DEMONSTRATIO MATHEMATICA
Vol. XXX No 2 1997

S. S. Kim, Y. J. Cho, T. D. Narang

SOME REMARKS ON ORTHOGONALITY AND BEST APPROXIMATION

1. Introduction

Let X be a real linear space of dimension greater than 1 and let $\|.,.\|$ be a real-valued function on $X \times X$ satisfying the following conditions:

(N_1) $\|x,y\| = 0$ if and only if x and y are linearly dependent,

(N_2) $\|x,y\| = \|y,x\|$,

(N_3) $\|\alpha x,y\| = |\alpha|\|x,y\|$, where α is real,

(N_4) $\|x+y,z\| \leq \|x,z\| + \|y,z\|$.

Then $\|.,.\|$ is called a *2-norm* on X and $(X,\|.,.\|)$ a *linear 2-normed space*. Some of the basic properties of the 2-norm are that they are non-negative and $\|x,y+\alpha x\| = \|x,y\|$ for every $x,y \in X$ and every real number α ([9]).

A concept which is closely related to linear 2-normed space is that of 2-inner product spaces. For a linear space X of dimension greater than 1, let $(.,.|.)$ be a real-valued function on $X \times X \times X$ which satisfies the following conditions:

(I_1) $(x,x|z) \geq 0$,

$(x,x|z) = 0$ if and only if x and z are linearly dependent,

(I_2) $(x,x|z) = (z,z|x)$,

(I_3) $(x,y|z) = (y,x|z)$,

(I_4) $(\alpha x,y|z) = \alpha(x,y|z)$, where α is real,

(I_5) $(x+x',y|z) = (x,y|z) + (x',y|z)$.

Then $(.,.|.)$ is called a *2-inner product* and $(X,(.,.|.))$ is a *2-inner product space* (or *2-pre-Hilbert space*).

1991 *AMS Subject Classification Code* : 46B15, 46B20.

Key words and phrases : 2-pre-Hilbertian, Gâteaux derivative, smoothness, strict convexity, orthogonality, best approximation.

This paper was supported by Non-Directed Research Fund of Korea Resaerch Foundation, 1995.

The concept of linear 2-normed spaces was introduced by S. Gähler ([9]) and that of 2-inner product spaces by C. Diminnie, S. Gähler and A. White ([3]). The concepts are 2-dimensional analogoues of the concepts of normed linear spaces and inner product spaces.

In [3], it is shown that the function

$$\|x, z\| = (x, x|z)^{1/2}$$

is a 2-norm on $(X, (.,.|.))$, i.e., every 2-inner product space is a linear 2-normed space but not every 2-norm arises in this way.

Some basic properties of the 2-inner product $(.,.|.)$ are the following ([3], [4]):

(1) For all $x, y, z \in X$,

$$|(x, y|z)| \leq \sqrt{(x, x|z)}\sqrt{(y, y|z)}.$$

(2) For all $x, y \in X, (x, y|y) = 0$.

(3) For all $x, y, z \in X$ and a real number α,

$$(x, y|\alpha z) = \alpha^2 (x, y|z).$$

A linear 2-normed space $(X, \|.,.\|)$ is said to be *2-pre-Hilbertian* if the norm of X is derived from a 2-inner product.

I. Franić ([8]), S.A. Mariadoss ([13]), R. Ravi ([19]) and others obtained some results in linear 2-normed spaces which are similar to the results concerning the best approximation theory in normed linear spaces ([1], [10], [14], [15], [17], [20] and [21]).

In this paper, motivated by some of the results of M.M. Day ([7]), R.C. James ([11]), P.L. Papini ([16]) and T. Precupanu ([18]), we give some conditions for a linear 2-normed space to be 2-pre-Hilbertian and also some relations between orthogonality and best approximation by using Gâteaux derivatives.

2. Gâteaux derivatives and orthogonality

Let $(X, \|.,.\|)$ be a linear 2-normed space and $V(x, y)$ be a subspace of X generated by x and y in X. For all $x, y \in X$, define

$$n(x, y|z)(t) = \frac{\|x + ty, z\| - \|x, z\|}{t}$$

for any real t and $z \notin V(x, y)$.

Then the functional $n(x, y|z)(t)$ is non-decreasing of the real positive variable t for any fixed x, y in X and for arbitrary z in X. Moreover, $\lim_{t \to 0^+} n(x, y|z)(t)$ exists ([2]).

The functional $n(x,z)(y)$ defined by

$$n(x,z)(y) = \lim_{t\to 0^+} n(x,y|z)(t)$$

is called the *Gâteaux derivative* of the 2-norm $\|.,.\|$ at (x,z) in the direction y. Further, the 2-norm $\|.,.\|$ is said to be *Gâteaux differentiable* at (x,z) in the direction y if this limit exists. Some properties of $n(.,.)(.)$ are given in the following:

THEOREM 2.1 ([2]). *For every x,y in X and $z \notin V(x,y)$, we have the following properties:*

(1) $|n(x,z)(y)| \le \|y,z\|$,
(2) $n(x,z)(y+y') \le n(x,z)(y) + n(x,z)(y')$,
(3) $n(\alpha x,z)(\beta y) = \beta n(x,z)(y)$ *for all reals* $\alpha > 0$ *and* $\beta \ge 0$,
(4) $n(x,z)(0) = 0$ *and* $n(0,z)(y) = \|y,z\|$,
(5) $-n(x,z)(-y) = \lim_{t\to 0^-}(\|x+ty,z\| - \|x,z\|)/t \le n(x,z)(y)$,
(6) $n(x,z)(\alpha x) = \alpha\|x,z\|$ *for all real* α.

DEFINITION 2.1. A linear 2-normed space $(X,\|.,.\|)$ is said to be *smooth* if the 2-norm is Gâteaux differentiable at every $(x,z) \in X \times X$ in the direction y.

It is easy to see that $(X,\|.,.\|)$ is smooth if and only if one of the following conditions holds:

(1) $n(x,z)(y+y') = n(x,z)(y) + n(x,z)(y')$,
(2) $-n(x,z)(-y) = n(x,z)(y)$.

An element x of a linear 2-normed space $(X,\|.,.\|)$ is said to be *orthogonal* (also called *B-orthogonal*, see [12]) to a given element y of X (write $x\perp_z y$) if

$$\|x+\alpha y,z\| \ge \|x,z\|$$

for all $\alpha \in R$ and $z \notin V(x,y)$. An element $x \in X$ is said to be *orthogonal* to a subset G of X (write $x\perp_z G$) if $x\perp_z y$ for all y in G.

A linear 2-normed space $(X,\|.,.\|)$ is said to be *strictly convex* ([5]) if the conditions $\|x,z\| = \|y,z\| = \|x+y,z\|/2$ and $z \notin V(x,y)$ imply that $x = y$.

Further characterizations of strict convexity in linear 2-normed space are given in [2] and [6].

It is easy to see that if a linear 2-normed space $(X,\|.,.\|)$ is 2-pre-Hilbertian, then the 2-inner product $(.,.|.)$ is given by $\|x,z\|n(x,z)(y)$ for all $x,y,z \in X$. In fact, if $(X,(.,.|.))$ is a 2-inner product space and

$\|x+ty,z\|+\|x,z\|\neq 0$, then we have

$$\begin{aligned}n(x,z)(y)&=\lim_{t\to 0}\frac{(\|x+ty,z\|-\|x,z\|)(\|x+ty,z\|+\|x,z\|)}{t(\|x+ty,z\|+\|x,z\|)}\\&=\lim_{t\to 0}\frac{\|x+ty,z\|^2-\|x,z\|^2}{t(\|x+ty,z\|+\|x,z\|)}\\&=\lim_{t\to 0}\frac{2t(x,y|z)+t^2\|y,z\|^2}{t(\|x+ty,z\|+\|x,z\|)}\\&=\frac{2(x,y|z)}{2\|x,z\|}.\end{aligned}$$

Therefore, $(x,y|z)=\|x,z\|n(x,z)(y)$. If $\|x+ty,z\|+\|x,z\|=0$, then $x=\alpha z$ for any real α and hence

$$n(x,z)(y)=n(\alpha z,z)(y)=\|y,z\|.$$

Therefore, we have

$$(x,y|z)=(\alpha z,y|z)=0=\|\alpha z,z\|\|y,z\|=\|x,z\|n(x,z)(y).$$

Moreover, a linear 2-normed space $(X,\|.,.\|)$ is 2-pre-Hilbertian if and only if $x\perp_z y$ implies $y\perp_z x$ for all $x,y\in X$ and $z\notin V(x,y)$ with $\|x,z\|=\|y,z\|=1$ ([2]).

LEMMA 2.2 ([2]). *Let $(X,\|.,.\|)$ be a linear 2-normed space. The following statements are equivalent*:

(1) *The condition $n(x,z)(y)=0$ implies that $n(y,z)(x)=0$ for all $x,y\in X$ and $z\notin V(x,y)$ with $\|x,z\|=\|y,z\|=1$;*

(2) *$(X,\|.,.\|)$ is smooth and condition $x\perp_z y$ implies that $y\perp_z x$ for all $x,y\in X$ and $z\notin V(x,y)$ with $\|x,z\|=\|y,z\|=1$.*

LEMMA 2.3 ([2]). *A smooth linear 2-normed space $(X,\|.,.\|)$ is 2-pre-Hilbertian if and only if $n(x,z)(y)=n(y,z)(x)$ for all $x,y\in X$ and $z\notin V(x,y)$ with $\|x,z\|=\|y,z\|=1$.*

By using Lemmas 2.2 and 2.3, we have the following:

THEOREM 2.4. *A linear 2-normed space $(X,\|.,.\|)$ is 2-pre-Hilbertian if and only if*

$$|n(x,z)(y)|=|n(y,z)(x)| \tag{A}$$

for all $x,y\in X$ and $z\notin V(x,y)$ with $\|x,z\|=\|y,z\|=1$.

Proof. If a linear 2-normed space $(X,\|.,.\|)$ is 2-pre-Hilbertian, then the result is obvious.

Conversely, since (A) implies (1) in Lemma 2.2, then $(X, \|.,.\|)$ is smooth. We have to show the impossibility of the condition $n(x,z)(y) = -n(y,z)(x) > 0$. Suppose that $n(x,z)(y) = -n(y,z)(x) \neq 0$ for all $x, y \in X$ and $z \notin V(x,y)$ with $\|x,z\| = \|y,z\| = 1$. Without loss of generality, we may assume that $n(y,z)(x) > 0$, what implies that $\|\alpha x + y, z\| > 1$ for $\alpha > 0$. Then, for $\alpha = -n(x,z)(y) > 0$, we have

$$n(x,z)(\alpha x + y) = 0.$$

Thus, by Lemma 2.2, $n(\alpha x + y, z)(x) = 0$ and so we obtain

$$\|\alpha x + y, z\| = n(\alpha x + y, z)(\alpha x + y) = n(\alpha x + y, z)(y) \leq \|y, z\| = 1,$$

which is a contradition. Therefore, by Lemma 2.3, $(X, \|.,.\|)$ is 2-pre-Hilbertian. This completes the proof.

3. Orthogonality and best approximations

Let $(X, \|.,.\|)$ be a linear 2-normed space with $\dim X \geq 3$ and $[x, G]$ be the subspace of X generated by x and the elements of G.

DEFINITION 3.1 ([18]). Let $(X, \|.,.\|)$ be a linear 2-normed space and G be a subspace of X. Fix $x \in X \setminus \overline{G}$ and take $z \in X \setminus [x, G]$. Then $g_o \in G$ is called *an element of best approximation* of x by G with respect to z iff the inequality

$$\|x - g_o, z\| \leq \|x - g, z\|$$

holds for all $g \in G$.

The set of all elements of best approximation of x by G with respect to z is denoted by $P_{G,z}(x)$; that is, for all $g \in G$ and $z \in X \setminus [x, G]$,

$$\begin{aligned} P_{G,z}(x) &= \{g_o \in G : \|x - g_o, z\| \leq \|x - g, z\|\} \\ &= \{g_o \in G : \|x - g_o, z\| = \inf_{g \in G} \|x - g, z\|\}. \end{aligned}$$

Remark 1. If $x \in G$, then $P_{G,z}(x) = \{x\}$ for each $z \in X \setminus [x, G]$.

EXAMPLE ([18]). Let $X = R^3$ with the vector addition and scalar multiplication defined componentwise and the 2-norm defined on X as follows: for $x = (a_1, b_1, c_1)$ and $y = (a_2, b_2, c_2) \in X$,

$$\|x, y\| = \max\{|a_1 b_2 - a_2 b_1|, |b_1 c_2 - b_2 c_1|, |a_1 c_2 - a_2 c_1|\}.$$

Let $G = \{(a, 0, 0) : a \in R\}$ be a subspace of X and $x = (0, 1, 0)$ be an element of $X \setminus G$. If $z \in \{(0, 0, k_3) : k_3 \in R \setminus \{0\}\}$, then

$$P_{G,z}(x) = \{g_o = (b, 0, 0) \in G : -1 \leq b \leq 1\}.$$

For further details on the best approximations in linear 2-normed spaces, we refer the reader to [8], [13] and [19] and, now we give our main results which extend the results of [16] to linear 2-normed spaces.

THEOREM 3.1. *Let* $(X, \|.,.\|)$ *be a strictly convex linear 2-normed space. Then* X *is 2-pre-Hilbertian if and only if*

$$\text{(B)} \qquad \|P_{V(x),z}(y)\| = \|P_{V(y),z}(x)\|$$

for all $x, y \in X$ *and* $z \notin V(x,y)$ *with* $\|x,z\| = \|y,z\| = 1$, *where* $\|P_{V(x),z}(y)\| = |(x,y|z)|$.

Proof. The strict convexity of the space X implies the uniqueness of best approximation and this makes (B) meaningful. The 'only if ' part is obvious and so we need to prove only the 'if ' part.

Suppose that the condition (B) holds. This means that

$$x \perp_z y \quad \text{implies} \quad y \perp_z x$$

for all $x, y \in X$ and $z \notin V(x,y)$ with $\|x,z\| = \|y,z\| = 1$. In fact, orthogonality $x \perp_z y$ implies $P_{V(y),z}(x) = \{0\}$. Then by (B), we have $P_{V(x),z}(y) = \{0\}$ and so $y \perp_z x$. Now if $\|x,z\| = \|y,z\| = 1$, then $P_{V(x),z}(y) = \{\alpha x\}$ is equivalent to $(y - \alpha x) \perp_z x$, thus also to $x \perp_z (y - \alpha x)$ and therefore $n(x,z)(y) = \alpha$. Since $\|y,z\| = 1$ then, by using (B) again, we obtain

$$|n(x,z)(y)| = \|P_{V(x),z}(y)\| = \|P_{V(y),z}(x)\| = |n(y,z)(x)|.$$

Therefore, from Theorem 2.4, $(X, \|.,.\|)$ is 2-pre-Hilbertian. This completes the proof.

The following lemma can be easily proved:

LEMMA 3.2. *Let* $(X, \|.,.\|)$ *be a linear 2-normed space,* C *be a convex subset of* X *and* $x \in X$. *Then the element* $x_o \in P_{C,z}(x)$ *are characterized by*

$$n(x - x_o, z)(x_o - y) \geq 0 \quad \text{for all} \quad y \in C.$$

Using Lemma 3.2, we have the following:

THEOREM 3.3. *Let* $(X, \|.,.\|)$ *be a linear 2-normed space satisfying the following condition*:

$$\text{(C)} \qquad \|\frac{x+y}{2}, z\| \leq \|tx + (1-t)y, z\|$$

for all $x, y \in X$ *and* $z \notin V(x,y)$ *with* $\|x,z\| = \|y,z\| = 1$ *and for all* $t \in [0,1]$. *Then* $(X, \|.,.\|)$ *is 2-pre-Hilbertian.*

Proof. Fix $x, y \in X$ and $z \notin V(x,y)$ with $\|x,z\| = \|y,z\| = 1$. By (C), $(x+y)/2 \in P_{C,z}(0)$, where C is the line segment joining x and y. Thus, by

Lemma 3.2, for all $t \in [0,1]$, we have

$$\text{(D)} \qquad 0 \le n\Big(-\frac{(x+y)}{2}, z\Big)\Big(\frac{x+y}{2} - (tx + (1-t)y\Big) = n\Big(\frac{x+y}{2}, z\Big)\Big((t-\frac{1}{2})(x-y)\Big).$$

For $t < 1/2$, from (D), we calculate $n(x+y,z)(x-y) \ge 0$, while we have $n(x+y,z)(-(x-y)) \ge 0$. Therefore, we have

$$-n(x+y,z)(y-x) \le 0 \le n(x+y,z)(x-y),$$

which means that $(x+y)\perp_z(x-y)$. Thus, from (C), it follows that the condition $\|x,z\| = \|y,z\| = 1$ implies, $(x+y)\perp_z(x-y)$. So $(X, \|.,.\|)$ is 2-pre-Hilbertian and this completes the proof.

References

[1] G. C. Ahuja, T. D. Narang and S. Trehar, *Best approximation on convex sets in metric linear spaces*, Math. Nachr. 78(1977), 125–130.

[2] Y. J. Cho and S. S. Kim, *Gâteaux derivatives and 2-inner product spaces*, Glasnik Mat. 27(47)(1992), 271–282.

[3] C. Diminnie, S. Gähler and A. White, *2-inner product spaces*, Demonstratio Math. 6(1973), 525–536.

[4] C. Diminnie, S. Gähler and A. White, *2-inner product spaces II*, Demonstratio Math. 10(1977), 169–188.

[5] C. Diminnie, S. Gähler and A. White, *Strictly convex linear 2-normed spaces*, Math. Nachr. 59(1974), 319–324.

[6] C. Diminnie, S. Gähler and A. White, *Remarks on strictly convex and strictly 2-convex 2-normed spaces*, Math. Nachr. 88(1979), 363–372.

[7] M. M. Day, *Normed linear spaces*, Springer–Verlag, Berlin, 1962.

[8] I. Franić, *An extension theorem for bounded linear 2-functionals and applications*, Math. Japon. 40(1994), 79–85.

[9] S. Gähler, *Lineare 2-normierte Räume*, Math. Nachr. 28(1965), 1–43.

[10] G. Godini, *Best approximation in certain classes of normed linear spaces*, J. Approx. Theory, 39(1983), 157–171.

[11] R. C. James, *Orthogonality and linear functionals in normed linear spaces*, Trans. Amer. Math. Soc. 61(1947), 265–292.

[12] A. Khan and A. Siddiqui, *B-orthogonality in 2-normed spaces*, Bull. Calcuta Math. Soc. 74(1982), 216–222.

[13] S. A. Mariadoss, *Orthogonality, Approximation and Fixed Points in Linear 2-Normed Spaces*, Doctoral Diss., Madras Univ., 1982.

[14] T. D. Narang, *Best approximation and strict convexity of metric spaces*, Arch. Math. 17(1981), 87–90.

[15] P. L. Papini, *Approximation and strong approximation in normed spaces via tangent functions*, J. Approx. Theory 22(1978), 111–118.

[16] P. L. Papini, *Inner product and norm derivatives*, J. Math. Anal. and Appl. 91(1983), 592–598.
[17] P. L. Papini and I. Singer, *Best coapproximation in normed linear spaces*, Mh. Math. 88(1979), 27–44.
[18] T. Precupanu, *Characterizations of hilbertian norms*, Boll. U.M.I. 15(1978), 161–169.
[19] R. Ravi, *Approximation in Linear 2-Normed Spaces and Normed Linear Spaces*, Doctoral Diss., Madras Univ., 1994.
[20] I. Singer, *Best approximation in normed linear spaces by elements of linear subspaces*, Springer–Verlag, New York, 1970.
[21] A. B. Thaheem, *Existence of best approximations*, Portug. Math. 42(1984), 435–440.

S. S. Kim
DEPARTMENT OF MATHEMATICS
DONGEUI UNIVERSITY
PUSAN 614-714, KOREA

Y. J. Cho
DEPARTMENT OF MATHEMATICS
GYEONGSANG NATIONAL UNIVERSITY
CHINJU 660-701, KOREA

T. D. Narang
DEPARTMENT OF MATHEMATICS
GURU NANAK DEL UNIVERSITY
AMRITSAR 143005, INDIA

Received November 28, 1995; revised version December 1996.