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# A COMMON FIXED POINT THEOREM FOR COMPATIBLE MAPPINGS

## 1. Introduction

In [1], the concept of compatible mappings was introduced as a generalization of commuting mappings. The utility of compatiblity in the context of fixed point theory was demonstrated by extending a theorem of Park and Bae [9]. Popa [8] proved a common fixed point for four self-mappings of a complete metric space using the concept of weak commutativity of Sessa [10]. His result improves and extends the recent results of Xieping Ding [12]. In this paper we extend the result of Popa [8] by employing compatible mappings in lieu of weakly commuting mappings.

The following definition is given in [1]:

DEFINITION 1.1. Let $f, g$ be mappings from a metric space $(X, d)$ into itself. Then $f$ and $g$ are said to be *compatible* if $\lim_{n\to\infty} d(fgx_n, gfx_n) = 0$ whenever $\{x_n\}$ is a sequence in $X$ such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = z \quad \text{for some} \quad z \quad \text{in} \quad X.$$

Thus, if $d(fgx_n, gfx_n) \to 0$ as $d(fx_n, gx_n) \to 0$, then $f$ and $g$ are compatible.

S. Sessa [10] generalized commuting mappings by calling mappings $A$ and $B$ from a metric space $(X, d)$ into itself *a weakly commuting pair* if $d(ABx, BAx) \le d(Ax, Bx)$ for all $x \in X$.

Clearly *commuting mappings* are *weakly commuting* and *weakly commuting pairs* are *compatible*, examples in [1], [10] show that neither converse is

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true. Several articles already in print, demonstrate that results can be generalized by using compatiblity in lieu of commutativity or weak commutativity (see for example [1]–[5] and [11]).

For our main theorem we need the following Lemma 1.1, which was proved by G.Jungck [1] .

LEMMA 1.1. *Let $S$ and $T$ be compatible mappings from a metric space $(X,d)$ into itself. Suppose that*

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$$

*for some $t \in X$. Then $\lim_{n\to\infty} TSx_n = St$ if $S$ is continuous.*

Throughout this paper, the function $\phi : [0,\infty)^5 \longrightarrow [0,\infty)$ satisfies the following conditions:

(i) $\phi$ is nondecreasing and upper semicontinuous in each coordinate variables,

(ii) For each $t > 0$

$$\Psi(t) = \max\{\phi(t,0,0,t,t), \phi(t,t,t,2t,0), \phi(t,t,t,0,2t)\} < t.$$

LEMMA 1.2 [7]. *Suppose $\Psi : [0,\infty) \longrightarrow [0,\infty)$ is nondecreasing and upper semicontinuous from the right. If $\Psi(t) < t$ for every $t > 0$, then $\lim_{n\to\infty} \Psi^n(t) = 0$, where $\Psi^n(t)$ denotes the composition of $\Psi(t)$ with $n$-times.*

Now, let $S, T, I$ and $J$ be mappings from a metric space $(X,d)$ into itself satisfying the following conditions

$$SX \supset JX \quad \text{and} \quad TX \supset IX, \tag{1.1}$$

$$d(Sx,Ty) \le \phi(d(Ix,Jy), d(Ix,Sx), d(Jy,Ty), d(Ix,Ty), d(Jy,Sx)), \tag{1.2}$$

for all $x, y \in X$, with $\phi$ satisfying (i)-(iii). Then for an arbitrary point $x_0 \in X$, by the assumption (1.1), we may choose a point $x_1 \in X$ such that $y_1 = Tx_1 = Ix_0$. For this point $x_1$ there exists an $x_2 \in X$ such that $y_2 = Sx_2 = Jx_1$. Continuing in this manner we get sequences $\{x_n\}$ and $\{y_n\}$ such that

$$y_{2n} = Sx_{2n} = Jx_{n-1}; \quad y_{2n+1} = Tx_{2n+1} = Ix_{2n}. \tag{1.3}$$

LEMMA 1.3. *Let $S$, $T$, $I$ and $J$ be mappings from a metric space $(X,d)$ into itself satisfying the conditions (1.1) and (1.2). Then $\{y_n\}$ defined by (1.3) is a Cauchy sequence in $X$.*

Proof. By (1.2) and (1.3), we have

$$\begin{aligned}d(y_{2n+1},y_{2n+2}) &= d(Tx_{2n+1},Sx_{2n+2})\\ &\leq \phi(d(Ix_{2n+2},Jx_{2n+1}),d(Ix_{2n+2},Sx_{2n+2}),d(Jx_{2n+1},Tx_{2n+1}),\\ &\qquad d(Ix_{2n+2},Tx_{2n+1}),d(Jx_{2n+1},Sx_{2n+2}))\\ &\leq \phi(d(y_{2n},y_{2n+1}),d(y_{2n},y_{2n+2}),d(y_{2n},y_{2n+1}),\\ &\qquad d(y_{2n+1},y_{2n+1}),d(y_{2n},y_{2n+2}))\\ &\leq \phi(d(y_{2n},y_{2n+1}),d(y_{2n+1},y_{2n+2}),d(y_{2n},y_{2n+1}),\\ &\qquad 0,d(y_{2n},y_{2n+1})+d(y_{2n+1},y_{2n+2})).\end{aligned}$$

If in the above inequality would be $d(y_{2n+1},y_{2n+2}) > d(y_{2n},y_{2n+1})$, then we would have

$$\begin{aligned}&d(y_{2n+1},y_{2n+2})\\ &\leq \phi(d(y_{2n+1},y_{2n+2}),d(y_{2n+1},y_{2n+2}),d(y_{2n+1},y_{2n+2}),0,2d(y_{2n+1},y_{2n+2}))\\ &\leq \Psi(d(y_{2n+1},y_{2n+2})) < d(y_{2n+1},y_{2n+2})\end{aligned}$$

which is a contradiction. Thus,

$$\begin{aligned}(1.4)\quad &d(y_{2n+1},y_{2n+2})\\ &\leq \phi(d(y_{2n},y_{2n+1}),d(y_{2n},y_{2n+1}),d(y_{2n},y_{2n+1}),0,2d(y_{2n},y_{2n+1}))\\ &\leq \Psi(d(y_{2n},y_{2n+1}).\end{aligned}$$

Similarly, we have

$$(1.5)\qquad d(y_{2n+2},y_{2n+3}) \leq \Psi(d(y_{2n+1},y_{2n+2}).$$

From (1.4) and (1.5) it follows that

$$(1.6)\qquad d_n = d(y_n,y_{n+1}) \leq \Psi(d(y_{n-1},y_n)) \leq \dots \leq \Psi^{n-1}(d(y_1,y_2)).$$

By (1.6) and Lemma 1.2 we obtain

$$(1.7)\qquad \lim_{n\to\infty} d_n = 0.$$

In order to show that $\{y_n\}$ is a Cauchy sequence, it is sufficient to show that $\{y_{2n}\}$ has this property. Suppose that $\{y_{2n}\}$ is not a Cauchy sequence. Then there is an $\epsilon > 0$ such that, for each even integer $2k$, there exist even integers $2m(k)$ and $2n(k)$ such that

$$(1.8)\qquad d(y_{2m(k)},y_{2n(k)}) > \epsilon$$

for $2m(k) > 2n(k) \geq 2k$. For each even integer $2k$, let $2m(k)$ be the least even integer exceeding $2n(k)$ satisfying (1.8), that is

$$(1.9)\qquad d(y_{2n(k)},y_{2m(k)-2}) < \epsilon \quad\text{and}\quad d(y_{2n(k)},y_{2m(k)}) > \epsilon.$$

Then for each even integer 2k,

$$\epsilon < d(y_{2n(k)}, y_{2m(k)}) \leq d(y_{2n(k)}, y_{2m(k)-2}) + d_{2m(k)-2} + d_{2m(k)-1}.$$

It follows from (1.7) and (1.8) that

$$\lim_{n\to\infty} d(y_{2n(k)}, y_{2m(k)}) = \epsilon. \tag{1.10}$$

By the triangle inequality,

$$|d(y_{2n(k)}, y_{2m(k)-1}) - d(y_{2n(k)}, y_{2m(k)})| \leq d_{2m(k)-1},$$

and

$$|d(y_{2n(k)+1}, y_{2m(k)-1}) - d(y_{2n(k)}, y_{2m(k)})| \leq d_{2m(k)-1} + d_{2n(k)}.$$

From (1.2) and (1.3), we have

$$\begin{aligned} d(y_{2n(k)}, y_{2m(k)}) &\leq d_{2n(k)} + d(y_{2n(k)+1}, y_{2m(k)}) \\ &\leq d_{2n(k)} + d(Sx_{2n(k)}, Tx_{2m(k)-1}) \\ &\leq d_{2n(k)} + \phi(d(y_{2n(k)}, y_{2m(k)-1}), d_{2n(k)}, d_{2m(k)-1}, \\ &\qquad d(y_{2n(k)}, y_{2m(k)-1}), d(y_{2n(k)-1}, y_{2m(k)+1})). \end{aligned}$$

Since $\phi$ is upper semicontinuous, then

$$\epsilon \leq \phi(\epsilon, 0, 0, \epsilon, \epsilon) < \epsilon \quad \text{as} \quad k \to \infty,$$

which is a contradiction. Hence $\{y_n\}$ is a Cauchy sequence.

## 2. A fixed point theorem

In a recent paper [8], the following theorem is proved:

THEOREM 2.1. *Let $S$, $T$, $I$ and $J$ be four self-mappings of $X$ such that*

$$TX = IX \quad \text{and} \quad SX = JX,$$

$$d(Sx, Ty) \leq \phi(d(Ix, Jy), d(Ix, Sx), d(Jy, Ty), d(Ix, Ty), d(Jy, Sx)).$$

*for all $x, y \in X$.*

If one of $S$, $T$, $I$ and $J$ is continuous and $S$ and $T$ weakly commute respectively with $I$ and $J$, then $S$, $T$, $I$, $J$ have a common fixed point $z$. Furthermore $z$ is the unique common fixed point of $S$ and $I$ and of $T$ and $J$.

Now, we prove a common fixed point theorem which improves and extends Theorem 2.1 for compatible mappings. Also our theorem improves Theorem 2.2 from [4].

THEOREM 2.2. *Let $S$, $T$, $I$ and $J$ be four self-mappings of $X$ satisfying the conditions* (1.1) *and* (1.2). *If one of $S$, $T$, $I$ and $J$ is continuous and $(S, I)$, $(T, J)$ are compatible pairs of $X$, then $S$, $T$, $I$ and $J$ have a common fixed point $z$. Furthermore $z$ is the unique common fixed point of $S$ and $I$ and of $T$ and $J$.*

Proof. By Lemma 1.3, $\{y_n\}$ defined by (1.3) is a Cauchy sequence and so it converges to a point $z$ in $X$. Consequently, the subseqences

$$\{Sx_{2n}\}, \quad \{Jx_{2n+1}\}, \quad \{Tx_{2n+1}\}, \quad \text{and} \quad \{Ix_{2n}\}$$

converge to $z$.

Let us first of all suppose that $I$ is continuous. Since $(S, I)$ is compatible on $X$, then Lemma 1.1 gives

$$I^2x_{2n} \longrightarrow Iz \quad \text{and} \quad SIx_{2n} \longrightarrow Iz \quad \text{as} \quad n \longrightarrow \infty.$$

By (1.2), we obtain

$$d(SIx_{2n}, Tx_{2n+1}) \leq \phi(d(I^2x_{2n}, Jx_{2n+1}), d(I^2x_{2n}, SIx_{2n}), d(Jx_{2n+1}, Tx_{2n+1}), d(I^2x_{2n}, Tx_{2n+1}), d(Jx_{2n+1}, SIx_{2n})).$$

Letting $n \longrightarrow \infty$, we have

$$d(Iz, z) \leq \phi(d(Iz, z), 0, 0, d(Iz, z), d(Iz, z)),$$

so that $Iz = z$.

By (1.2), we also have

$$d(Sz, Tx_{2n+1}) \leq \phi(d(Iz, Jx_{2n+1}), d(Iz, Sz), d(Jx_{2n+1}, Tx_{2n+1}), d(Iz, Tx_{2n+1}), d(Jx_{2n+1}, Sz)).$$

Letting $n \longrightarrow \infty$, we have

$$d(Sz, z) \leq \phi(0, d(Sz, z), 0, 0, d(z, Sz)),$$

so that $Sz = z$. Since $SX = JX$, $z \in JX$ and hence there exists a point $u$ in $X$ such that $z = Sz = Ju$,

$$\begin{aligned} d(z, Tu) &= d(Sz, Tu) \\ &\leq \phi(d(Iz, Ju), d(Iz, Sz), d(Ju, Tu), d(Iz, Tu), d(Ju, Sz)) \\ &\leq \phi(0, 0, d(z, Tu), d(z, Tu), 0). \end{aligned}$$

which implies that $Tu = z$. Since $(T, J)$ is compatible on $X$ and $Tu = Ju = z$, then $d(JTu, TJu) = 0$ and hence

$$Tz = TJu = JTu = Jz.$$

Moreover, by (1.2), we obtain

$$d(z, Tz) = d(Sz, Tz) \leq \phi(d(z, Tz), 0, 0, d(z, Tz), d(z, Tz)),$$

so that $Tz = z$. Therefore $z$ is a common fixed point of $S$, $T$, $J$ and $I$. Similarly we can complete the proof when $J$ is continuous.

Next, suppose that $S$ is continuous. Since $(S, I)$ is continuous on $X$, it follows from Lemma 1.1 that

$$S^2x_{2n} \longrightarrow Sz, \quad ISx_{2n} \longrightarrow Sz \quad \text{as} \quad n \longrightarrow \infty.$$

By (1.2), we have

$$d(S^2x_{2n}, Tx_{2n+1}) \le \phi(d(ISx_{2n}, Jx_{2n+1}), d(ISx_{2n}, S^2x_{2n}), d(Jx_{2n+1}, Tx_{2n+1}), d(ISx_{2n}, Tx_{2n+1}), d(Jx_{2n+1}, S^2x_{2n})).$$

Letting $n \longrightarrow \infty$, we have

$$d(Sz, z) \le \phi(d(Sz, z), 0, 0, d(Sz, z), d(Sz, z)),$$

so that $Sz = z$. Hence there exists a point $v$ in $X$ such that $z = Sz = Jv$.

$$d(S^2x_{2n}, Tv) \le \phi(d(ISx_{2n}, Jv), d(ISx_{2n}, S^2x_{2n}), d(Jv, Tv), d(ISx_{2n}, Tv), d(Jv, S^2x_{2n})).$$

By letting $n \longrightarrow \infty$, we have

$$d(z, Tv) \le \phi(0, 0, d(z, Tv), d(z, Tv), 0),$$

so that $Tv = z$. Since $(T, J)$ is compatible on $X$ and

$$Tv = Jv = z, \quad d(JTv, TJv) = 0$$

and hence

$$Tz = TJv = JTv = Jz,$$

$$d(Sx_{2n}, Tz) \le \phi(d(Ix_{2n}, Jz), d(Ix_{2n}, Sx_{2n}), d(Jz, Tz), d(Ix_{2n}, Tz), d(Jz, Sx_{2n})).$$

By letting $n \longrightarrow \infty$, we have

$$d(z, Tz) \le \phi(d(z, Tz), 0, 0, d(z, Tz), d(z, Tz)),$$

so that $Tz = z$. Since $TX = IX$, there exists a point $w$ in $X$ such that $z = Tz = Iw$.

By (1.2), we have

$$d(Sw, z) = d(Sw, Tz) \le \phi(0, d(Sw, z), 0, 0, d(Sw, z)),$$

so that $Iw = Sw$, $d(ISw, SIw) = 0$ and hence

$$Iz = ISw = SIw = Sz.$$

Therefore $z$ is a common fixed point of $S$, $T$, $I$ and $J$. Similarly we can complete the proof when $T$ is continuous.

Finally we prove the uniqueness of $z$. Indeed let $z$ and $w$, are common fixed points of $S$, $T$, $I$ and $J$. Therefore

$$d(z, w) = d(Sz, Tw) \le \phi(d(z, w), 0, 0, d(z, w), d(z, w)),$$

which implies that $z = w$.

In conclusion, we wish to present an example which shows that our Theorem 2.2 is indeed a generalization of Theorem 2.1.

EXAMPLE. Let $X = [1, \infty)$ with the Euclidean metric $d$. Define $S$, $T$, $I$ and $J : X \longrightarrow X$ by

$$Sx = x^2, \quad Tx = x^4$$
$$Ix = 2x^4 - 1, \quad Jx = 2x^8 - 1, x \geq 1.$$

Now

$$SX = TX = IX = JX = X$$

and $S, T, I$ and $J$ are continuous. Moreover, since

$$d(Sx_n, Ix_n) = |x_n^2 - 2x_n^4 + 1| \longrightarrow 0 \quad \text{iff} \quad x_n \longrightarrow 1,$$
$$d(SIx_n, ISx_n) = |(2x_n^4 - 1)^2 - 2(2x_n^4 - 1)^4 + 1| \longrightarrow 0 \quad \text{as} \quad x_n \longrightarrow 1.$$

Thus S and I are compatible on X. Likewise

$$d(Tx_n, Jx_n) = |x_n^4 - 2x_n^8 + 1| \longrightarrow 0 \quad \text{iff} \quad x_n \longrightarrow 1.$$
$$d(TJx_n, JTx_n) = |(2x_n^8 - 1)^4 - 2(x_n^4)^8 + 1| \longrightarrow 0 \quad as \quad x_n \longrightarrow 1$$

and so $T$ and $J$ are compatible on $X$.

Take

$$\phi(t_1, t_2, t_3, t_4, t_5) = \frac{t_1}{2}$$

for every $t_i \in R^5$, i=1,2,3,4,5.

For any $x, y \in X$ we have

$$\begin{aligned} d(Sx, Ty) &= |x^2 - y^4| \leq |x^2 - y^4||x^2 + y^4| \\ &= |x^4 - y^8| = 2\frac{|x^4 - y^8|}{2} \\ &\leq \phi(d(Ix, Jy), d(Ix, Sx), d(Jy, Ty), d(Ix, Ty), d(Jy, Sx)). \end{aligned}$$

Since $\phi$ obviously satisfies the conditions (i) and (ii) then all the assumptions of Theorem 2.2 are verified and 1 is the unique common fixed point of $S$, $T$, $I$ and $J$.

Note Theorem 2.1 is not applicable even if $S = T$ and $I = J$ because $S$ and $I$ are not weakly commuting mappings at $x = 2$.

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