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DUAL OF NON-COMMUTATIVE $L^{p\infty}$-SPACES WITH $0 < p \le 1$

1. Preliminaries

Let τ be a faithful normal semifinite trace on a semifinite von Neumann algebra $\mathcal{M}$. Under the strong sum, strong product, and the adjoint operation, the set of all closed densely defined τ-measurable operators affiliated with $\mathcal{M}$ forms a *-algebra $\overline{\mathcal{M}}$ [7]. If $a \in \overline{\mathcal{M}}$, the function

$$a(\cdot) : (0,\infty) \mapsto [0,\infty),$$

defined by setting

$$a(t) = \inf\{s \ge 0 : \tau(e_s^{\perp}) \le t\}, \qquad t > 0$$

where $|a| = (a^*a)^{\frac{1}{2}} = \int_0^\infty s\, de_s$ is the spectral decomposition of $|a|$, is called the decreasing rearrangement of a. We define $\overline{\mathcal{M}}_0 = \{a \in \overline{\mathcal{M}} : a(t) \to 0$ as $t \to \infty\}$.

DEFINITION 1. For $0 < p < \infty$, $a \in \overline{\mathcal{M}}$, we define

$$\|a\|_{p\infty} = \sup\{t^{\frac{1}{p}} a(t) : t > 0\}.$$

The non-commutative $L^{p\infty}(M) = L^{p\infty}$-space (non-commutative weak L^p-space) is the collection of all $a \in \overline{\mathcal{M}}_0$ such that

$$\|a\|_{p\infty} < \infty.$$

$L^{p\infty}$ is a quasi-Banach space (see [2], Th. 2.1).

2. Dual of non-commutative $L^{p\infty}$-spaces with $0 < p < 1$

THEOREM 1. *For* $0 < p < 1$,

$$L^{p\infty}(\mathcal{M})^* = \{0\}$$

if $\mathcal{M}$ *has no minimal projection.*

P r o o f. Suppose that h is a nonzero continuous linear functional on $L^{p\infty}$. We take a nonzero selfadjoint element $a \in L^{p\infty}$ such that $h(a) \neq 0$. Then there exist a commutative von Neumann subalgebra $\mathcal{N}$ of $\mathcal{M}$ such that $a \in \overline{\mathcal{N}}_0$ and a *-isomorphism U from $\overline{\mathcal{N}}$ onto $\overline{\mathcal{A}}$ where $\mathcal{A} = L^\infty(0, \tau(\operatorname{supp}|a|))$ such that $(Ub)(t) = b(t)$ for any $t > 0$ and $b \in \overline{\mathcal{N}}$ (see [1], Lemma 1.3). The restriction of the *-isomorphism U to $L^{p\infty}(\mathcal{N})$ is an isometry onto $L^{p\infty}(\mathcal{A})$. Moreover, $h(U^{-1}\cdot) \in L^{p\infty}(\mathcal{A})^*$, $h(U^{-1}Ua) = h(a) \neq 0$, which is a contradiction since $L^{p\infty}(\mathcal{A})^* = \{0\}$ [3]. ∎

Suppose that there exists a minimal projection e in $\mathcal{M}$. Putting $h(e)$ as an arbitrary non-zero complex number, we define $h(\lambda e) = \lambda h(e)$, $\lambda \in \mathbb{C}$ and $h(a) = h(eae)$, $a \in L^{p\infty}$. Then h is a non-zero linear functional on $L^{p\infty}$. Since $|h(a) - h(a_n)| = |h(e(a - a_n)e)| = |\lambda_n||h(e)|$ and $|\lambda_n|\tau(e)^{\frac{1}{p}} = |\lambda_n|\|e\|_{p\infty} = \|e(a - a_n)e\|_{p\infty} \le \|a - a_n\|_{p\infty}$, h is continuous.

Let $\mathcal{H}$ be an arbitrary Hilbert space over the field $\mathbb{C}$ of complex numbers $\mathcal{B}(\mathcal{H})$ the von Neumann algebra of all bounded linear operators from $\mathcal{H}$ into $\mathcal{H}$. To show finally that $L^{p\infty}(\mathcal{B}(\mathcal{H}))^* = \mathcal{B}(\mathcal{H})$ is a simple deduction from the observation that

$$\operatorname{tr}(|a|) = \int_0^\infty a(t)\,dt = \sum_{n=0}^\infty \int_n^{n+1} a(t)\,dt \le \|a\| + \|a\|_{p\infty} \sum_{n=1}^\infty n^{-\frac{1}{p}}$$
$$\le \|a\|_{p\infty}\left(1 + \zeta\left(\frac{1}{p}\right)\right)$$

and from the fact that the Banach envelope of $L^{p\infty}(\mathcal{B}(\mathcal{H}))$ is isometric to $L^1(\mathcal{B}(\mathcal{H}))$ — the ideal of trace class operators equipped with the usual trace norm (see [6], [9]). Any linear continuous functional h on $L^{p\infty}(\mathcal{B}(\mathcal{H}))$ is of the form $h(a) = \operatorname{tr}(ab) = \operatorname{tr}(ba)$ for some $b \in \mathcal{B}(\mathcal{H})$.

R e m a r k 1. The space $F(\mathcal{H})$ of all finite-dimensional operators in $\mathcal{B}(\mathcal{H})$ is not dense in $L^{p\infty}(\mathcal{B}(\mathcal{H}))$ (see [2], Prop. 2.7). However, using the above represenation of $h \in L^{p\infty}(\mathcal{B}(\mathcal{H}))^*$, we see that $F(\mathcal{H})$ is dense in $L^{p\infty}(\mathcal{B}(\mathcal{H}))$ in its weak topology.

3. Dual of non-commutative $L^{1\infty}$-space

DEFINITION 2 (see [2], [4]). For $0 < r \le 1$, $a \in \overline{\mathcal{M}}$, we define

$$a(t, r) = \begin{cases} \sup\{\tau(p)^{-\frac{1}{r}}\|pa\|_r : t \le \tau(p) < \infty\}, & 0 < t \le \tau(1), \\ 0, & t > \tau(1), \end{cases}$$

where

$$\|pa\|_r = \tau(|pa|^r)^{\frac{1}{r}} = \left\{ \int_0^\infty (pa)^r(t)\,dt \right\}^{\frac{1}{r}}.$$

DEFINITION 3 (see [2], [4]). For any $a \in \overline{\mathcal{M}}$ and $0 < r \le 1$, we define

$$\tilde{a}(t,r) = \left\{\frac{1}{t}\int_0^t a^r(s)\,ds\right\}^{\frac{1}{r}}, \quad t > 0.$$

It is not hard to prove the following inequalities ([2], Prop. 2.3)

$$a(t) \le a(t,r) \le \tilde{a}(t,r), \quad t > 0,\ a \in \overline{\mathcal{M}}_0.$$

We define four continuous non-trivial semi-norms $N_0(\tilde{N}_0)$, $N_\infty(\tilde{N}_\infty)$ on $L^{1\infty}$ by

DEFINITION 4 (cf. [4]). For $a \in L^{1\infty}$, we define

(i) $N_0(a) = \limsup_{t\to 0} t^2 a(t, 1-t)$,

(ii) $N_\infty(a) = \limsup_{t\to\infty} a\big(t, 1 - \frac{1}{t}\big)$,

(iii) $\tilde{N}_0(a) = \limsup_{t\to 0} t^2 \tilde{a}(t, 1-t)$,

(iv) $\tilde{N}_\infty(a) = \limsup_{t\to\infty} \tilde{a}\big(t, 1 - \frac{1}{t}\big)$.

For $a, b \in \overline{\mathcal{M}}_0$, $0 < r < 1$, we have ([5], 4.9, th. 4.7)

$$\|p(a+b)\|_r \le 2^{\frac{1}{r}-1}(\|pa\|_r + \|pb\|_r)$$

and

$$\widetilde{(a+b)}(t,r) \le 2^{\frac{1}{r}-1}(\tilde{a}(t,r) + \tilde{b}(t,r)).$$

As a consequence, we obtain the subadditivity of $N_0(N_\infty)$, $\tilde{N}_0(\tilde{N}_\infty)$. The homogeneity is also satisfied.

Since

$$\begin{aligned} t^2 a(t, 1-t) &\le t^2 \tilde{a}(t, 1-t) = t^2\left\{\frac{1}{t}\int_0^t (a(s))^{1-t}\,ds\right\}^{\frac{1}{1-t}} \\ &\le \|a\|_{1\infty} t^2 \left\{\frac{1}{t}\int_0^t s^{t-1}\,ds\right\}^{\frac{1}{1-t}} = \|a\|_{1\infty} t^{\frac{-t}{1-t}}, \end{aligned}$$

$$\begin{aligned} a\left(t, 1-\frac{1}{t}\right) &\le \tilde{a}\left(t, 1-\frac{1}{t}\right) = \left\{\frac{1}{t}\int_0^t (a(s))^{1-\frac{1}{t}}\,ds\right\}^{\frac{t}{t-1}} \\ &\le \|a\|_{1\infty}\left\{\frac{1}{t}\int_0^t s^{\frac{1}{t}-1}\,ds\right\}^{\frac{t}{t-1}} = \|a\|_{1\infty} t^{\frac{1}{t-1}}. \end{aligned}$$

We have continuity of $N_0(\tilde{N}_0)$ and $N_\infty(\tilde{N}_\infty)$ in $L^{1\infty}$. The seminorms $N_0(\tilde{N}_0)$ are non-trivial (see [4] and Remark 3). If $\tau(1) = \infty$, then $N_\infty(\tilde{N}_\infty)$ are also non-trivial (see Remark 3).

An important application of the above considerations is contained in the following

THEOREM 2. *Dual of the space $L^{1\infty}(\mathcal{M})$ is non-trivial.*

Proof. By the Hahn-Banach theorem and the above constructions of $N_0(\widetilde{N}_0)$, $N_\infty(\widetilde{N}_\infty)$.

Remark 2. If h is a continuous linear functional on $L^{1\infty}(\mathcal{M})$, then there exists a unique element $b \in \mathcal{M}+L^1(\mathcal{M})$ such that $h(a) = \tau(ba) = \tau(ab)$ for

$$a \in \overline{F_\tau(\mathcal{M})} = \{a \in L^{1\infty}(\mathcal{M}) : \lim ta(t) = 0 \text{ as } t \to 0 \text{ or } t \to \infty\}$$

(see [2], [10]). It is clear that $\widetilde{N}_0(a) = 0$, $\widetilde{N}_\infty(a) = 0$, $a \in \overline{F_\tau(\mathcal{M})}$.

4. Haagerup's $L^p(\mathcal{M})$-spaces and weak L^p-spaces

We now assume that $\mathcal{M}$ is a general von Neumann algebra (not necessarily semi-finite) and let $\mathcal{N}$ be the crossed product of $\mathcal{M}$ by the modular automorphism group of a fixed weight on $\mathcal{M}$. The Haagerup's $L^p(\mathcal{M})$-spaces are contained in $L^{p\infty}(\mathcal{N})$, $0 < p < \infty$, and we have

$$a(t) = \frac{1}{t^{\frac{1}{p}}}\|a\|_p, \qquad t > 0,\ a \in L^p(\mathcal{M}),$$

$\|\cdot\|_p$-the norm (quasi-norm) in the $L^p(\mathcal{M})$-space where $a(t)$ is relative to the canonical trace τ on $\mathcal{N}$ (see [5], [8]).

Remark 3. For $a \in L^1(\mathcal{M})$, we have

$$(1) \qquad \widetilde{N}_0(a) = \widetilde{N}_\infty(a) = \|a\|_1.$$

Moreover, it is known that $L^1(\mathcal{M})$ is order isomorphic to the predual $\mathcal{M}_*$ and $h \in L^1(\mathcal{M})^*$ iff $h(a) = \mathrm{tr}(ab) = \mathrm{tr}(ba)$, $a \in L^1(\mathcal{M})$, for some $b \in \mathcal{M}$, where $\mathrm{tr}(\cdot)$ is a positive linear functional on $L^1(\mathcal{M})$ and $\|h\| = \|b\|$ (see [8]). By the Hahn-Banach theorem and equality (1) there is an $\widetilde{h} \in L^{1\infty}(\mathcal{N})^*$ such that $\widetilde{h}(a) = h(a) = \mathrm{tr}(ab)$ for $a \in L^1(\mathcal{M}) \subset L^{1\infty}(\mathcal{N})$ and some $b \in \mathcal{M}$, $\widetilde{h}(a) = 0$, $a \in \overline{F_\tau(\mathcal{N})}$.

Secondly, we consider the Haagerup's L^p-space and weak L^p with $p > 1$.

DEFINITION 5. For $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $a \in \overline{\mathcal{N}}$, we define

$$\|a\|_{p1} = \frac{1}{p}\int_0^\infty t^{-\frac{1}{q}} a(t)\,dt,$$

$$\|a\|_{p1}^{\#} = \frac{1}{p}\int_0^\infty t^{-\frac{1}{q}} \widetilde{a}(t,1)\,dt.$$

The non-commutative $L^{p1}(\mathcal{N}) = L^{p1}$-space is the collection of all $a \in \overline{\mathcal{N}}_0$ such that $\|a\|_{p1} < \infty$. L^{p1}, $p > 1$ are Banach spaces, $\|\cdot\|^{\#}_{p1}$, $p > 1$ is a norm (see [2], Prop. 2.4) and

$$\|a\|_{p1} \leq \|a\|^{\#}_{p1} \leq q\|a\|_{p1},$$

L^{p1} is the Köthe dual of $L^{q\infty}$ (see [2], Th. 2.4). By that, if $b \in L^{p1}$, then the linear functional h_b

$$a \rightarrow h_b(a) = \tau(ab) = \tau(ba), \quad a \in L^{q\infty},$$

is continuous, h_b is called normal.

If $h \in L^{q\infty *}$ and $h(a) = 0$ for

$$a \in \overline{F_{\tau,q}} = \{a \in L^{q\infty} : \lim t^{\frac{1}{q}} a(t) = 0 \text{ as } t \rightarrow 0 \text{ or } t \rightarrow \infty\},$$

then h is called singular (h_b is singular iff $b = 0$!).

We define two continuous non-trivial semi-norms K_0 (K_∞) on $L^{q\infty}(\mathcal{N})$ by

DEFINITION 6. For all $a \in L^{q\infty}(\mathcal{N})$, we define

(i) $K_0(a) = \limsup_{t\rightarrow 0} t^{\frac{1}{q}} \widetilde{a}(t,1)$,
(ii) $K_\infty(a) = \limsup_{t\rightarrow \infty} t^{\frac{1}{q}} \widetilde{a}(t,1)$.

It is clear that $K_0(a) = 0$ ($K_\infty(a) = 0$) if $a \in \overline{F_{\tau,q}}$, by that, if the linear functional h is continuous with respect to K_0 (K_∞), then h is singular. Moreover, if $a \in L^q(\mathcal{M})$, $q > 1$, then we have

$$K_0(a) = K_\infty(a) = p\|a\|_q.$$

Let now $h_b \in L^{q\infty}(\mathcal{N})^*$ for some $b \in L^{p1}(\mathcal{N})$. Then there exists a unique element $T(b) \in L^p(\mathcal{M})$ such that (see [8])

$$h_b(a) = \tau(ab) = \mathrm{tr}(aT(b)), \quad a \in L^q(\mathcal{M}).$$

The mapping $T : L^{p1}(\mathcal{N}) \rightarrow L^p(\mathcal{M})$ is linear and continuous. On the other hand, for $a \in L^q(\mathcal{M})$,

$$|\,\mathrm{tr}(aT(b))| \leq \|a\|_q\|Tb\|_p = \frac{1}{p}K_0(a)\|Tb\|_p\left(\frac{1}{p}K_\infty(a)\|Tb\|_p\right).$$

As a consequence we obtain:

There exists a singular functional $h \in L^{q\infty}(\mathcal{N})^*$ such that

$$h(a) = \mathrm{tr}(aT(b)) = \tau(ab) = h_b(a), \quad a \in L^q(\mathcal{M})$$

for some $b \in L^{p1}(\mathcal{N})$.

Finally, let us assume that $0 < p < 1$.

Remark 4. If $\mathcal{M}$ has a minimal projection, then there exists a non-zero continuous linear functional on $L^p(\mathcal{M})$ with $0 < p < 1$. On the other hand, $L^{p\infty}(\mathcal{N})^* = \{0\}$ since $\mathcal{N}$ has no minimal projection. As a consequence, $L^p(\mathcal{M})$ has no topological complement in $L^{p\infty}(\mathcal{N})$, $0 < p < 1$.

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