DEMONSTRATIO MATHEMATICA
Vol. XXX No 2 1997

Bao Jundong

OSCILLATORY BEHAVIOUR OF FIRST ORDER LINEAR NEUTRAL DIFFERENTIAL EQUATIONS WITH DISTRIBUTED DELAYS

In this paper, the oscillatory and nonoscillatory behavior of solutions of a class of the first order neutral differential equations with distributed delays is discussed, and the oscillatory and nonoscillatory criteria are obtained. Furthermore, the sufficient and necessary conditions for oscillation are given. In the end we also obtained a comparison theorem.

1. Introduction

In our paper, we consider the oscillatory behavior of a class of first order neutral differential eqations with distributed delays of the form

$$\frac{d}{dt}[x(t) - px(t-\tau)] + \int_0^{\sigma} x(t-s)dg(t,s) = 0 \tag{1.1}$$

and

$$\frac{d}{dt}[x(t) - px(t-\tau)] + \sum_{i=1}^{n} \int_0^{\sigma_i} x(t-s)dg_i(t,s) = 0, \tag{1.2}$$

where $p \in [0,1]$, τ, σ and σ_i are positive constants, $i = 1,2,\cdots,n$; $g(t,s)$, $g_i(t,s)$ are functions of bounded variation in $s \in [0,\sigma]$ or $s \in [0,\sigma_i]$, $i = 1,2,\cdots,n$, and they are continuous with respect to argument $t \in [t_0,\infty)$ for every fixed s.

Let us denote by $r = \max\{\tau,\sigma\}$ (or $r = \max\{\tau,\sigma_1,\sigma_2,\cdots,\sigma_n\}$) and $L = \{\tau,\sigma\}$ (or $L = \{\tau,\sigma_1,\sigma_2,\cdots,\sigma_n\}$). By a solution of Eq. (1.1) (or Eq. (1.2)), we mean that $x \in C([t_0 - r,\infty),R)$ for some $t_0 \in R$, such that $x(t) - px(t-\tau)$ is continuously differentiable for $t \geq t_0$ and such that Eq. (1.1) (or Eq. (1.2)) is satisfied for all $t \geq t_0$.

As it is customary, a solution x(t) is called to be oscillatory if it has arbitrarily large zeros; otherwise it is called to be nonoscillatory. Eq. (1.1) (or Eq. (1.2)) is said to be oscillatory if all of its solutions are oscillatory.

Recently many authors have considered the oscillation characteristics of the first neutral differential equations [1-15] and give necessary and sufficient conditions for oscillation. Although they obtained many strong resultes, most of them have concentrated their attention on the equations with discrete delays, and only a few authors (see [7-12]) concentrated on the equations with distributed delays. In [1-6] the authors discussed the equations of the following form

$$\frac{d}{dt}[x(t) - px(t-\tau)] + \sum_{i=1}^{n} q_i(t)x(t-\sigma_i) = 0 \tag{1.3}$$

with $p \in [0,1]$ and with distinct conditions for $q_i(t)$, $i = 1, 2, \cdots, n$. Especially L.H.Erbe and Q.Kong [1] substantially improved the results of [2-6] and obtained good and sharp results for Eq. (1.3). In our paper, we solved more generallized neutral differential equations with distributed delays Eq. (1.1) and Eq. (1.2). The results we obtained included [1]. It is well-known that equations with discrete delays are the special case of the one with distributed delays. Thus, here we have successfully extended the results of [1] to the equations with distributed delays. Hence, we have further discovered the general characters and common rules between the special case and his general form.

First we cite the following Lemma from [1]:

LEMMA 1.1. *Let $a > 0, b > 0$ and $f(t) \geq 0$ be a locally integrable function on $[0, \infty)$. Assume that both the limits*

$$I_1 = \lim_{t\to\infty} \frac{1}{a} \int_t^{t+a} f(s)ds \quad \text{and} \quad I_2 = \lim_{t\to\infty} \frac{1}{b} \int_t^{t+b} f(s)ds$$

exist and are finite. Then $I_1 = I_2$.

2. Oscillation criteria

Let us now consider the equation

$$\frac{d}{dt}[x(t) - px(t-\tau)] + \int_0^{\sigma} x(t-s)dg(t,s) = 0 \tag{2.1}$$

where $p \in [0,1]$, $\tau, \sigma \in R_+, g(t,s)$ is a function of bounded variation on $s \in [0,\sigma]$, $g(t,s) > 0$ and continuous in $t \in [t_0, \infty]$ for every fixed $s \in [0,\sigma]$ and satisfies

$$dg(t-\tau, s) \leq dg(t,s), \quad s \in [0,\sigma], \quad t \geq t_0 + \tau. \tag{2.2}$$

Denote by $L = \{\tau, \sigma\}$ and assume that

$$V(t,\sigma) = V_{s=0}^{\sigma}\, g(t,s) > v > 0, \quad \text{for any } t \in [t_0, \infty),$$

where v is a positive costant. First we give the following lemma:

LEMMA 2.1. *Let $x(t)$ be an eventually positive solution of Eq.* (2.1) *and let*

$$z(t) = x(t) - px(t-\tau).$$

Then we have $z(t) > 0$, $z'(t) < 0$ and

$$z'(t) - pz'(t-\tau) + \int_0^\sigma z(t-s)dg(t,s) \leq 0. \tag{2.3}$$

Proof. From (2.1) it is easy to see that $z(t) > 0$, $z'(t) < 0$. Because of (2.2) we have

$$z'(t) - pz'(t-\tau) = -\int_0^\sigma x(t-s)dg(t,s) + p\int_0^\sigma x(t-\tau-s)dg(t-\tau,s)$$
$$\leq -\int_0^\sigma [x(t-s) - px(t-\tau-s)]dg(t,s) = -\int_0^\sigma z(t-s)dg(t,s).$$

So (2.3) is true.

Our main results are as follows:

THEOREM 2.1. *If for all $\mu > 0$ and $l \in L$*

$$\liminf_{t\to\infty}\left[pe^{\mu\tau} + \frac{1}{l\mu}\int_t^{t+l}\int_0^\sigma e^{\mu s}dg(\theta,s)d\theta\right] > 1, \tag{2.4}$$

then Eq. (2.1) *is oscillatory.*

Proof. Suppose that (2.1) has an eventually positive solution $x(t) > 0$, $t \geq t_1$, t_1 is large enough. By Lemma (2.1), $z(t) > 0$, $z'(t) < 0$, and (2.3) holds for $t \geq t_1$. Let $w(t) = -\frac{z'(t)}{z(t)}, t \geq t_1$. Then $w(t) > 0$ and from (2.3) we have

$$w(t) \geq pw(t-\tau)\exp\Big(\int_{t-\tau}^t w(s)ds\Big) + \int_0^\sigma \exp\Big(\int_{t-s}^t w(\theta)d\theta\Big)dg(t,s), \quad \text{for} \quad t \geq t_1 + r. \tag{2.5}$$

Let $\{w_k(t)\}$ be a sequence of functions, for $t \geq t_1$, $\{\mu_k\}$ defined by

$$\begin{cases} w_1 \equiv 0, \ t \geq t_1 \\ w_{k+1}(t) = pw_k(t-\tau)\exp\Big(\int_{t-\tau}^t w_k(s)ds\Big) \\ \qquad + \int_0^\sigma \exp\Big(\int_{t-s}^t w_k(\theta)d\theta\Big)dg(t,s), \quad k = 1, 2, \cdots t \geq t_1 + kr \end{cases} \tag{2.6}$$

and consider a sequence $\{\mu_k\}$ given by

$$(2.7)\quad \begin{cases} \mu_1 = 0 \\ \mu_{k+1} = \inf_{t\geq t_1} \min_{l\in L}\{p\mu_k e^{\mu_k \tau} + \frac{1}{l}\int_t^{t+l}\int_0^{\sigma} e^{\mu_k s} dg(\theta,s)d\theta\},\ k = 1,2,\cdots \end{cases}$$

Now we claim that the following statements are true for $\{w_k\}$ and $\{\mu_k\}$ given by (2.6) and (2.7)

i) $0 = \mu_1 < \mu_2 < \cdots$ and $\lim_{k\to\infty}\mu_k = +\infty$;
ii) $w_k(t) < w(t)$ for $t \geq t_1 + (k-1)r$ and $k = 1,2,\cdots$,
iii) $\frac{1}{l}\int_t^{t+l} w_k(s)ds \geq \mu_k$ for $t \geq t_1 + (k+1)r$, $k = 1,2,\cdots, l \in L$.

To see i), observe that for $k = 1$ and $k = 2$, we have

$$\mu_2 = \inf_{t\geq t_1} \min_{l\in L}\{p\mu_1 e^{\mu_1 \tau} + \frac{1}{l}\int_t^{t+l}\int_0^{\sigma} e^{\mu_1 s} dg(\theta,s)d\theta\} = \frac{1}{l}\int_t^{t+l} V(\theta,\sigma)d\theta > 0 = \mu_1.$$

If we assume that $\mu_k > \mu_{k-1}$, then

$$\begin{aligned}\mu_{k+1} &= \inf_{t\geq t_1} \min_{l\in L}\{p\mu_k e^{\mu_k \tau} + \frac{1}{l}\int_t^{t+l}\int_0^{\sigma} e^{\mu_k s} dg(\theta,s)d\theta\} \\ &> \inf_{t\geq t_1} \min_{l\in L}\{p\mu_{k-1} e^{\mu_{k-1} \tau} + \frac{1}{l}\int_t^{t+l}\int_0^{\sigma} e^{\mu_{k-1} s} dg(\theta,s)d\theta\} = \mu_k\end{aligned}$$

i.e. $\mu_{k+1} > \mu_k$. So by the induction argument, μ_k is an increasing sequence. By (2.4) and (2.7) one can easily see that $\lim_{k\to\infty}\mu_k = +\infty$.

ii) Since $0 = w_1(t) < w(t)$, for $t \geq t_1$ then

$$\begin{aligned}w_2(t) &= pw_1(t-\tau)\exp\Big(\int_{t-\tau}^{t} w_1(s)ds\Big) + \int_0^{\sigma} \exp\Big(\int_{t-s}^{t} w_1(\theta)d\theta\Big)dg(t,s) \\ &= V(t,\sigma) > 0 = w_1(t), \text{ for } t \geq t_1 + r.\end{aligned}$$

If we assume that $w_{k-1}(t) < w(t)$ for $t \geq t_1 + (k-2)r$, then

$$\begin{aligned}w_k(t) &= pw_{k-1}(t)\exp\Big(\int_{t-\tau}^{t} w_{k-1}(s)ds\Big) + \int_0^{\sigma} \exp\Big(\int_{t-s}^{t} w_{k-1}(\theta)d\theta\Big)dg(t,s) \\ &< pw(t-\tau)\exp\Big(\int_{t-\tau}^{t} w(s)ds\Big) + \int_0^{\sigma} \exp\Big(\int_{t-s}^{t} w(\theta)d\theta\Big)dg(t,s) \\ &\leq w(t), \text{ for } t \geq t_1 + (k-1)r.\end{aligned}$$

So by the induction argument for any $k = 1, 2, \cdots$, we have $w_k(t) < w(t)$, for $t \geq t_1 + (k-1)r$.

iii) It is clear that iii) is true for $k = 1$. Now let us assume that iii) is true for some k. Then from (2.6) and (2.7) we conclude that

$$\frac{1}{l}\int_t^{t+l} w_{k+1}(s)ds$$

$$= \frac{p}{l}\int_t^{t+l} w_k(s)\exp\Big(\int_{s-\tau}^{s} w_k(\theta)d\theta\Big)ds + \frac{1}{l}\int_t^{t+l}\int_0^{\sigma} \exp\Big(\int_{s-\theta}^{s} w_k(\delta)d\delta\Big)dg(s,\theta)ds$$

$$\geq p\mu_k e^{\mu_k \tau} + \frac{1}{l}\int_t^{t+l}\int_0^{\sigma} e^{\mu_k s}dg(\theta, s)d\theta$$

$$\geq \inf_{t\geq t_1}\min_{l\in L}\{p\mu_k e^{\mu_k \tau} + \frac{1}{l}\int_t^{t+l}\int_0^{\sigma} e^{\mu_k s}dg(\theta, s)d\theta\} = \mu_{k+1}.$$

Hence iii) holds.

Now from i), ii) and iii) we deduce that

$$\lim_{t\to\infty}\int_t^{t+\sigma} w(s)ds = \infty.$$

Integrating $w(t) = -\frac{z'(t)}{z(t)}$ from t to $t + \frac{\sigma}{2}$, we get

$$\frac{z(t)}{z(t+\frac{\sigma}{2})} = \exp\int_t^{t+\frac{\sigma}{2}} w(s)ds$$

and thus

$$\limsup_{t\to\infty}\frac{z(t)}{z(t+\frac{\sigma}{2})} = \limsup_{t\to\infty}\exp\int_t^{t+\frac{\sigma}{2}} w(s)ds = \infty. \tag{2.8}$$

Now let us observe that

$$z'(t) = -\int_0^{\sigma} x(t-s)dg(t,s) \leq -\int_0^{\sigma} z(t-s)dg(t,s) \leq -V(t,\sigma)z(t-\sigma)$$
$$\leq -vz(t-\sigma).$$

Integrating both sides from $t + \frac{\sigma}{2}$ to $t + \sigma$, we find that

$$0 < z(t+\sigma) \leq z(t+\frac{\sigma}{2}) - \frac{v\sigma}{2}z(t).$$

Hence $\frac{z(t)}{z(t+\frac{\sigma}{2})} < \frac{2}{v\sigma}$, which contradicts (2.8). This completes the proof.

COROLLARY 2.1. *Assume that $l \in L$*

$$\liminf_{t\to\infty} \sum_{k=0}^{\infty} \frac{1}{l} \int_t^{t+l} \int_0^{\sigma} p^k(k\tau + s)dg(\theta, s)d\theta > \frac{1}{e}. \tag{2.9}$$

Then Eq. (2.1) *is oscillatory.*

Proof. By Theorem 2.1, it is enough to show that (2.4) is true.

It is true if for all $\mu > 0$, $pe^{\mu\tau} \geq 1$. So we may assume that $pe^{\mu\tau} < 1$. But

$$\frac{1}{\mu l} \int_t^{t+l} \int_0^{\sigma} e^{\mu s}(1 - pe^{\mu\tau})^{-1} dg(\theta, s)d\theta = \frac{1}{\mu l} \sum_{k=0}^{\infty} \int_t^{t+l} \int_0^{\sigma} p^k e^{\mu(k\tau+s)} dg(\theta, s)d\theta$$
$$\geq \sum_{k=0}^{\infty} \frac{1}{l} \int_t^{t+l} \int_0^{\sigma} p^k e(k\tau + s)dg(\theta, s)d\theta.$$

Thus from (2.9) we obtain

$$\liminf_{t\to\infty} \frac{1}{\mu l} \int_t^{t+l} \int_0^{\sigma} e^{\mu s}(1 - pe^{\mu\tau})^{-1} dg(\theta, s)d\theta > 1$$

i.e.

$$\liminf_{t\to\infty} \Big[pe^{\mu\tau} + \frac{1}{\mu l} \int_t^{t+l} \int_0^{\sigma} e^{\mu s} dg(\theta, s)d\theta\Big] > 1.$$

So (2.4) is true for all $\mu > 0$. This completes the proof.

In the following theorem we will give a criterion for the existence of nonoscillatory solution.

THEOREM 2.2. *Assume that there exist $\mu^* > 0$ and $t_1 \geq t_0$ for $l \in L$ such that*

$$\sup_{t\geq t_1} \Big[pe^{\mu^*\tau} + \frac{1}{\mu^* l} \int_t^{t+l} \int_0^{\sigma} e^{\mu^*\theta} dg(s, \theta)ds\Big] \leq 1. \tag{2.10}$$

Then Eq. (2.1) *has at least one positive solution on* $[t_1 + r, \infty)$.

Proof. First let us observe that the integral equation

$$u(t) = pu(t - \tau)\exp\Big(\int_{t-\tau}^{t} u(s)ds\Big) + \int_0^{\sigma} \exp\Big(\int_{t-s}^{t} u(\theta)d\theta\Big) dg(t, s) \tag{2.11}$$

possesses a positive solution on $[t_1 + r, \infty)$. To see this, let us make up a sequence $\{u_k(t)\}$. Setting

$$(2.12)\quad \begin{cases} u_1(t) \equiv 0, \quad t \geq t_1 \\ u_{k+1}(t) = \begin{cases} p u_k(t-\tau)\exp\Big(\int\limits_{t-\tau}^{t} u_k(s)ds\Big) \\ \quad + \int\limits_0^{\sigma} \exp\Big(\int\limits_{t-s}^{t} u_k(\theta)d\theta\Big)dg(t,s), \quad t \geq t_1 + r \\ \beta_{k+1}(t), \quad t_1 \leq t < t_1 + r, \end{cases} \end{cases}$$

where $\{\beta_k(t)\}$ is given sequence satisfying

i) $\beta_k \in C^2\Big([t_1, t_1 + r), [0,\infty)\Big)$ with $\beta_k'(t) \geq 0$ and $\beta_k''(t) \geq 0$, $t \in [t_1, t_1 + r)$, $k = 1, 2, \cdots$,

ii) $\beta_k(t) = 0, t \in [t_1, t_1 + r - \tau), \beta_k(t_1 + r) = u_k(t_1 + r)$ and $\beta_k(t)$ are increasing in k for $t \in [t_1 + r - \tau, t_1 + r)$, $k = 1, 2, \cdots$,

iii) for $l \in L, k = 1, 2, \cdots$

$$\int\limits_t^{t_1+r} \beta_k(s)ds \leq \int\limits_{t+l}^{t_1+r+l} u_k(s)ds, t \in [t_1 + r - l, t_1 + r).$$

It is clear that $u_1(t) \leq u_2(t) \leq \cdots$. We claim that for $k = 1, 2, \cdots$, and $l \in L$

$$(2.13)\qquad \frac{1}{l}\int\limits_t^{t+l} u_k(s)ds \leq \mu^*, \quad t \geq t_1.$$

In fact, (2.13) is true for k=1. Assume (2.13) is true for some k. Then from (2.10) and (2.12), we have

$$(2.14)\quad \frac{1}{l}\int\limits_t^{t+l} u_{k+1}(s)ds$$

$$= \frac{p}{l}\int\limits_t^{t+l} u_k(s-\tau)\exp\Big(\int\limits_{s-\tau}^{s} u_k(\theta)d\theta\Big) + \frac{1}{l}\int\limits_t^{t+l}\int\limits_0^{\sigma} \exp\Big(\int\limits_{s-\theta}^{s} u_k(\delta)d\delta\Big)dg(s,\theta)ds$$

$$\leq p\mu^* e^{\mu_* \tau} + \frac{1}{l}\int\limits_t^{t+l}\int\limits_0^{\sigma} e^{\mu^*\theta}dg(s,\theta)ds \leq \mu^*, \qquad t \geq t_1 + r.$$

For $t \in [t_1 + r - l, t_1 + r), l \in L$, from (2.14) and condition iii) we have

$$\frac{1}{l}\int\limits_t^{t+l} u_{k+1}(s)ds = \frac{1}{l}\Big[\int\limits_t^{t_1+r} \beta_{k+1}(s)ds + \int\limits_{t_1+r}^{t+l} u_{k+1}(s)ds\Big]$$

$$\leq \frac{1}{l}\Big[\int_{t+l}^{t_1+r+l} u_{k+1}(s)ds + \int_{t_1+r}^{t+l} u_{k+1}(s)ds\Big] = \frac{1}{l}\int_{t_1+r}^{t_1+r+l} u_{k+1}(s)ds \leq \mu^*.$$

From the monotonic property of $\beta_{k+1}(t)$ with respect to t we can see that (2.13) also holds for $t \in [t_1, t_1 + r - l)$, $l \in L$.

Now let $u(t) = \lim_{k\to\infty} u_k(t)$. Then $u(t) \equiv 0, t \in [t_1, t_1 + r - \tau)$, $u(t)$ is increasing on $[t_1 + r - \tau, t_1 + r)$ and for $t \geq t_1$ and $l \in L$

$$\frac{1}{l}\int_t^{t+l} u(s)ds \leq \mu^*.$$

Let $k \to \infty$ on both sides of (2.12). By the Lebesgue monotone convergence theorem, we see that $u(t)$ satisfies (2.11) for $t \geq t_1 + r$. It is also easy to see that $u(t)$ is well-defined on $[t_1, \infty)$. In fact, by condition ii) of $\{\beta_k\}$

$$\begin{aligned} u(t_1 + r) &= \int_0^\sigma \exp\Big(\int_{t_1+r-s}^{t_1+r} u(\theta)d\theta\Big)dg(t_1 + r, s) \\ &\leq \int_0^\sigma e^{\mu^* s} dg(t_1 + r, s) \leq e^{\mu^*\sigma} V(t_1 + r, \sigma) < \infty \end{aligned}$$

and hence $u(t)$ is bounded for $t \in [t_1, t_1+r]$. If $u(t^*) = \infty$ for some $t^* > t_1+r$, then choose an integer m such that $t^* - m\tau \in [t_1 + r - \tau, t_1 + r)$. By (2.11) we have $u(t^* - m\tau) = \infty$, this is imposible. Furthermore, from i) we get that $u(t)$ is continuous on $[t_1, t_1 + r]$, so in view of (2.11) we see that $u(t)$ is continuous on $[t_1, \infty)$. Thus $u(t)$ is a positive solution of (2.11) on $[t_1 + r, \infty)$. Set

$$x(t) = \exp\Big(-\int_{t_1+r}^t u(s)ds\Big).$$

We can verify that $x(t)$ is a positive solution of (2.1). In fact, notice that $u(t)$ is a solution of (2.11), we have

$$\frac{d}{dt}[x(t) - x(t-\tau)] = -u(t)\exp\Big(-\int_{t_1+r}^t u(s)ds\Big) + pu(t-\tau)\exp\Big(-\int_{t_1+r}^{t-\tau} u(s)ds\Big) =$$

$$-\Big[pu(t-\tau)\exp\Big(\int_{t-\tau}^t u(s)ds\Big) + \int_0^\sigma \exp\Big(\int_{t-s}^t u(\theta)d\theta\Big)dg(t,s)\Big]\exp\Big(-\int_{t_1+r}^t u(\theta)d\theta$$

$$-pu(t-\tau)\exp\Big(-\int_{t_1+r}^{t-\tau} u(s)ds\Big)$$

$$= -\int_0^\sigma \exp\Big(-\int_{t_1+\tau}^{t-s} u(\theta)d\theta\Big)dg(t,s) = -\int_0^\sigma x(t-s)dg(t,s).$$

This completes the proof.

THEOREM 2.3. *Assume that*

$$\int_t^{t+l}\int_0^\sigma e^{\mu\theta}dg(s,\theta)ds$$

is a nondecreasing function in t for $l \in L$. *Then Eq.* (2.1) *is oscillatory if and only if for all* $\mu > 0$ *and* $l \in L$

$$\lim_{t\to\infty}\Big[pe^{\mu\tau} + \frac{1}{l\mu}\int_t^{t+l}\int_0^\sigma e^{\mu\theta}dg(s,\theta)ds\Big] > 1.$$

Proof. Denote

$$f(t,\mu,l) = pe^{\mu\tau} + \frac{1}{l\mu}\int_t^{t+l}\int_0^\sigma e^{\mu\theta}dg(s,\theta)ds.$$

Since $\int_t^{t+l}\int_0^\sigma e^{\mu\theta}dg(s,\theta)ds$ is nondecreasing, so we conclude that $\lim_{t\to\infty} f(t,\mu,l)$ exists for $l \in L$. By Lemma 1.1 we have

$$\lim_{t\to\infty} f(t,\mu,\tau) = \lim_{t\to\infty} f(t,\mu,\sigma)$$

and for any $l \in L$

$$\lim_{t\to\infty} f(t,\mu,l) = \liminf_{t\to\infty} f(t,\mu,l) = \sup_{t\geq t_1} f(t,\mu,l).$$

Then Theorems 2.1 and 2.3 immediately complete the proof.

Now we give out a corollary as a special case of Theorem 2.4.

COROLLARY 2.2. *If there exists* $c > 0$ *such that* $\tau = m_0c$, $\sigma = m_1c$, m_0, m_1 *are integers, and*

$$g(t,s) = V_1(t,s) + V_2(t,s),$$

where $V_i(t,s)$ $(i=1, 2)$ *are also functions of bounded variation in* $s \in [0,\sigma]$, *and* $V_1(t,s)$ *is c-periodic function with respect to t, and* $\frac{1}{l}\int_t^{t+l} d_\theta V_1(\theta,s) = V_1(s)$, $V_2(t,s)$ *satisfies* $\lim_{t\to\infty} V_2(t,s) = V_2(s)$ *is a bounded variation function and also*

$$\lim_{t\to\infty}\int_0^\sigma e^{\mu s}dV_2(t,s) = \int_0^\sigma e^{\mu s}dV_2(s),$$

then Eq. (2.1) *is oscillatory if and only if for all* $\mu > 0$

$$pe^{\mu\tau} + \frac{1}{\mu}\int_0^\sigma e^{\mu s}d(V_1(s) + V_2(s)) > 1.$$

In the following let's consider the equation

$$\frac{d}{dt}[x(t) - px(t-\tau)] + \sum_{i=1}^{n} \int_0^{\sigma_i} x(t-s)dg_i(t,s) = 0 \tag{2.15}$$

where $p, \tau, \sigma_i, g_i(t,s), (i = 1,2,\cdots,n)$ satisfy the same condition as p, τ, σ, $g(t,s)$ which we mentioned above. Here we only need to redefine $r = \max\{\tau, \sigma_1, \sigma_2, \cdots, \sigma_n\}$ and $L = \{\tau, \sigma_1, \sigma_2, \cdots, \sigma_n\}$. Since the proof of the following Theorems are similar to those which we proved above, we will only give the theorems without proofs.

LEMMA 2.1*. *Let $x(t)$ be an eventually positive solution of Eq.* (2.15), *and let $z(t) = x(t) - px(t-\tau)$. Then we have $z(t) > 0, z'(t) < 0$ and*

$$z'(t) - pz'(t-\tau) + \sum_{i=1}^{n} \int_0^{\sigma_i} z(t-s)dg_i(t,s) \le 0. \tag{2.3*}$$

THEOREM 2.1*. *If for all $\mu > 0$, and $l \in L$*

$$\lim_{t\to\infty} inf\left[pe^{\mu\tau} + \frac{1}{l\mu}\int_t^{t+l} \sum_{i=1}^{n} \int_0^{\sigma_i} e^{\mu s} dg_i(\theta,s)d\theta\right] > 1. \tag{2.4*}$$

Then Eq. (2.15) *is oscillatory.*

COROLLARY 2.1*. *If for all $l \in L$*

$$\lim_{t\to\infty} inf \sum_{i=1}^{n} \sum_{k=0}^{\infty} \frac{1}{l} \int_t^{t+l} \int_0^{\sigma_i} p^k(k\tau + s)dg_i(\theta,s)d\theta > \frac{1}{e}, \tag{2.9*}$$

then Eq. (2.15) *is oscillatory.*

THEOREM 2.2*. *Assume that there exists $\mu^* > 0$ and $t_1 \ge t_0$ for $l \in L$ such that*

$$\sup_{t\ge t_1}\left[pe^{\mu^*\tau} + \frac{1}{\mu^* l}\int_t^{t+l} \sum_{i=1}^{n} \int_0^{\sigma_i} e^{\mu^*\theta} dg_i(s,\theta)ds\right] \le 1.$$

Then Eq. (2.15) *has at least one positive solution on $[t_1 + r, \infty)$.*

THEOREM 2.3*. *Assume that $\int_t^{t+l}\sum_{i=1}^n \int_0^{\sigma_i} e^{\mu\theta}dg_i(s,\theta)ds$ is a nondecreasing function in t for $l \in L$. Then Eq. (2.15) is oscillatory if and only if for all $\mu > 0$ and $l \in L$*

$$\lim_{t\to\infty} \left[pe^{\mu\tau} + \frac{1}{l\mu}\int_t^{t+l} \sum_{i=1}^{n} \int_0^{\sigma_i} e^{\mu\theta} dg_i(s,\theta)ds\right] > 1.$$

COROLLARY 2.2*. *If there exists $c > 0$ such that $\tau = m_0 c$, $\sigma_i = m_i c$, m_0, m_i, $(i = 1, 2, \cdots, n)$ are integers, and*

$$g_i(t,s) = V_{i1}(t,s) + V_{i2}(t,s)$$

where $V_{ij}(t,s), (i = 1,2,\cdots,n, j = 1,2)$ are also functions of bounded variation and $V_{i1}(t,s), (i = \overline{1,n})$ are c-periodic functions with respect to t, and $\frac{1}{l}\int_t^{t+l} d_\theta V_{i1}(\theta,s) = V_{i1}(s), V_{i2}(t,s)$ satisfy $\lim_{t\to\infty} V_{i2}(t,s) = V_{i2}(s)$ are bounded variation functions and also

$$\lim_{t\to\infty} \int_0^{\sigma_i} e^{\mu s} dV_{i2}(t,s) = \int_0^{\sigma_i} e^{\mu s} dV_{i2}(s).$$

Then Eq. (2.15) is oscillatory if and only if

$$pe^{\mu\tau} + \frac{1}{\mu}\sum_{i=1}^{n} \int_0^{\sigma_i} e^{\mu s} d(V_{i1}(s) + V_{i2}(s)) > 1.$$

3. Comparison theorem

Consider the following pair of equations

$$\frac{d}{dt}[x(t) - p_1 x(t-\tau_1)] + \int_0^{\sigma_1} x(t-s) dg_1(t,s) = 0 \tag{3.1}$$

and

$$\frac{d}{dt}[x(t) - p_2 x(t-\tau_2)] + \int_0^{\sigma_2} x(t-s) dg_2(t,s) = 0, \tag{3.2}$$

$p_i \in [0,1], \tau_i, \sigma_i \in R_+, g_i(t,s)$ satisfy the same conditions as $g(t,s)$ mentioned in Section 2, i=1, 2. We also assume that the conditions in Corollary 2.2 hold for $g_1(t,s)$. Then we have the following comparison theorem

THEOREM 3.1. i) *Suppose that Eq. (3.1) is oscillatory, $p_2 \geq p_1, \tau_2 \geq \tau_1$ and for all $\mu > 0, l = \{\tau_2, \sigma_2\}$*

$$\liminf_{t\to\infty} \frac{1}{l} \int_t^{t+l} \int_0^{\sigma_2} e^{\mu s} dg_2(\theta,s) d\theta \geq \int_0^{\sigma_1} e^{\mu s} d(V_1(s) + V_2(s)). \tag{3.3}$$

Then Eq. (3.2) is oscillatory.

ii) *Suppose that Eq. (3.2) is oscillatory, $p_2 \leq p_1, \tau_2 \leq \tau_1$, and for all $\mu > 0, l = \{\tau_2, \sigma_2\}$, there exists $T \geq t_0$ such that*

$$\sup_{t\geq T} \frac{1}{l} \int_t^{t+l} \int_0^{\sigma_2} e^{\mu s} dg_2(\theta,s) d\theta \leq \int_0^{\sigma_1} e^{\mu s} d(V_1(s) + V_2(s)). \tag{3.4}$$

Then Eq. (3.1) is oscillatory.

iii) *Suppose that Eq.* (3.2) *is nonoscillatory*, $p_2 \geq p_1, \tau_2 \geq \tau_1$, *and for all* $\mu > 0, l = \{\tau_2, \sigma_2\}$ (3.3) *holds. Then Eq.* (3.1) *has at least one nonoscillatory solution.*

iv) *Suppose that Eq.* (3.1) *is nonoscillatory*, $p_2 \leq p_1, \tau_2 \leq \tau_1$, *and there exists* $T \geq t_0$ *such that for all* $\mu > 0, l = \{\tau_2, \sigma_2\}$ *(3.4) holds. Then Eq. (3.2) has at least one nonoscillatory solution.*

Proof. i) Since Eq. (3.1) is oscillatory, by Corollary 2.2, we have for all $\mu > 0$

$$p_1 e^{\mu\tau_1} + \frac{1}{\mu}\int_0^{\sigma_1} e^{\mu s} d(V_1(s) + V_2(s)) > 1$$

and from (3.3) we obtain that

$$\lim_{t\to\infty} inf\left[p_2 e^{\mu\tau_2} + \frac{1}{l\mu}\int_t^{t+l}\int_0^{\sigma_2} e^{\mu s} dg_2(\theta, s)d\theta\right] > 1. \tag{3.5}$$

By Theorem 2.1, (3.5) implies that Eq. (3.2) is oscillatory.

ii) If Eq. (3.1) is nonoscillatory, then by Corollary 2.2, there must exist $\mu^* > 0$ such that

$$pe^{\mu^*\tau_1} + \frac{1}{\mu^*}\int_0^{\sigma_1} e^{\mu^* s} d(V_1(s) + V_2(s)) \leq 1.$$

Hence from (3.4)

$$\lim_{t\to\infty} inf\left[pe^{\mu^*\tau_2} + \frac{1}{l}\int_t^{t+l}\int_0^{\sigma_2} e^{\mu^* s} dg_2(\theta, s)d\theta\right] \leq 1. \tag{3.6}$$

Thus, by Theorem 2.2, (3.6) implies that Eq. (3.2) has nonoscillatory solution, it contradicts assumption.

iii) and iv) are the converses of i) and ii). This completes the proof.

For the following pair of equations

$$\frac{d}{dt}[x(t) - p_1 x(t-\tau_1)] + \sum_{i=1}^{n}\int_0^{\sigma_{i1}} x(t-s)dg_{i1}(t,s) = 0 \tag{3.7}$$

and

$$\frac{d}{dt}[x(t) - p_2 x(t-\tau_2)] + \sum_{i=1}^{n}\int_0^{\sigma_{i2}} x(t-s)dg_{i2}(t,s) = 0 \tag{3.8}$$

where p_j, τ_j, σ_{ij}, are constants satisfing the corresponding conditions in Theorem 2.3*, $g_{i1}(t,s), (i = 1, 2, \cdots, n; j = 1, 2)$ also satisfy the condition as $g_2(t,s)$ in Theorem 3.1. Then we can give out the comparison theorem between Eq. (3.7) and Eq. (3.8) without proof.

THEOREM 3.2. i) *Suppose that Eq.* (3.7) *is oscillatory,* $p_2 \geq p_1$, $\tau_2 \geq \tau_1$ *and for all* $\mu > 0$, $l = \{\tau_2, \sigma_{12}, \sigma_{22}, \cdots, \sigma_{n2}\}$

$$(3.9) \quad \lim_{t\to\infty} inf \frac{1}{l} \int_t^{t+l} \sum_{i=1}^{n} \int_0^{\sigma_{i2}} e^{\mu s} dg_{i2}(\theta, s) d\theta \geq \sum_{i=1}^{n} \int_0^{\sigma_{i1}} d(V_{i1}(s) + V_{i2}(s)).$$

Then Eq. (3.8) *is oscillatory.*

ii) *Suppose that Eq.* (*3.8*) *is oscillatory,* $p_2 \leq p_1, \tau_2 \leq \tau_1$ *and for all* $\mu > 0, l = \{\tau_2, \sigma_{12}, \sigma_{22}, \cdots, \sigma_{n2}\}$, *there exists* $T \geq t_0$ *such that*

$$(3.10) \quad \sup_{t\geq T} \frac{1}{l} \int_t^{t+l} \sum_{i=1}^{n} \int_0^{\sigma_{i2}} e^{\mu s} dg_{i2}(\theta, s) d\theta \leq \sum_{i=1}^{n} \int_0^{\sigma_{i1}} e^{\mu s} d(V_{i1}(s) + V_{i2}(s)).$$

Then Eq. (3.7) *is oscillatory.*

iii) *Suppose that* (3.8) *is nonoscillatory,* $p_2 \geq p_1, \tau_2 \geq \tau_1$, *and for all* $\mu > 0, l = \{\tau_2, \sigma_{12}, \sigma_{22}, \cdots, \sigma_{n2}\}$ (*3.9*) *holds. Then Eq.* (3.7) *has at least one nonoscillatory solution.*

iv) *Suppose that Eq.* (3.7) *is nonoscillatory,* $p_2 \leq p_1, \tau_2 \leq \tau_1$, *and there exists* $T \geq t_0$ *such that for all* $\mu > 0, l = \{\tau_2, \sigma_{12}, \sigma_{22}, \cdots, \sigma_{n2}\}$ (*3.10*) *holds. Then Eq.* (3.8) *has a nonoscillatory solution.*

References

[1] L. H. Erbe and Q. Kong, *Oscillation and nonoscillation properties of neutral differential equations*, Canad. J. Math. 46(2) (1994), 284–297.

[2] O. Arino and I. Győri, *Necessary and sufficient condition for oscillation of a neutral differential system with several delays*, J. Differential Equations, 81(1989), 98–105.

[3] Q. Chuanxi, G. Ladas, B. G. Zhang, and T. Zhao, *Sufficient conditions for oscillation and existence of positive solutions*, Appl. Anal. 35(1990), 187–194.

[4] E. A. Grove, M. R. S. Kulenovic and G. Ladas, *Sufficient conditions for oscillation and nonoscillation of neutral equations*, J. Differential Equations, 68(1987), 373–382.

[5] Q. Huang and S. Chen, *Oscillation of neutral differential equations with periodic coefficients*, Proc. Amer. Math. Soc. 110(1990), 997–1001.

[6] G. Ladas and Y. G. Sficas, *Oscillation of higher-order neutral equations*, J. Austral. Math. Soc. Ser(B) 27(1986), 502–511.

[7] D. Tsvetkov, *Oscillation in linear differential-difference equations*, Math. Balkanica, (1995) to appear.

[8] J. Graf, M. Grammatikopoulos and P. Spikes, *On the asymptotic behaviour of solutions of a second-order nonlinear neutral delay differential equations*, J. Math. Anal. and Appl. 156(1991) 23–39.

[9] I. Győri, G. Ladas, *Positive solutions of integro-differential equations with unbounded delay*, J. Integral Equations Appl. 4(3)(1992), 377–390.

[10] I. Győri, B. S. Lalli, E. Thandapani, *Oscillatory conditions for a Stieltjes type neutral integro-differential equations*, Differential Equations Dynam. Systems. 1(3)(1993), 245–256.

[11] D. Bainov and V. Petrov, *Necessary and sufficient conditions for oscillation of autonomous neutral differential equations with distributed delay*, J. Math. Anal. Appl., 182(1)(1994), 202–220.

[12] D. Bainov and V. Petrov, *Asymptotic properties of the nonoscillatory solutions of second-order neutral equations with a deviating argument*, J. Math. Anal. Appl., 190(3)(1995), 645–653.

[13] G. Ladas, C. Qian, *A comparison theorem for odd-order neutral differential equations*, J. Nigerian Math. Soc. 10(1991), 99–104.

[14] Pitambar Das, *Oscillation criteria for odd order neutral equations*, J. Math. Anal. Appl., 188(1)(1994), 245–257.

[15] Qian Chuanxi, G. Ladas, *Oscillation of systems of first order neutral differential equations*, Ann. Differential Equations, 8(2)(1992), 139–147.

DEPARTMENT OF MATHEMATICS
INNER MONGOLIA NORMAL UNIVERSITY
HUHHOT 010022, CHINA

Received October 17, 1995.