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# ON THE STRONG COMPONENTWISE STABILITY AND H-MATRICES

## 1. Introduction

An algorithm for solving a system of linear equations is said to be numerically stable if a solution $\widetilde{x}$, obtained by that method, satisfy a relation $(A+E)\widetilde{x}=b$, where $\|E\|$ is of order $\varepsilon\,\|A\|$, $\varepsilon$ is the relative computer precision. If $E=[e_{ij}]$, $A=[a_{ij}]$ and $|e_{ij}|$ are of order $\varepsilon|a_{ij}|$, then an algorithm is numerically stable in a componentwise sense. Such problems are considered in [2], [1], [11] etc.

An algorithm for solving linear equations is strongly stable for a class of matrices $C$ if for each $A\in C$, the computed solution to $Ax=b$ satisfies $\widetilde{A}\widetilde{x}=b$, where $\widetilde{A}\in C$ and $\widetilde{A}$ is close to $A$.

Bunch, Demmel and van Loan, [3], show that any stable algorithm on the class of nonsingular symmetric matrices is also strongly stable on the same matrix class. Smoktunowicz [11], considers the class of diagonally dominant symmetric matrices and obtain the strong componentwise stability.

In this paper we consider the class of symmetric H-matrices, which includes the class of symmetric diagonally dominant matrices, investigated by Smoktunowicz, and show that if an algorithm is stable for some matrix from that class it is also strongly stable in a componentwise sense. Motivation for such investigation lies in the fact that systems of linear equation with an H-matrix arise frequently in practise. Also, some well-known algorithms are stable for some subclasses of H-matrices, for example Gaussian elimination without pivoting (LU decomposition) on column diagonally dominant matrices etc. For discussion about stability one can see [2], [3], [9] and referenced cited there.

## 2. H-matrices

For $A=[a_{ij}]$ let $M(A)$ denote its modular matrix, i.e. $M(A)=[m_{ij}]$,

$$m_{ij} = \begin{cases} |a_{ii}|, & i = j, \\ -|a_{ij}|, & i \neq j. \end{cases}$$

DEFINITION 1. Matrix $A \in R^{n,n}$ is said to be an M-matrix if $A$ is regular, $a_{ij} \leq 0$, $i \neq j$ and $A^{-1} \geq 0$.

DEFINITION 2. Matrix $A \in C^{n,n}$ is said to be an H-matrix if $M(A)$ is an M-matrix.

DEFINITION 3. Matrix $A = [a_{ij}]$ is an SDD (strictly diagonally dominant) matrix if

$$|a_{ii}| > \sum_{j \neq i} |a_{ij}|, \quad i = 1, \ldots n.$$

THEOREM 1 [4]. *Matrix $A$ is an H-matrix if and only if there exists a regular diagonal matrix $W$ such that $AW$ is an SDD matrix.*

Let $N = \{1, 2, \ldots, n\}$, $N(i) = N \setminus \{i\}$, and for $A = [a_{ij}]$, let

$$P_i(A) = \sum_{j \neq i} |a_{ij}|, \quad P_i'(A) = \sum_{j=1}^{i-1} |a_{ij}|, \quad Q_i(A) = \max_{j \in N(i)} |a_{ij}|, \quad i \in N.$$

DEFINITION 4. Matrix $A \in C^{n,n}$ is called lower semistrictly diagonally dominant if and only if

$$|a_{ii}| \geq P_i(A), \quad i \in N,$$
$$|a_{ii}| > P_i'(A), \quad i \in N.$$

For some subclasses of H-matrices, matrix $W$ such that $AW$ is an SDD matrix can be found as in the next theorem.

THEOREM 2 [7]. *Let one of the following conditions for a matrix $A$ and a diagonal matrix $W = diag(w_1, w_2, \ldots w_n)$ be satisfied:*

(i) *$A$ is a lower semistrictly diagonally dominant and*

$$1 > w_n > \frac{P_n'(A)}{a_{nn}},$$
$$1 > w_i > \frac{P_i'(A) + \sum_{j=i+1}^{n} w_j |a_{ij}|}{a_{ii}}, \quad i = n-1, \ldots, 1;$$

(ii) *$A$ is not an SDD matrix and*

$$a_{ii} > 0, \quad a_{ii}a_{jj} > P_i(A)P_j(A), \quad i \in N, \quad j \in N(i).$$

*Then there exists exactly one $p \in N$ such that $a_{pp} \leq P_p(A)$. Let*

$$w_i = 1, \quad i \in N(p),$$

$$w_p > \frac{P_p(A)}{a_{pp}}, \quad a_{ip} = 0, \ i \in N(p),$$

*or*

$$w_p \in \left( \frac{P_p(A)}{a_{pp}}, \ 1 + \min \left\{ \frac{a_{ii} - P_i(A)}{|a_{ip}|}; \quad i \in N(p), \quad a_{ip} \neq 0 \right\} \right);$$

(iii) *There exists $i \in N$ such that*

$$a_{ii} \left( a_{jj} - P_j(A) + |a_{ji}| \right) > P_i(A) |a_{ji}|, \ j \in N(i),$$

*and $w_j = 1$, $j \in N(i)$,*

$$w_i > \frac{P_i(A)}{a_{ii}}, \quad \text{if} \quad a_{ji} = 0, \quad j \in N(i),$$

*or*

$$w_i \in \left( \frac{P_i(A)}{a_{ii}}, \ 1 + \min \left\{ \frac{a_{jj} - P_j(A)}{|a_{ji}|}; \quad j \in N(i), \quad a_{ji} \neq 0 \right\} \right).$$

*Then the matrix $AW$ is an SDD matrix.*

## 3. Strong stability for symmetric H-matrices

Using a simple generalization of technique given in [11] we obtain more general result given in the next theorem.

THEOREM 1. *Let $A$ be a symmetric H-matrix, $W = \operatorname{diag}(w_1, w_2, \ldots, w_n)$ regular diagonal matrix such that $AW$ is an SDD matrix and*

$$|a_{ii}||w_i| (1 - 3\varepsilon) \geq (1 + \varepsilon) \sum_{j \neq i} |a_{ij}||w_j|.$$

*If $(A + E)\tilde{x} = b$, where $|E| \leq \varepsilon |A|$ and $\tilde{x} \neq 0$, then there exists a matrix $F = F^T$ such that $(A + F)\tilde{x} = b$, $|F| \leq 3\varepsilon |A|$ and $A + F$ is a symmetric H-matrix.*

Proof. Let us introduce vector $\tilde{y}$ as

$$\tilde{x} = W\tilde{y}.$$

Then, the system $(A + E)\tilde{x} = b$ becomes

$$(AW + EW)\tilde{y} = b,$$

where $AW$ is an SDD matrix.

Case 1:

$$|\widetilde{y}_1| \le |\widetilde{y}_2| \le \ldots \le |\widetilde{y}_n|.$$

For the matrix $F = [f_{ij}]$ we can choose

$$f_{11} = e_{11},$$
$$f_{ij} = e_{ij}, \quad i = 1, 2, \ldots, n, \quad j = i+1, \ldots, n,$$

and

$$f_{ii} = e_{ii} + \sum_{j=1}^{i-1} (e_{ij} - e_{ji}) \frac{w_j \widetilde{y}_j}{w_i \widetilde{y}_i}.$$

If $\widetilde{y}_i = 0$ then $\widetilde{y}_1 = \widetilde{y}_2 = \ldots = \widetilde{y}_{i-1} = 0$ and $f_{ii} = 0$. Otherwise

$$f_{ii}\widetilde{x}_i = e_{ii}\widetilde{x}_i + \sum_{j=1}^{i-1} (e_{ij} - e_{ji}) \widetilde{x}_j,$$

so, in both cases equality $E\widetilde{x} = F\widetilde{x}$ is satisfied.

Case 2: Let $P$ be a permutation matrix such that

$$P\widetilde{y}^T = [\widetilde{y}_{p1}, \widetilde{y}_{p2}, \ldots, \widetilde{y}_{pn}]^T$$

and

$$|\widetilde{y}_{p1}| \le |\widetilde{y}_{p2}| \le \ldots |\widetilde{y}_{pn}|.$$

Let $\widehat{x} = P\widetilde{x} = PW\widetilde{y}$, $\widehat{A} = PAP^T$, $\widehat{E} = PEP^T$, $\widehat{b} = Pb$. Matrix $\widehat{A}$ is a symmetric H-matrix and

$$\left(\widehat{A} + \widehat{E}\right)\widehat{x} = \widehat{b}.$$

From the previous case we can conclude that there exists a symmetric matrix $\widehat{F}$ such that $\left(\widehat{A} + \widehat{F}\right)\widehat{x} = \widehat{b}$ and $\left|\widehat{F}\right| \le 3\varepsilon \left|\widehat{A}\right|$. If we take $F = P^T \widehat{F} P$, then $F$ is a symmetric and $(A + F)\widetilde{x} = b$. Also,

$$|F| = \left|P^T \widehat{F} P\right| \le 3\varepsilon\, |P|^T \left|\widehat{A}\right| |P| = 3\varepsilon\, |A|.$$

So, we have proved that there exists a symmetric matrix $F$ with the property $|F| \le 3\varepsilon\, |A|$ which satisfies

$$(A + F)\widetilde{x} = b.$$

To complete the proof, we have to show that $A + F$ is an H-matrix. It is enough to prove that $(A + F)W$ is an SDD matrix

$$\sum_{j\neq i} |a_{ij} + f_{ij}|\,|w_j| \leq \sum_{j\neq i} (|a_{ij}| + |f_{ij}|)\,|w_j| \leq (1+\varepsilon)\sum_{j\neq i} |a_{ij}|\,|w_j|\,.$$

From the inequality given in the theorem we have

$$(1+\varepsilon)\sum_{j\neq i} |a_{ij}|\,|w_j| \leq (1-3\varepsilon)\,|a_{ii}|\,|w_i|\,.$$

As $|f_{ii}| \leq 3\varepsilon\,|a_{ii}|$, we have

$$\sum_{j\neq i} |a_{ij} + f_{ij}|\,|w_j| \leq (|a_{ii}| - |f_{ii}|)\,|w_i| \leq |a_{ii} + f_{ii}|\,|w_i|\,,$$

which means that $(A+F)\,W$ is an SDD matrix and the theorem is proved.

In the paper [11] inequality

$$|a_{ii}||w_i|\,(1-3\varepsilon) \geq (1+\varepsilon)\sum_{j\neq i} |a_{ij}||w_j|.$$

is replaced with the inequality

$$|a_{ii}|\,(1-3\varepsilon) \geq (1+3\varepsilon)\sum_{j\neq i} |a_{ij}|,$$

for a SDD matrix $A$. As $\varepsilon$ is supposed to be a relative computer precision and for any SDD matrix $AW$ we have

$$|a_{ii}||w_i| > \sum_{j\neq i} |a_{ij}||w_j|,$$

neither of these conditions doesn't seem too restrictive.

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