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ON AUTOMORPHISMS OF PRIME RINGS WITH INVOLUTION

1. Introduction

Throughout this note R denotes a ring. We assume that R contains the unit element 1, although some results may hold without this assumption. Following Laradji and Thaheem [10], an element $a \in R$ is said to be a dependent element of a mapping α of R into itself if $\alpha(x)a = ax$ for all $x \in R$. Obviously, 0 is a dependent element of α. If α is an inner automorphism of R induced by an element $a \in R$ (i.e. $\alpha(x) = axa^{-1}$ for all $x \in R$), then a is a dependent element of a. If 0 is the only dependent element of α then α is said to be freely acting on R. Dependent elements were first introduced by Choda, Kasahara and Nakamoto [8] for automorphisms of C^*-algebras in the process of generalization of the notion of free action of automorphisms of von Neumann algebras (due to von Neumann and Murray [15, 16], see also Kallman [9]) to C^*-algebras. Several other authors have also studied dependent elements in operator algebras (see e.g. [5, 7]). Dependent elements have also been discussed in the book of Strătilă [13]. Laradji and Thaheem [10] have further generalized this notion to rings and have proved some basic results on dependent elements in rings analogous to those in [8]. In this paper we consider automorphisms on prime rings with involution rather than general mappings on arbitrary rings. Thus following [10], if α is a dependent element of α, then $\alpha(a) = a$. In case R has involution (*) and α is a *-automorphism, then $a^*a = aa^* \in Z(R)$, the center of R, and a^* is the dependent element of α^{-1}. That is, $\alpha^{-1}(x)a^* = a^*x$ for all $x \in R$. Our main results include showing (Theorem 3.1) that any centralizing *-automorphism α that is not freely acting on a prime ring with involution satisfies the functional equation $\alpha + \alpha^{-1} = 2$. We also show (Proposition 3.2) that if R is not of characteristic 2 then such an automorphism is the identity automorphism. Thus combining the two results we obtain

THEOREM A. *Let R be a prime ring with involution such that the characteristic of R is not equal to 2. Then any centralizing *-automorphism which is not freely acting on R is the identity automorphism.*

We remark that the functional equation $\alpha + \alpha^{-1} = 2$ is a particular case of widely studied functional equation $\alpha + \alpha^{-1} = \beta + \beta^{-1}$ for automorphisms α, β on C^*-algebras and prime and semiprime rings (see e.g. [2, 3, 6, 14]). Theorem A offers a new proof of a result of [10, Remark following Proposition 3] and also generalizes a result of Miers [12, Remark 2, p. 63] which states that any centralizing inner automorphism of an operator algebra is the identity automorphism. We may further remark that Luh [11] proved that any commuting automorphism on a prime ring is the identity automorphism. Since any commuting automorphism is centralizing, therefore Theorem A provides an alternate proof of Luh's result for automorphisms that are not freely acting. These and other results are contained in section 3 while section 2 contains some technical preliminaries required for our results. We remark that our results in this note are purely algebraic. The concepts like von Neumann algebras and C^*-algebras appear only in the historical perspective of the problem.

2. Preliminaries

Let R be a ring. Then R is said to be prime if $axb = 0$ for all $x \in R$ implies $a = 0$ or $b = 0$. A von Neumann algebra is prime if and only if it is a factor (i.e. its center consists of the scalar multiples of the identity). R is said to be semiprime if $axa = 0$ for all $x \in R$ implies $a = 0$. A prime ring is semiprime and also any C^*-algebra is also semiprime. An automorphism α of a ring R with involution is a *-automorphism if $\alpha(x^*) = \alpha(x)^*$ for all $x \in R$. A mapping α of a ring R into itself is said to be centralizing if $[\alpha(x), x] \in Z(R)$ for all $x \in R$. In the special case when $[\alpha(x), x] = 0$ for all $x \in R$, then α is said to be commuting, where $[x, y] = xy - yx$.

3. Results

Let α be an automorphism of a prime ring R and a be a nonzero dependent element of α. Then a is neither a right zero divisor nor a left zero divisor. Indeed, if a is a right zero divisor then there is $b \neq 0$ such that $ba = 0$. Then $0 = bax = b\alpha(x)a$ for all x in R. Since α is onto and R is prime we get $a = 0$ or $b = 0$, a contradiction. Thus a is not a right zero divisor. Similarly a is not a left zero divisor.

THEOREM 3.1. *Let α be a centralizing *-automorphism of a prime ring R with involution. Assume that α is not freely acting on R. Then*

(a) *α satisfies the equation $\alpha + \alpha^{-1} = 2$.*

(b) *If in addition the characteristic of R is not equal to 2, then $\alpha = id$ (the identity automorphism).*

Proof. By [1, Lemma 2], α is commuting and a linearization of $[\alpha(x), x] = 0$ implies $[\alpha(x), y] = [x, \alpha(y)]$ for all $x, y \in R$. Since α is not freely acting, therefore we assume that a nonzero element a is a depending element of α. Substituting a for y and using $\alpha(a) = a$, we get

(1) $$[\alpha(x), a] = [x, \alpha(a)] = [x, a] \text{ for all } x \in R.$$

That is,

(2) $$\alpha(x)a - a\alpha(x) - xa + ax = 0 \text{ for all } x \in R.$$

Since $\alpha(x)a = ax$, we get from (2)

(3) $$a(x - \alpha(x)) + ax - xa = 0 \text{ for all } x \in R.$$

Multiplying (3) by a^* on the left, we get

(4) $$a^*a(x - \alpha(x)) + a^*ax - a^*xa = 0 \text{ for all } x \in R.$$

Since $\alpha^{-1}(x)a^* = a^*x$, we get from (4)

(5) $$a^*a(x - \alpha(x)) + a^*ax - \alpha^{-1}(x)a^*a = 0 \text{ for all } x \in R.$$

Since $a^*a \in Z(R)$, we get from (5) that $a^*a(x - \alpha(x) + x - \alpha^{-1}(x)) = 0$ for all $x \in R$. But a is not a zero divisor implies a^*a is not a zero divisor and hence $\alpha(x) + \alpha^{-1}(x) - 2x = 0$ for all $x \in R$. That is, $\alpha + \alpha^{-1} = 2$. This completes the proof of (a). The proof of (b) follows from (a) and the Proposition 3.3 below. □

The equation $\alpha + \alpha^{-1} = 2$ is a particular case of the more general equation $\alpha + \alpha^{-1} = \beta + \beta^{-1}$ for automorphisms α, β on prime rings (see e.g. [2]). The proposition below follows from Brešar's result [3, Corollary 3] whose proof depends on several technical lemmas. In this particular case we give a direct and simple proof.

PROPOSITION 3.2. *Let α be an automorphism of a prime ring R of characteristic not equal to 2 such that $\alpha + \alpha^{-1} = 2$. Then α is the identity automorphism.*

Proof. Put $d = (\alpha - 1)$. Then d is an additive mapping of R into R and satisfies $d(xy) = \alpha(x)d(y) + d(x)y$ for all $x, y \in R$. That is, d is an α-derivation. Also,

$$d^2(x) = (\alpha - 1)(\alpha - 1)x = (\alpha^2 - 2\alpha + 1)x = 0 \text{ for all } x \in R.$$
$$\begin{aligned} d^2(xy) &= d(d(xy)) = d(\alpha(x)d(y) + d(x)(y)) = d(\alpha(x)d(y)) + d(d(x)y) \\ &= \alpha^2(x)d^2(y) + d(\alpha(x))d(y) + \alpha(d(x))d(y) + d^2(x)y. \end{aligned}$$

Since $d^2(xy) = d^2(x) = d^2(y) = 0$ and α commutes with d, therefore $2d(\alpha(x))d(y) = 0$ for all $x, y \in R$. Since characteristic of R is not equal to 2, therefore $d(\alpha(x))d(y) = 0$ for all $x, y \in R$. Replacing y by zx, we get $0 = d(\alpha(x))d(zx) = d(\alpha(x))(\alpha(z)d(x) + d(z)x) = d(\alpha(x))\alpha(z)d(x) + d(\alpha(x))d(z)x$ for all $x, z \in R$. Since $d(\alpha(x))d(z)x = 0$, therefore $d(\alpha(x))\alpha(z)d(x) = 0$ for all $x, z \in R$. α is onto implies $d(\alpha(x))Rd(x) = 0$ and R being prime implies $d(\alpha(x)) = 0$ or $d(x) = 0$. In any case $d = 0$ because α is onto. Thus we obtain that $\alpha = 1$. □

We conclude the note with the following characterization of the identity automorphism. This generalizes a result of [4] for C^*-algebras.

PROPOSITION 3.3. *Suppose that α is an inner automorphism of a prime ring R with involution. Then α is the identity automorphism if and only if*

$$(\alpha(x^*))^* = \alpha(\alpha(x)) \text{ for all } x \in R. \tag{6}$$

Proof. Assume that (6) holds. If α is a *-automorphism (not necessarily inner) then $\alpha(x) = \alpha(\alpha(x))$. This implies $(\alpha - 1)\alpha(x) = 0$ for all $x \in R$ and α being onto gives $\alpha - 1 = 0$ or $\alpha = 1$.

Now assume that $\alpha(x) = axa^{-1}$ for all $x \in R$, $a \in R$. Then by assumption, $a^2xa^{-2} = a^{*^{-1}}xa^*$ for all $x \in R$. This implies that $a^*a^2x = xa^*a^2$ or $a^*a\,ax - xa^*a\,a = 0$ for all $x \in R$. Since $a^*a \in Z(R)$, therefore rewriting the preceding equation we get $a^*a(ax - xa) = 0$ for all $x \in R$. Since a^*a is not a zero divisor, therefore $ax = xa$ for all $x \in R$ and hence a is a central element. This proves that $\alpha = 1$. The converse is simple. This completes the proof.□

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