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SOME FIXED POINT THEOREMS IN QUASI-METRIC SPACES

1. Introduction

A number of papers have been written involving fixed point theorems in quasi-complete metric spaces. (See, for example [2], [3], [5], and [6]). It is natural to ask what is the most general fixed point theorem available in such spaces. The purpose of this note is to partially answer that question by proving three general fixed point theorems.

By a quasi-metric on a space X we mean a mapping $d : X \times X \to \mathbf{R}^+$ satisfying $d(x, y) = 0$ iff $x = y$ and $d(x,z) \leq d(x,y) + d(y,z)$ for each x, y, z in X. We say that (X, d) is complete if every d-Cauchy sequence $\{x_n\}$ is left d-convergent; i.e., $d(x_n, x_m) \to 0$ as $m, n \to \infty$ implies that $\lim_{n\to\infty} d(x, x_n) = 0$ for some $x \in X$. The mapping $\overline{d}$ defined by $\overline{d}(x,y) := d(y,x)$ for $x, y \in X$ is called the conjugate of d. Then $(X, \overline{d})$ is complete if every d-Cauchy sequence is right d-convergent. If one defines the metric d^* on X by the condition

$$(1) \qquad d^*(x,y) = \max\{d(x,y), d(y,x)\},$$

then $(X,\ d^*)$ becomes a metric space. However, it is possible for $(X,\ d)$ to be complete without having $(X,\ d^*)$ complete. An example of such a situation appears in [6].

2. Main theorems

Our main results are extensions and generalizations of results of Rhoades and Watson [4] to quasi-metric spaces. Throughout this note $Q : \mathbf{R}^+ \to \mathbf{R}^+$

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is a nondecreasing function satisfying

$$\text{(lim)} \qquad \lim_{n\to\infty} Q^n(s) = 0 \text{ for each } s \in \mathbf{R}^+, \text{ and } \lim_{n\to\infty} (s - Q(S)) = \infty.$$

THEOREM 1. *Let T be a selfmap of a d-complete quasi-metric space $(X,\ d)$, p a fixed positive integer, Q satisfies* (lim) *and, for each x, y in X,*

$$(2) \qquad d(T^px,\ Ty) \le Q(\max\{d(y,\ T^ix),\ d(T^ix,\ Ty),\ d(T^ix,\ T^jx),\ d(y,\ Ty) : 0 \le i,j \le p\}).$$

Then T has a unique fixed point z, for each x in X, $\{T^nx\}$ is left and right d-convergent to z, and z is the only limit of this sequence.

Proof. Let d^* be the metric on X defined by (1). In (2) replace y with $T^{p-1}y$ to obtain

$$\begin{aligned}(3) \qquad d(T^px,\ T^py) &\le Q(\max\{d(T^{p-1}y,\ T^ix),\ d(T^ix,\ T^py),\\ &\qquad d(T^ix,\ T^jx),\ d(T^{p-1}y,\ T^py) : 0 \le i,j \le p\})\\ &\le Q(\max\{d(T^jy,\ T^ix),\ d(T^ix,\ T^jy),\\ &\qquad d(T^ix,\ T^jx),\ d(T^iy,\ T^jy) : 0 \le i,j \le p\})\\ &\le Q(\max\{d^*(T^jy,\ T^ix),\ d^*(T^ix,\ T^jy),\\ &\qquad d^*(T^ix,\ T^jx),\ d^*(T^iy,\ T^jy) : 0 \le i,j \le p\}).\end{aligned}$$

Interchanging the roles of x and y in (1) yields

$$(4) \qquad d(T^py,\ T^px) \le Q(\max\{d^*(T^jx,\ T^iy),\ d^*(T^ix,\ T^jy),\ d^*(T^ix,\ T^jx) : 0 \le i,j \le p\}).$$

Combining (3) and (4) gives

$$(5) \qquad d^*(T^px,\ T^py) \le Q(\max\{d^*(T^jy,\ T^ix),\ d^*(T^ix,\ T^jx),\ d^*(T^iy,\ T^jy) : 0 \le i,j \le p\}).$$

Then T satisfies (1) of Jachymski [1]. Therefore by Lemma 4 of Jachymski [1], $\{T^nx\}$ is a Cauchy sequence in $(X,\ d^*)$, hence Cauchy in $(X,\ d)$. Since $(X,\ d)$ is d−complete, $\{T^nx\}$ is left convergent to a point z in X; i.e., $\lim\ d(z,\ T^nx) = 0$.

Using the triangular inequality,

$$d(z,\ Tz) \le d(z,\ T^{n+p}x) + d(T^{n+p}x,\ Tz),$$

from (2), we have

$$d(T^{n+p}x,\ Tz) \le Q(\max\{d(z,\ T^{n+i}x),\ d(T^{n+j}x,\ Tz),\ d(T^{n+i}x,\ T^{n+j}x),\ d(z,\ Tz) : 0 \le i,j \le p\}).$$

Define $a_n = d(T^n x,\ Tz)$, $b_n := \max\ \{d(z,\ T^{n+i}x),\ d(T^{n+i}x,\ T^{n+j}x) : 0 \le i,j \le p\}$, $r := d(z,\ Tz)$. Then

$$(6) \qquad a_{n+p} \le Q(\max\ \{a_n,\ a_{n+1}, \ldots, a_{n+p},\ b_n, r\}).$$

If $a_{n+p} = 0$, then (6) is trivially true with a_{n+p} removed from the right hand side of the inequality. If $a_{n+p} > 0$, then, if the maximum of the quantity in braces is a_{n+p} we have $a_{n+p} \le Q(a_{n+p}) < a_{n+p}$, a contradiction. Since $\lim\ b_n = 0$, for all n sufficiently large, it follows that

$$a_{n+p} \le Q(\max\ \{a_n,\ a_{n+1}, \ldots, a_{n+p-1},\ r\}.$$

From Lemma 1 of Jachymski [1], lim sup $a_n \le Q(r)$. Suppose that $d(z,\ Tz) \neq 0$. Since $d(z,\ Tz) \le d(z,\ T^{n+p}x) + a_{n+p}$, then taking the lim sup yields $d(z,\ Tz) \le Q(r) = Q(d(z,\ Tz)) < d(z,\ Tz)$, a contradiction. Therefore z is a fixed point of T. Since $Q(0) = 0$, $a_n \ge 0$, and $\lim\ a_n = 0$. Therefore lim $d(T^n x,\ z) = 0$ and $\{T^n x\}$ is both left and right d-convergent to z.

To prove uniqueness, suppose that z and w are fixed points of T. Then, from (2),

$$\begin{aligned} d(z,\ w) = d(T^p z,\ Tw) &\le Q(\max\{d(w,\ T^i z), \\ &\quad d(T^i z,\ Tw),\ d(T^i z,\ T^j z),\ d(w,\ Tw) : 0 \le i,j \le p\}) \\ &= Q(\max\ \{d(w,\ z),\ d(z,\ w),\ 0,\ 0\}) = Q(d^*(z,\ w)). \end{aligned}$$

Similarly, $d(w,\ z) = d(T^p w, Tz) \le Q(d^*(z,\ w))$. Therefore $d^*(z,\ w) \le Q(d^*(z,\ w))$, which implies that $z = w$.

Remarks

1. Theorem 1 of Jachymski [2] is the special case of Theorem 1 by choosing $Q(s) = hs$.

2. Theorem 1 of Romaguera and Checa [6] is a special case of Theorem 1 by choosing $Q(s) = hs$, $p = 1$, and omitting the terms $d(y, T^i x)$ and $d(T^i x,\ Ty)$ in (2).

3. Theorem 2 of Romaguera [5], with $Q(s) = hs$, is also a special case of Theorem 1.

THEOREM 2. *Let T be a selfmap of a d-complete quasi-metric space $(X,\ d)$ such that $(X,\ \bar{d})$ is complete, p a fixed positive integer, Q satisfies (lim) and, for each x, y in X,*

$$d(Tx,\ T^p y) \le Q(\max\ \{d(x,\ T^i y),\ d(T^i y,\ Tx),\ d(T^i y,\ T^j y),\ d(x,\ Tx) : 0 \le i,j \le p\}).$$

Then T has a unique fixed point z and, for each x in X, $\{T^n x\}$ is left and right d-convergent to z, and z is the only limit of this sequence.

Proof. Apply Theorem 1 with d replaced by its conjugate $\bar{d}$.

THEOREM 3. *Let T be a continuous selfmap of a Hausdorff d-complete quasi-metric space $(X,\ d)$, p, q fixed positive integers, and, for each x, y in X,*

$$(7)\quad d(T^p x,\ T^q y) \le Q(\max\{d(T^i x,\ T^r y),\ d(T^i x,\ T^j x),\ d(T^r y,\ T^s y),\ d(T^r y,\ T^i x) : 0 \le i,j \le p,\ 0 \le r,s \le q\}).$$

Then T has a unique fixed point z and, for each x in X, $\{T^n x\}$ is both left and right d-convergent and z is the only limit of this sequence.

Proof. Define $t = \max\{p,\ q\}$. In (7) set $x = T^{t-p}x$, $y = T^{t-q}y$ to obtain

$$(8)\quad \begin{aligned} d(T^t x,\ T^t y) &\le Q(\max\{d(T^{i+t-p}x,\ T^{r+t-q}y),\ d(T^{i+t-p}x,\ T^{j+t-p}x),\\ &\qquad d(T^{r+t-q}y,\ T^{s+t-q}y),\ d(T^{t-q}y,\ T^{i+t-p}x) :\\ &\qquad 0 \le i,j \le p, 0 \le r,s \le q\})\\ &= Q(\max\{d(T^i x,\ T^r y),\ d(T^i x,\ T^j x),\\ &\qquad d(T^r y,\ T^s y),\ d(T^r y, T^i x) : t-p \le i,j \le t, t-q \le r,s, \le t\}\\ &\le Q(\max\{d^*(T^i x,\ T^r y),\ d^*(T^i x,\ T^j x),\ d^*(T^r y,\ T^s y) :\\ &\qquad 0 \le i,j,\ r,s \le t\}. \end{aligned}$$

Interchanging the roles of x and y in (8) we obtain

$$(9)\quad d(T^t y,\ T^t x) \le Q(\max\{d^*(T^i y,\ T^r x),\ d^*(T^i y,\ T^j y),\ d^*(T^r x,\ T^s x) : 0 \le i,j,r,s \le t\},$$

which implies that

$$d^*(T^t y,\ T^t x) \le Q(\max\{d^*(T^i y,\ T^r x),\ d^*(T^i y,\ T^j y),\ d^*(T^r x,\ T^s x) : 0 \le i,j,r,s \le t\}.$$

By Lemma 4 of Jachymski [1], $\{T^n x\}$ is Cauchy in $(X,\ d^*)$, where d^* is defined by (1). Hence $\{T^n x\}$ is Cauchy in $(X,\ d)$. Since $(X,\ d)$ is d-complete, $\{T^n x\}$ converges to some point z in X. Since T is continuous and X is Hausdorff, $z = \lim\ T^{n+1}x = T(\lim\ T^n x) = Tz$, and z is a fixed point of T. As in the proof of Theorem 1 of this paper, $\{T^n x\}$ is both left and right d-convergent.

Suppose that z and w are fixed points of T. Then, from (7),

$$\begin{aligned} d(z,\ w) &= d(T^p z,\ T^q w)\\ &\le Q(\max\{d(T^i z,\ T^r w),\ d(T^i z,\ T^j z),\ d(T^r w,\ T^s w).\\ &\qquad d(T^r w,\ T^i z) : 0 \le i,j \le p, 0 \le r,s \le q\})\\ &= Q(\max\{d(z,\ w),\ 0, 0, d(w,\ z)\}) = Q(d^*(z,\ w)). \end{aligned}$$

Similarly, $d(w, z) \leq Q(d^*(w, z))$. Therefore $d^*(z, w) \leq Q(d^*(z, w))$, which implies that $z = w$.

Remark 4. Theorems 1 and 4 of Romaguera [5] and Theorem 2 of Romaguera and Checa [6] are special cases of Theorem 3.

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