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SOME REMARKS ON 2-ELEMENTS AND 2-SUBGROUPS OF FINITE GROUPS

The notations and terminology are standard (see for example [1]). All groups will be finite.

DEFINITION. i) A rational group is a group all whose irreducible characters are rational valued.

ii) An ambivalent group is a group all whose irreducible characters are real valued.

iii) A 2-ambivalent group is a group all whose irreducible characters are real valued on the 2-elements.

PROPOSITION 1 (see [5]). *A group G is rational iff*

$$\mathrm{Aut}(K) \simeq N_G(K)/C_G(K)$$

for all cyclic subgroups K of G.

THEOREM 2. *Let G be a rational group such that*

$$\mathrm{Aut}(K) \simeq N_G(K)/C_G(K)$$

for all 2-subgroups K of G. Then: $G \simeq G'\mathcal{Z}_2$ where G' is a 3-group and $\mathcal{Z}_2$ inverts all elements of G'.

Proof. We shall prove first that the Sylow 2-subgroups of G are isomorphic to $\mathcal{Z}_2$. Suppose G has a 2-subgroup K of order $2^n, n \geq 2$. By Gaschutz's theorem (see [2]) $Out(K)$ has an element of order 2. Let $A = L/C_G(K)$ be a Sylow 2-subgroup of $N_G(K)/C_G(K)$. The order of A is less than the order of $N_G(K)/C_G(K)$, hence L contains a 2-subgroup of order strictly greather than the order of K. So, by induction we can construct 2-subgroups of G of arbitrarily large order. This is a contradiction.

Now, by Walter's theorem [7] G has a normal subgroup $N \geq O_{2'}(G)$ such that G/N has odd order and $N/O_{2'}(G) \simeq S \times P$ where S is a 2-group and P is a direct product of simple groups of the form $L_2(k), k > 3, k \equiv 3, 5 (\mathrm{mod}\ 8)$

or $k = 2^p$, or the Janko simple group J(11), or is of Ree type. Since G is a rational group, $N = G$ and hence $G/O_{2'}(G) \simeq S \times P$. The simple groups listed before are not rational, hence $G/O_{2'}(G) \simeq S$. Therefore G is 2-nilpotent and $S \in \mathrm{Syl}_2(G)$. For every rational group G, $O_{2'}(G) \subset G'$ therefore $G' = O_{2'}(G)$.

We shall prove now by induction on $|G|$ that G' is a 3-group. Let L be a minimal normal subgroup of G. Then L is an elementary abelian p-group. Suppose $p = 2$. Then $L \simeq S \in \mathrm{Syl}_2(G)$ is normal in G, therefore $L = M = G$.

Suppose now $p \neq 2$. Then $L \not\leq G'$, $(G/L)'$ is a 3-group by induction and $|G| = 2.3^a.p^b$. Let $x \in G$ of order p and $X =< x >$. Then $N_G(X) > S \in Syl_2(G)$ and $N_G(X)' < C_G(x) < C_G(x)S < N_G(X)$. Since $C_G(x)S$ is selfnormalizing in G it follows that $C_G(x)S = N_G(X)$. Then $\mathrm{Aut}(X) \simeq N_G(X)/C_G(x) \simeq S/(S \cap C_G(x)) \simeq \mathcal{Z}_2$ and thus $p = 3$.

COROLLARY 3. *Let G be a group such that $Aut(K) \simeq N_G(K)/C_G(K)$ for all subgroups of G. Then $G \simeq S_1, S_2$ or S_3.*

Proof. Clearly G is a rational group by Prop. 1. Analogously in view of the first part of the proof of Theorem 2, the Sylow 3-groups of G must have order 3.

PROPOSITION 4. *A group G is 2-ambivalent iff for every 2 element $x \in G$ there is an element $z \in G$ such that $x^z = x^{-1}$.*

Proof. Analogous to the proof of the similar proposition for ambivalent groups (see [2]).

It is easy to prove the following:

PROPOSITION 5. *Factor groups of 2-ambivalent groups are 2-ambivalent groups.*

DEFINITION. Let $\langle H, Y\rangle$ be a permutation group on the set y and $x \in H$. The cyclic group $\langle x\rangle$ acts on Y and one obtains a decomposition of Y into tranzitive constituents, the orbit of Y. Denote $O\langle x, y\rangle$ the orbit of $y \in Y$. We say that $\langle H, Y\rangle$ is 2-ambivalent transversal iff for every 2-element $x \in H$ there is some element $z \in H$ such that $x^2 = x^{-1}$ and $zO, x, y = O < x, y >$ for every $y \in Y$.

Let $\langle f, x\rangle \in G\omega r H$. Define $x^* : G^Y \to G^Y$ by $x^* f\langle y\rangle = f_x\langle y\rangle .. f_{x^{s-1}}\langle y\rangle$ where $s = |O\langle x, y\rangle|$. This product is called the *cycleproduct* (see [4] pp. 40).

THEOREM 6. *If $G\omega r H$ is a 2-ambivalent group then both G and H are 2-ambivalent groups.*

Proof. H is a factor group of $G\omega r H$, hence by Prop. 5 H is 2-ambivalent.

Let $g \in G$ be a 2 element. Define $f : Y \to G$ setting $f(y) = g$ for every $y \in Y$. Then $1^*(f)(y) = f(y) = g$ for every y, therefore $1^*(f) = f$. Hence

$|(f;1)| = |g|$. Since $G\omega rH$ is an 2-ambivalent group, there is an element $(h;z) \in G\omega rH$ such that $(h;z)(f;1)(h;z)^{-1} = f(y)^{-1} = g^{-1}$ hence G is 2-ambivalent.

Remark. Hence to construct new 2-ambivalent groups by wreathing it is to consider only 2-ambivalent groups. In general, it is not true that the wreath product of two 2-ambivalent groups is an 2-ambivalent group.

THEOREM 7. *Let G be a 2-ambivalent group and $< H, Y >$ a 2 ambivalent transversal group. Then $G\omega r\langle H, Y\rangle$ is 2-ambivalent.*

Proof. Let $(f;x) \in G\omega rH$ be a 2-element. We have to show that $(f;x)^{-1} \simeq (f;x)$. By the definition of the wreath product x is a 2-element. Clearly $(f;x)^{-1} = ({f^{-1}}_{x^{-1}z};x)$. Since H is 2 ambivalent transversal, there is an element $x \in H$ such that $x^z = x^{-1}$ and $zO(x,y) = O(x,y)$ for every $y \in Y$. Then

$$(1;z)(f;x)^{-1}(1;z)^{-1} = ({f^{-1}}_{x^{-1}z};x).$$

Denote ${f^{-1}}_{x^{-1}}$ by g. We shall prove now that $(g_z;x) \simeq (f;y)$.

It is straithforward to prove that $x^*(g_z)(y) = (x^*(f)(z^{-1}(y)))^{-1}$ for every $y \in Y$. Since $z0(x,y) = O(x;y)$, then $z^{-1}(y) \in O(x;y)$ and therefore $(x^*(f)(z^{-1}(y)))^{-1} \simeq x^*(f)(y)$. Hence $x^*(g_z)(y) \simeq (x^*(f)(y))^{-1}$. Since G is 2-ambivalent group and $x^*(f)(y) \in G$ is a 2-element then $(x^*(f)(y))^{-1} \simeq x^*(f)(y)$. Therefore $x^*(g_z)(y) \simeq x^*(f)(y)$.

We shall construct now a $w : Y \to G$ such that $(w;1)(g_z)(w;1)^{-1} = (f;x)$.

Let $Y = O(x,y_1)\bigcup\ldots\bigcup O(x,y_q)$ be the orbit decomposition of the action of $\langle x\rangle$ on Y with pairwise disjoint factors. Then

$$O(x,y_i) = \{y_i, x^{-1}(y_i), \ldots, x^{-(s_i-1)}(y_i)\}.$$

Let $w(y_i)$ be an element of G such that $w(y_i)x^*(g_z)(y_i)w(y_i)^{-1} = x^*(f)(y_i)$ for $i = 1, \ldots, q$. By the previous such an element exists.

For $1 \leq k \leq s_i - 1$, set

$$w(x^{-1}(y_i)) = \{f(y_i)f_x(y_i)\ldots f_{x^{k-1}}(y_i)\}^{-1}w(y_i)\{g_z(y_i)\ldots g_z(x^{x^{-(k-1)}}(y_i))\}$$

so w is defined on all Y. It remains to verify that $(w;1)(g_z;x)(w;1)^{-1} = (f;x)$. This follows if we prove that $wg_zw_x^{-1} = f$ or equivalently $w(y)g_z(y)w(x^{-1}(y))^{-1} = f(y)$ for every $y \in Y$.

For $y \in y_i$ it is easy to see that $w(x^{-1}(y_i)) = f(y_i)^{-1}w(y_i)g_z(y_i)$. In general, we write $y = x^{-k}(y_i)$ and the statement follows immediately.

Remark. Since the symetric group S_n and the alternating groups A_n are 2-ambivalent transversal groups on $Y = \{1, \ldots, n\}$ then $GwrS_n$ and $GwrA_n$ are 2-ambivalent groups iff G is a 2-ambivalent group.

DEFINITION. Let G be a finite group. A subgroup H of G is said to be *strongly embedded in* G if the following conditions are satisfied:

(1) H is a proper subgroup of even order.

(2) For any element $x \in G - H$, the order $H \bigcap H^*$ is odd (see [6] p. 391).

THEOREM 8 (see [6] p. 391). *Let* G *be a group having a strongly embedded subgroup* H. *Then, we have of the following alternatives*:

(1) *Every Sylow* 2*-subgroup of* G *contains exactly one element of order* 2. *Thus a Sylow* 2*-subgroup of* G *is either a cyclic group or generalized quaternion group.*

(2) *The group* G *posseses a normal series* $G > L > M > \{1\}$ *such that both* G/L *and* m *are groups of odd order, and such that the factor group* L/M *is isomorphic to one of the simple groups* $PSL(2,q)$, $Sz(q)$, *or* $PSU(3,q)$, *where* q *is a power of* 2.

In the first case (1), *let* t *be any element of order two. Then,* $C_G(t)$ *is a proper subgroup of* G *and any proper subgroup of* G *containing* $C_G(t)$ *is strongly embedded in* G. *In the second case* (2), *every strongly embedded subgroup* H *of* G *is of the form* $H = N_G(S)O_{2'}(G)$ *for some* 2*-Sylow subgroup* S *of* G.

THEOREM 9 (see [3] p. 393). *Let* H *be a strongly embedded subgroup of a group* G. *Let* u *be an element of* $I(H) = \{x \in H \mid x^2 = 1,\ x \neq 1\}$. *Then, the folowing proposition hold.*

(1) *The set* $I(G) = \{x \in G \mid x^2 = 1, x \neq 1\}$ *is a conjugacy class of* G. *In other words, all involutions of* G *are conjugate.*

(2) *The set* $I(H)$ *is a conjugacy class of* H. *Futhermore, if* $b = a^x$ *for* $A, B \in I(H)$ *and* $x \in G$, *then we have* $x \in H$.

THEOREM 10. *Suppose the assumptions of Theorem* 8(2). *Then a Sylow* 2*-subgroup of* G *is either homocyclic or a Suzuki* 2*-group and* $G/O_{2'}(G)$ *is a* 2*-normal group.*

Proof. It is clear that we can suppose that $O_{2'}(G)$ is trivial. Let H be a strongly embedded subgroup of G. Because $H \bigcap H^*$ is odd for any $x \in G - H$,$S \bigcap T = \{1\}$ for any distinct S, T Sylow 2-subgroups of G. Let S be a Sylow 2-subgroup of G such that $S < H$. Then, by Theorem 8(2) $H = N_G(S)$ and by Theorem 9 $I(H)$ is a conjugacy class of H. On the other hand, since S is a normal, nilpotent subgroup of H and H/S has odd order, H is solvable. By Thompson Theorem (see [2], p. 511), S is either homocyclic or a Suzuki 2-Group.

If S is homocyclic then $S = Z(S)$ and $H = N_G(Z(S)) = N_G(S)$ so that $G/O_{2'}(G)$ is 2-normal.

If S is a Suzuki 2-group, then (see [3], p. 313) $S' = Z(S) = I(H) \cup 1$. Evidently, $N_G(S) < N_G(Z(S))$. Let $x \in N_G(Z(S)$. Since $S^x \bigcap S > Z(S)$ it follows that $N_G(S) = N_G(Z(S))$ and $G/O_{2'}(G)$ is 2-normal.

THEOREM 11. *Let G be a solvable rational group having a strongly embedded subgroup. Then a Sylow* 2*-subgroup of G is isomorphic with the quaternion group of order* 8 Q_8.

Proof. We can suppose that $O_{2'}(G)$ is trivial. Let $A = \langle I(G) \rangle$. Then A is a normal subgroup of G and since G is solvable it contains an abelian minimal normal subgroup of G. Since $O_{2'}(G)$ is trivial it follows that $A = I(G)$ and A is abelian. Let H be a strongly embedded subgroup of G such that $A < H$. Then G has only one 2-Sylow subgroup S. Then G/S is also a rational group so that G=S and S contain only one involution. By Theorem 10, then S is cyclic or quaternion group. Since S is also a rational group it follows that S is isomorphic to Z_2 or to Q_S.

THEOREM 12. *Let G be a solvable rational group having a strongly embedded subgroup. Then G is isomorphic to* E_3Z_2 *where* E_3 *is an elementary abelian* 3*-group and* Z_2 *inverts all elements of* E_3. *G is involutory.*

Proof. By Theorem 11, a Sylow 2-subgroup S of G is Z_2 or Q_8. By Corollary 36 of [4], if S is Z_2 we have the asserts. If S is Q_8 then (see [6]) $Z(G)$ contains an involution, therefore G cannot have a strongly embedded subgroup.

COROLLARY 13. *The only groups G having a strongly embedded subgroup which can be embedded without fusion in a symmetric group* S_n *are the groups* $G = E_3Z_2$ *of Theorem* 12.

Proof. By [1] a group G can be embedded without fusion in a symmetric group if G is a Q-group.

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