Phi_\nu(x,y,\{u_\nu\}_{\nu\in T})$, $\nu\in T$, are defined in the domain $\{(x,y,u_\nu) : x\in\Omega,\ y\in\Omega,\ u_\nu\in\Pi\}$, where Π denotes here a plane of complex variable, and

$$(14)\qquad |\Phi_\nu(x,y,\{u_\gamma\}_{\gamma\in T})| \le \frac{M'}{|y-y_s|^\alpha} + M\sup_{\nu\in T}|u_\nu|, \qquad \nu\in T,$$

$$(15)\qquad |\Phi_\nu(x,y,\{u_\gamma\}_{\gamma\in T}) - \Phi_\nu(\widetilde{x},\widetilde{y},\{\widetilde{u}_\gamma\}_{\gamma\in T})|$$
$$\le \frac{K'[|x-\widetilde{x}|^{h_1} + |y-\widetilde{y}|^h]}{|y-y_s|^{\alpha+h}} + K\sup_{\nu\in T}|u-\widetilde{u}|, \qquad \nu\in T,$$

where

$$(16)\qquad 0<h<h_1\le 1, \quad h\le h_N\le 1, \quad \alpha>0, \quad \alpha+h<1,$$

M', M, K', K are positive constants; x and $\widetilde{x}$ are arbitrary points situated in one of the regions $\Omega_1,\Omega_2,\ldots,\Omega_{p+1}$ (both in the same). Moreover, we assume that $|x - x_s| \leq |\tilde{x} - \tilde{x}_s|$. The same assumption we made with respect to pair of the points y,$\tilde{y}$.

3. Proof of the existence of a solution of (9)

Consider a set $C(\Omega)$ consisting of all complex functions $\varphi(x)$ defined in the set Ω, continuous in every region Ω_j, $j = 1, 2, \ldots, p+1$, and satisfying the condition

$$\sup_{\Omega} |x - x_s|^{\alpha+h} |\varphi(x)| < \infty, \tag{18}$$

where α and h are constants appearing in the assumptions (14), (15).

In the usual way we define the sum of two elements of this set and the product of an element of this set and a real number. Hence the set $C(\Omega)$ is a linear space. Consider the set $\Lambda = \prod_{\nu \in T} \Lambda_\nu$ being a Cartesian product of the spaces Λ_ν, where $\Lambda_\nu = C(\Omega)$ for each $\nu \in$T. The set Λ consists of all systems $\{\varphi_\nu(x)\}_{\nu \in T}$ of complex functions, defined and continuous on Ω.

In the space Λ_ν, $\nu \in T$, we define the norm $\|\varphi(x)\|$ of its point $\varphi(x)$ by the formula

$$\|\varphi(x)\| = \sup_{x \in \Omega} [|x - x_s|^{\alpha+h} |\varphi(x)|]. \tag{19}$$

The space $\Lambda_\nu, \nu \in T$, is a locally convex Hausdorff space (see e.g. [1], p. 116).

We define a topology in the Cartesian product Λ by taking as the basis of open sets all the sets of the form $\prod_{\nu \in T} D_\nu$, where D_ν is an open set in Λ_ν and $D_\nu = \Lambda_\nu$ for almost all ν (see e.g. [2], p.252). The so-defined linear topological space Λ is a locally convex Hausdorff space (see [1], pp.31, 116).

For every $\nu \in T$ let Z_ν denote the subset of all points in the space Λ_ν satisfying the conditions

$$|x - x_s|^{\alpha} |\varphi(x)| \leq R, \tag{20}$$

$$|x - x_s|^{\alpha+h} |\varphi(x) - \varphi(\tilde{x})| \leq R |x - \tilde{x}|^h, \tag{21}$$

where R is a positive constant, x and $\tilde{x}$ are situated in one of the regions $\Omega_1, \Omega_2, \ldots, \Omega_{p+1}$(both in the same) and satisfy the condition $|x - x_s| \leq |\tilde{x} - \tilde{x}_s|$. Obviously, the set Z_ν, $\nu \in T$, is convex.

Let

$$F(x) = \begin{cases} |x - x_s|^{\alpha+h} \varphi(x), & x \in \Omega, \\ 0, & x \in \sum_{k=0}^{p} S_k, \end{cases}$$

where $\varphi(x) \in \mathcal{H}_\alpha^h$. The so defined function $F(x)$ is uniformly bounded and uniformly continuous in a bounded domain $\overline{\Omega}$. So the set Z_ν, $\nu \in T$, is compact by virtue of Arzela theorem.

Consider in the space Λ the set $Z = \prod_{\nu \in T} Z_\nu$. This set consists of all systems $\{\varphi_\nu(x)\}_{\nu \in T}$ of functions satisfying for every $\nu \in T$ conditions (20) and (21). The set Z, being a Cartesian product of convex, compact sets, is itself convex and compact. (cf. e.g. [2], pp. 158–159, 250–252).

Basing on (9) we define on the set Z the following transformation

$$\psi_\nu(x) = f_\nu(x) + \int_\Omega N_\nu(x-y,x)\Phi_\nu[x,y,\{\varphi_\alpha(y)\}_{\alpha\in T}]dy, \qquad \nu \in T, \tag{22}$$

which associates with each point $\{\varphi_\nu(x)\}_{\nu\in T}$ of the set Z a point $\{\psi_\nu(x)\}_{\nu\in T}$ in the space Λ. Now we shall show that the constant R in the assumptions (20), (21) can be chosen in such a way that the operation (22) would transform the set Z into itself.

Note that, because of the assumptions (14), (15) and the conditions (20), (21), we have

$$|y-y_s|^\alpha|\Phi_\nu[x,y,\{\varphi_\nu(y)\}_{\nu\in T}]| \le M' + MR, \qquad \nu \in T, \tag{23}$$

$$\begin{aligned}|y-y_s|^{\alpha+h}|\Phi_\nu[x,y,\{\varphi_\gamma(y)\}_{\gamma\in T}] - \Phi_\nu[x,y,\{\varphi_\gamma(\overline{y})\}_{\gamma\in T}]| \le \\ \le (K'+KR)[|x-\overline{x}|^h + |y-\overline{y}|^h], \qquad \nu\in T,\end{aligned} \tag{24}$$

where y and $\overline{y}$ (x and $\overline{x}$) are arbitrary points situated in any of the regions $\Omega_1, \Omega_2, \ldots, \Omega_{p+1}$(both in the same), satisfying the condition $|y-y_s| \le |\overline{y}-\overline{y}_s|$ ($|x-x_s| \le |\overline{x}-\overline{x}_s|$). Thus, basing on (13), (23), (24) and the above mentioned theorem of Pogorzelski, we have

$$|x-x_s|^\alpha|\psi_\nu(x)| \le C_1(M'+MR) + C_2(K'+KR) + M_f, \qquad \nu \in T, \tag{25}$$

and

$$\begin{aligned}&|x-x_s|^{\alpha+h}|\psi_\nu(x) - \psi_\nu(\overline{x})| \\ &\qquad \le [C_1'(M'+MR) + C_2'(K'+KR) + K_f]|x-\overline{x}|^h, \qquad \nu\in T,\end{aligned} \tag{26}$$

where $|x-x_s| \le |\overline{x}-\overline{x}_s|$ and C_1, C_2, C_1', C_2' are positive constants independent of f_ν, Φ_ν, N_ν.

We see that the operation (22) transforms the set Z into itself, if the constant R from (20) and (21) satisfies the inequalities

$$\begin{cases} M_f + C_1(M'+MR) + C_2(K'+KR) \le R, \\ K_f + C_1'(M'+MR) + C_2'(K'+KR) \le R. \end{cases} \tag{27}$$

Simple calculation leads to the conclusion that if

$$\begin{cases} C_1'M + C_2'K < 1, \\ C_1M + C_2K < 1, \end{cases} \tag{28}$$

then we can choose R in such a way that the system (27) holds. Namely, we can take

$$(29) \qquad R = \max\left(\frac{M_f + C_1 M' + C_2 K'}{1 - C_1 M - C_2 K}, \frac{K_f + C_1' M' + C_2' K'}{1 - C_1' M - C_2' K}\right).$$

Hence, if the constants of the problem satisfy the conditions (28), then with R defined by (29) the operation (22) transforms the set Z into itself.

Now we shall show that the transformation (22) is continuous in the space Λ. Let M^* denote a directed set ([2], p. 150) and T_1 be any finite subset of T. Consider an arbitrary net $U^{(m)} = \{\varphi_\nu^{(m)}(x)\}_{\nu \in T}$, $m \in M^*$, of point of the set Z, convergent to a point $U = \{\varphi_\nu(x)\}_{\nu \in T}$. We have to prove that the net of transformated points $\overline{U}^{(m)} = \{\psi_\nu^{(m)}(x)\}_{\nu \in T}$ is convergent to the point $\overline{U} = \{\psi_\nu(x)\}_{\nu \in T}$, being the image of U under the transformation (22). Accordingly, it suffices to show that for every neighbourhood W of the point $\overline{U}$ there exists $m_0 \in M^*$ such that $\overline{U}^{(m)} \in W$, for all $m \succ m_0$ ($\succ$ denotes here the order relation in the set M^*). Since the neighbourhood W is an open set, we can assume that it is of the form $\prod_{\nu \in T} W_\nu$, where W_ν is an open set in Λ_ν and $W_\nu = \Lambda_\nu$ for almost all ν. Hence it suffices to check that for every $\nu \in T_1$ the net $\psi_\nu^{(m)}(x)$ is convergent to the point $\psi_\nu(x)$. For a fixed ν convergence of the net $\psi_\nu^{(m)}(x)$ to the point $\psi_\nu(x)$ is understood in the sense of the metric in the space Λ_ν, i.e.

$$(30) \quad \lim_m \|\overline{U}^{(m)}(x) - \overline{U}(x)\| = \lim_m \sup_\Omega |x - x_s|^{\alpha+h} |\psi_\nu^{(m)}(x) - \psi_\nu(x)| = 0, \qquad \nu \in T.$$

Consider the difference

$$(31) \quad \psi_\nu^{(m)}(x) - \psi_\nu(x) = \int_\Omega N_\nu(x-y,x)\Phi_\nu[x,y,\{\varphi_\nu^{(m)}(y)\}_{\nu \in T}]dy - \int_\Omega N_\nu(x-y,x)\Phi_\nu[x,y,\{\varphi_\nu(y)\}_{\nu \in T}]dy, \qquad \nu \in T, \qquad m \in M^*.$$

To estimate (31) let us consider a neighbourhood $\widetilde{K}(x, r_\epsilon)$ of the point $x \in \Omega$ and of radius $r_\epsilon > 0$. Let

$$(32) \qquad \psi_\nu^{(m)}(x) - \psi_\nu(x) = I_\nu^K(x) + I_\nu^{\Omega - K}(x), \qquad \nu \in T, \qquad m \in M^*,$$

where $K = \Omega \cap \widetilde{K}(x, r_\epsilon)$. If $|x - x_s| \geq r_\epsilon$, then, by (12),

(33) $I_\nu^K(x)$

$$= \int_{\widetilde{K}} N_\nu(x-y,x)\{\Phi_\nu[x,y,\{\varphi_\nu^{(m)}(y)\}_{\nu\in T}] - \Phi_\nu[x,x,\{\varphi_\nu^{(m)}(x)_{\nu\in T}\}]\}dy$$

$$- \int_{\widetilde{K}} N_\nu(x-y,x)\{\Phi_\nu[x,y,\{\varphi_\nu(y)\}_{\nu\in T}] - \Phi_\nu[x,x,\{\varphi_\nu(x)\}_{\nu\in T}]\}dy.$$

Hence, by (15) and [4], p. 37, we can state that for every $\epsilon > 0$ there exists such a radius r_ϵ that

$$|x - x_s|^{\alpha+h}|I_\nu^K(x)| < \frac{\epsilon}{2}, \quad \nu \in T. \tag{34}$$

If $|x - x_s| < r_\epsilon$, then we have

(35) $I_\nu^K(x)$

$$= \int_{K_s} N_\nu(x-y,x)\{\Phi_\nu[x,y,\{\varphi_\nu^{(m)}(y)\}_{\nu\in T}] - \Phi_\nu[x,x,\{\varphi_\nu^{(m)}(x)\}_{\nu\in T}]\}dy$$

$$+ \int_{K_s} N_\nu(x-y,x)\{\Phi_\nu[x,y,\{\varphi_\nu(y)\}_{\nu\in T}] - \Phi_\nu[x,x,\{\varphi_\nu(x)\}_{\nu\in T}]\}dy$$

$$+ \int_{K-K_s} N_\nu(x-y,x)\{\Phi_\nu[x,y,\{\varphi_\nu^{(m)}(y)\}_{\nu\in T}] - \Phi_\nu[x,y,\{\varphi_\nu(y)\}_{\nu\in T}]\}dy,$$

where K_s denotes a sphere with center x and radius $|x - x_s|$. Basing on the investigations given in [4] (pp. 11, 37), we can state that in this case also we have

$$|x - x_s|^{\alpha+h}|I_\nu^K(x)| < \frac{\epsilon}{2}, \quad \nu \in T. \tag{36}$$

Now, by (15), we have

(37) $|I_\nu^{\Omega-K}(x)|$

$$\le \sup_{x\in\Omega}\sup_{\nu\in T}[|y-y_s|^{\alpha+h}|\varphi_\nu^{(m)}(y) - \varphi_\nu(y)| \int_{\Omega-K} \frac{KR}{|x-y|^h|y-y_s|^{\alpha+h}}dy.$$

By the uniform convergence of the net $U^{(m)} = \{\varphi_\nu^{(m)}(x)\}$ to the point $U = \{\varphi_\nu(x)\}$, $\nu \in T$, and the estimation of the last integral given in [4] (p. 11), we state that for every $\epsilon > 0$ there exists $m' \in M^*$ such that

$$\sup_{x\in\Omega}[|x - x_s|^{\alpha+h} I_\nu^{\Omega-K}(x)| < \frac{\epsilon}{2}, \quad \text{if } m \succ m'. \tag{38}$$

Combining (36) and (38), we have the result

$$\lim_m \sup_{x\in\Omega}[|x - x_s|^{\alpha+h}|\psi_\nu^{(m)} - \psi_\nu(x)| = 0, \quad \nu \in T,$$

which completes the proof of continuity of the transformation (22).

Hence all assumptions of Schauder–Tikhonov theorem are satisfied. By this theorem, the system (9) has at least one solution $U^*(x) = \{\varphi_\nu^*(x)\}_{\nu\in T}$ in the set Z. So we have the following result.

THEOREM. *If assumptions* I-IV *and condition* (27) *hold, then there exists a system of functions* $\{\varphi_\nu(x)\}_{\nu\in T}$, *being a solution of system* (9) *and satisfying conditions* (20), (21) *with* R *given by* (29).

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