DEMONSTRATIO MATHEMATICA
Vol. XXX No 2 1997

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RATE OF CONVERGENCE OF MODIFIED BASKAKOV OPERATORS

1. Introduction

Modified Baskakov operator [4] is defined as

$$B_n(f,x) = (n-1)\sum_{k=0}^{\infty} p_{n,k}(x) \int_0^{\infty} p_{n,k}(t)f(t)\,dt, \qquad x \in [0,\infty)$$

where $p_{n,k}(x) = \frac{(n+k-1)!}{k!(n-1)!}x^k(1+x)^{-n-k}$.

Gupta and Agrawal [2] estimated the rate of convergence of modified Szász-Mirakyan operators for functions of bounded variation. Sahai and Prasad [3] gave correct and improved estimate for modified Szász operators for functions of bounded variation. In the present paper, we study the analogous problem for modified Baskakov operators $B_n(f,x)$ for functions of bounded variation, using some results of probability theory considering the functions bounded on $[0,\infty)$.

2. Auxiliary results

To prove the main theorem, we shall need the following results:

LEMMA 1 [5]. *If $\{\xi_i\}$, $i = 1,2,\ldots$ are independent random variables with same geometric distribution*

$$P\{\xi_i = k\} = \left(\frac{x}{1+x}\right)^k \frac{1}{1+x}, \ x > 0; \quad i = 0,1,\ldots$$

then $E(\xi_i) = x$, $\rho_2 = E(\xi_i - E(\xi_i))^2 = x(1+x)$ and $\eta_n = \sum_{i=1}^{n} \xi_i$ is a random variable with distribution

$$(1) \qquad P(\eta_n = k) = \binom{n+k-1}{k}\frac{x^k}{(1+x)^{n+k}} = p_{n,k}(x).$$

1991 *Mathematics Subject Classification*: 41A30, 41A36.

LEMMA 2. *If y is a positive valued random variable with a non-degenerate probability distribution, then we have by Schwarz's inequality $E(y^3) \leq (E(y^2).\ E(y^4))^{1/2}$ provided $E(y^2)$, $E(y^3)$ and $E(y^4) < \infty$.*

LEMMA 3. *For every $k \in N$, $x \in (0, \infty)$, we have*

$$p_{n,k}(x) \leq \frac{2\{1+9x(1+x)\}^{1/2}+1}{2\sqrt{nx(1+x)}}. \tag{2}$$

Proof. By (1) we have

$$\begin{aligned} p_{n,k}(x) &= P(\eta_n = k) = P(k-1 < \eta_n \leq k) \\ &= P\left(\frac{k-1-nx}{\sqrt{nx(1+x)}} < \frac{\eta_n - nx}{\sqrt{nx(1+x)}} \leq \frac{k-nx}{\sqrt{nx(1+x)}}\right). \end{aligned}$$

Using Berry Esseen theorem [1] with $a_1 = x$ and $b_1 = \sqrt{x(1+x)}$ (from Lemma 1), we have

$$\left| P(\eta_n = k) - \frac{1}{\sqrt{2\pi}} \int_{\frac{k-1-nx}{\sqrt{nx(1+x)}}}^{\frac{k-nx}{\sqrt{nx(1+x)}}} e^{-t^2/2}\, dt \right| < \frac{2(0.409)\rho_3}{\sqrt{n} b_1^3}. \tag{3}$$

We now calculate ρ_3. By an easy computation, we can show that

$$\begin{cases} T_0(x) = \sum\limits_{k=0}^{\infty} \left(\frac{x}{1+x}\right)^k \frac{1}{1+x} = 1, \\ T_1(x) = \sum\limits_{k=0}^{\infty} k\left(\frac{x}{1+x}\right)^k \frac{1}{1+x} = x, \\ T_2(x) = \sum\limits_{k=0}^{\infty} k^2\left(\frac{x}{1+x}\right)^k \frac{1}{1+x} = (1+2x)x, \\ T_3(x) = \sum\limits_{k=0}^{\infty} k^3\left(\frac{x}{1+x}\right)^k \frac{1}{1+x} = x[1+6x(1+x)], \\ T_4(x) = \sum\limits_{k=0}^{\infty} k^4\left(\frac{x}{1+x}\right)^k \frac{1}{1+x} = x[1+14x+36x^2+24x^3]. \end{cases} \tag{4}$$

Also, if $M_r(x) = \sum_{k=0}^{\infty}(k-x)^r(\frac{x}{1+x})^k\frac{1}{(1+x)}$ stands for the central moment of order r about the mean $T_1(x)$, i.e. x, it is easily checked

$$M_2(x) = \sum_{k=0}^{2} \binom{2}{j}(-1)^j T_{2-j}(x)(T_1(x))^j = x(1+x)$$

and

$$M_4(x) = \sum_{j=0}^{4} \binom{4}{j} (-1)^j T_{4-j}(x)(T_1(x))^j$$
$$= x(1 + 10x + 18x^2 + 9x^3), \quad \text{using (2.4)}.$$

Next, in view of Lemma 2, we have

$$\rho_3 \le (M_2(x) \cdot M_4(x))^{1/2} \le x(1+x)\{1 + 9x(1+x)\}^{1/2}.$$

So the right hand side of (3) is less than $\frac{1}{\sqrt{n}}(9 + \frac{1}{x(1+x)})^{1/2}$ as $0.818 < 1$ and

$$\frac{1}{\sqrt{2\pi}} \int\limits_{\frac{k-1-nx}{\sqrt{nx(1+x)}}}^{\frac{k-nx}{\sqrt{nx(1+x)}}} e^{-t^2/2}\, dt < \frac{1}{\sqrt{2\pi nx(1+x)}} < \frac{1}{2\sqrt{nx(1+x)}}.$$

Hence

$$p_{n,k}(x) = P(\eta_n = k) \le \frac{2\{1 + 9x(1+x)\}^{1/2} + 1}{2\sqrt{nx(1+x)}}.$$

LEMMA 4. *For $x = (0, \infty)$, we have*

$$(n-1) \int\limits_{x}^{\infty} p_{n,k}(t)\, dt = \sum_{j=0}^{k} p_{n-1,j}(x).$$

LEMMA 5 [4]. *Let the m-th order moment be defined as*

$$T_{n,m}(x) = (n-1) \sum_{k=0}^{\infty} p_{n,k}(x) \int\limits_{0}^{\infty} p_{n,k}(t)(t-x)^m\, dt.$$

Then

$$T_{n,1}(x) = \frac{1+2x}{n-2},\ n > 2,\ T_{n,2}(x) = \frac{2(n-1)x(1+x) + 2(1+2x)^2}{(n-2)(n-3)},\ n > 3$$

and there holds the recurrence relation

$$(5)\quad (n-m-2)T_{n,m+1}(x) = x(1+x)(T^{(1)}_{n,m}(x) + 2mT_{n,m-1}(x)) + (m+1)(1+2x)T_{n,m}(x), \quad n > m+2.$$

Remark. In particular, we have

$$(6)\qquad T_{n,2}(x) \cong \frac{2x(1+x)}{n}.$$

Also, for all $x \in [0, \infty)$, (5) leads us to $T_{n,m}(x) = 0(n^{-[(m+1)/2]})$, where $[\alpha]$ denotes the integral part of α.

LEMMA 6. *Let $K_n(x,t) = (n-1)\sum_{k=0}^{\infty} p_{n,k}(x)p_{n,k}(t)$. Then, for sufficiently large n*

(i) *For $0 \le y < x$, we have $\int_0^y K_n(x,t)\, dt \le \frac{2x(1+x)}{n(x-y)^2}$.*

(ii) *For* $x < z < \infty$, *we have* $\int_z^\infty K_n(x,t)dt \le \frac{2x(1+x)}{n(z-x)^2}$.

The proof of this lemma follows easily by using (6) from Lemma 5 along the lines of [3].

LEMMA 7. *For* $k \ge 0$ *and sufficiently large* n, *we have*

$$\left|\sum_{j=0}^{k} p_{n,j}(x) - \sum_{j=0}^{k} p_{n-1,j}(x)\right| \le \frac{2\{1+9x(1+x)\}^{1/2}+x}{2\sqrt{nx(1+x)}}.$$

Proof. We have

$$\left|\sum_{j=0}^{k} p_{n,j}(x) - \sum_{j=0}^{k} p_{n-1,j}(x)\right|$$

$$= \left|\sum_{j=0}^{k} p_{n,j}(x) - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{(j-nx)/\sqrt{nx(1+x)}} e^{-t^2/2}\,dt\right.$$

$$+ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{(j-(n-1)x)/\sqrt{(n-1)x(1+x)}} e^{-t^2/2}\,dt$$

$$\left. - \frac{1}{\sqrt{2\pi}} \int_{(j-nx)/\sqrt{nx(1+x)}}^{(j-(n-1)x)/\sqrt{(n-1)x(1+x)}} e^{-t^2/2}\,dt - \sum_{j=0}^{k} p_{n-1,j}(x)\right|$$

$$\le \left|\sum_{j=0}^{k} p_{n,j}(x) - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{(j-nx)/\sqrt{nx(1+x)}} e^{-t^2/2}\,dt\right|$$

$$+ \left|\sum_{j=0}^{k} p_{n-1,j}(x) - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{(j-(n-1)x)/\sqrt{(n-1)x(1+x)}} e^{-t^2/2}\,dt\right|$$

$$+ \left|\frac{1}{\sqrt{2\pi}} \int_{(j-nx)/\sqrt{nx(1+x)}}^{(j-(n-1)x)/\sqrt{(n-1)x(1+x)}} e^{-t^2/2}\,dt\right|$$

$$\le \frac{2(0.409)}{\sqrt{n}}\left(9 + \frac{1}{x(1+x)}\right)^{1/2} + \frac{x}{\sqrt{2\pi nx(1+x)}}$$

$$\le \frac{2\{1+9x(1+x)\}^{1/2}+x}{2\sqrt{nx(1+x)}}.$$

3. Main Theorem

In this section, we shall state and prove the main result.

THEOREM. *Let f be a function of bounded variation on every finite subinterval of $[0,\infty)$ and let $V_a^b(g_x)$ be the total variation of g_x on $[a,b]$. Then for sufficiently large n, we have*

$$\left|B_n(f,x)-\frac{1}{2}\{f(x+)+f(x-)\}\right| \le \frac{(4+5x)}{nx}\sum_{k=1}^{n} V_{x-x/\sqrt{k}}^{x+x/\sqrt{k}}(g_x) + \left[\frac{6[1+9x(1+x)]^{1/2}+(2x+1)}{4\sqrt{nx(1+x)}}\right]|f(x+)-f(x-)|,$$

where, for any fixed $x \in (0,\infty)$, we define g_x as

$$g_x(t)=\begin{cases} f(t)-f(x+), & x<t<\infty \\ 0, & t=x \\ f(t)-f(x-), & 0\le t<x. \end{cases}$$

Proof. We have

$$\left|B_n(f,x)-\frac{1}{2}\{f(x+)+f(x-)\}\right| \le |B_n(g_x,x)|+\frac{1}{2x}|f(x+)-f(x-)|\cdot|B_n(\mathrm{Sign}(t-x),x)|.$$

In order to obtain the result, we need the estimate for $B_n(g_x,x)$ and $B_n(\mathrm{Sign}(t-x),x)$. Now,

$$B_n(\mathrm{Sign}(t-x),x)=\int_0^\infty \mathrm{Sign}(t-x)K_n(x,t)\,dt$$

and

$$\int_x^\infty K_n(x,t)\,dt-\int_0^x K_n(x,t)\,dt = A_n(x)-B_n(x), \text{ say.}$$

Using Lemma 4, we have

$$A_n(x)=\int_x^\infty K_n(x,t)\,dt=(n-1)\sum_{k=0}^{\infty}p_{n,k}(x)\int_x^\infty p_{n,k}(t)\,dt = \sum_{k=0}^{\infty}p_{n,k}(x)\sum_{j=0}^{k}p_{n-1,j}(x).$$

Next, using Lemma 7, we have

$$(7)\qquad \begin{aligned}\Big|A_n(x) - \sum_{k=0}^{\infty} p_{n,k}(x)\sum_{j=0}^{k} p_{n,j}(x)\Big| &= \Big|\sum_{k=0}^{\infty} p_{n,k}(x)\Big(\sum_{j=0}^{k} p_{n-1,j}(x) - \sum_{j=0}^{k} p_{n,j}(x)\Big)\Big| \\ &\le \frac{2\{1+9x(1+x)\}^{1/2}+x}{2\sqrt{nx(1+x)}}.\end{aligned}$$

Let

$$\begin{aligned}S &= \Big(\sum_{k=0}^{\infty} p_{n,k}(x)\sum_{j=0}^{k} p_{n,j}(x)\Big) \\ &= P_{n,0}P_{n,0} + P_{n,1}(P_{n,0}+P_{n,1}) + \ldots + P_{n,k}(P_{n,0}+P_{n,1}+\ldots+P_{n,k}) + \ldots\end{aligned}$$

Since

$$\begin{aligned}1 &= (P_{n,0}+P_{n,1}+P_{n,2}+\ldots)(P_{n,0}+P_{n,1}+P_{n,2}+\ldots) \\ &= P_{n,0}(P_{n,0}+P_{n,1}+\ldots) + P_{n,1}(P_{n,0}+P_{n,1}+\ldots) + \ldots \\ &\quad + P_{n,k}(P_{n,0}+P_{n,1}) + \ldots + \ldots).\end{aligned}$$

We have

$$1 - S = P_{n,0}(P_{n,1}+\ldots) + P_{n,1}(P_{n,2}+\ldots) + \ldots$$

and

$$2S - 1 = P_{n,0}^2 + P_{n,1}^2 + \ldots = \sum_{k=0}^{\infty} P_{n,k}^2(x).$$

Using Lemma 3, we obtain

$$(8)\qquad |S - 1/2| \le \frac{\{1+2(1+9x(1+x)\}^{1/2}}{4\sqrt{nx(1+x)}}.$$

By (7) and (8) it follows that

$$|A_n(x) - 1/2| \le \frac{6\{1+9x(1+x)\}^{1/2}+(2x+1)}{4\sqrt{nx(1+x)}}.$$

Now as

$$A_n(x) + B_n(x) = \int_0^{\infty} K_n(x,t)\,dt = 1,$$

$$(9)\quad |A_n(x) - B_n(x)| = |2A_n(x) - 1| \le \frac{6\{1+9x(1+x)\}^{1/2}+(2x+1)}{2\sqrt{nx(1+x)}}.$$

Next, to estimate $B_n(g_x, x)$, we decompose $[0,\infty)$ into three parts as follows:
$I_1 = [0, x - x/\sqrt{n}]$, $\quad I_2 = [x - x/\sqrt{n}, x + x/\sqrt{n}]$, $\quad I_3 = [x + x/\sqrt{n}, \infty)$,

$$B_n(g_x, x) = \int_0^\infty g_x(t)K_n(x,t)\,dt$$
$$= \Big(\int_{I_1} + \int_{I_2} + \int_{I_3}\Big) g_x(t)K_n(x,t)\,dt = R_1 + R_2 + R_3, \quad \text{say.}$$

Suppose $\lambda_n(x,t) = \int_0^t K_n(x,u)\,dt$. We first estimate R_1. Let $y = x - x/\sqrt{n}$. Using partial integration we have

$$R_1 = \int_0^y g_x(t)K_n(x,t)\,dt = \int_0^y g_x(t)\,d_t(\lambda_n(x,t))$$
$$= g_x(y+)\lambda_n(x,y) - \int_0^y \lambda_n(x,t)d_t(g_x(t)).$$

Since $|g_x(y+)| = |g_x(y+) - g_x(x)| \le V_{y+}^x(g_x)$, then by using Lemma 7, we have

$$|R_1| \le V_{y+}^x(g_x)\lambda_n(x,y) + \int_0^y \lambda_n(x,t)\,d_t(-V_t^x(g_x))$$
$$\le V_{y+}^x(g_x)\frac{2x(1+x)}{n(x-y)^2} + \frac{2x(1+x)}{n}\int_0^y \frac{1}{(x-t)^2}d_t(-V_t^x(g_x)).$$

Integrating by parts leads to the following

$$\int_0^y \frac{1}{(x-t)^2}\,d_t(-V_t^x(g_x) = \frac{-V_{y+}^x(g_x)}{(x-y)^2} + \frac{V_0^x(g_x)}{x^2} + 2\int_0^y \frac{\widehat{V}_t^x(g_x)}{(x-t)^3}\,dt,$$

where $\widehat{V}_t^x$ is the normalized form of $V_t^x(g_x)$ and $\widehat{V}_t^x(G_x) = V_t^x(g_x)$. Consequently, we have

$$|R_1| \le V_{y+}^x(g_x)\frac{2x(1c+x)}{n(x-y)^2}$$
$$+ \frac{2x(1+x)}{n}\left[\frac{-V_{y+}^x(g_x)}{(x-y)^2} + \frac{V_0^x(g_x)}{x^2} + 2\int_0^y \frac{V_t^x(g_x)}{(x-t)^3}\right]dt$$
$$= \frac{2x(1+x)}{n}\left[\frac{V_0^x(g_x)}{x^2} + 2\int_0^y \frac{V_t^x(g_x)}{(x-t)^3}\,dt\right].$$

Replacing the variable y in the last integral by $x - x/\sqrt{n}$, we get

$$\int_0^{x-x/\sqrt{n}} V_t^x(g_x)\frac{dt}{(x-t)^3} = \frac{1}{2x^2}\int_1^n V_{x-x/\sqrt{n}}^x(g_x)\,dt \le \frac{1}{2x^2}\sum_{k=1}^n V_{x-x/\sqrt{k}}^x(g_x).$$

Therefore.

$$|R_1| \le \frac{2x(1+x)}{n}\left[\frac{V_0^x(g_x)}{x^2} + \frac{1}{x^2}\sum_{k=1}^{n} V_{x-x/\sqrt{k}}^{x}(g_x)\right] \tag{10}$$

$$\le \frac{4(1+x)}{nx}\sum_{k=1}^{n} V_{x-x/\sqrt{k}}^{x}(g_x).$$

We, next estimate R_2. For $x \in I_2$, we have

$$|g_x(t)| = |g_x(t) - g_x(x)| \le V_{x-x/\sqrt{n}}^{x+x/\sqrt{n}}(g(x).$$

Since $\int_a^b d_t\lambda_n(x,t) \le 1$ for all $(a,b) \subset [0,\infty]$ therefore,

$$|R_2| \le V_{x-x/\sqrt{n}}^{x+x/\sqrt{k}}(g_x) \le \frac{1}{n}\sum_{k=0}^{n} V_{x-x/\sqrt{k}}^{x+x/\sqrt{k}}(g_x). \tag{11}$$

The estimate of R_3 is similar as that of R_1. By using Lemma 7, we have

$$|R_3| \le \frac{4(1+x)}{nx}\sum_{k=1}^{n} V_x^{x+x/\sqrt{k}}(g_x). \tag{12}$$

Combining (9) to (12), we get the required result.

Remark. Exactly similarly as in [3], we can show that the estimate in our main theorem is essentially the best asymptotically.

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Received January 10, 1996; revised version May 29, 1996.