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THE LATTICE OF ROUGH SUBSETS OF A ROUGH SET. THE CATEGORY OF ROUGH SETS

Introduction

The notions of a rough set and of a Heyting algebra valued set have originated from attempts to describe complex phenomena or loosely defined concepts which are intractable to the methods of conventional mathematics.

The notion of a rough set was introduced by Pawlak [6] in 1981. The theory of these sets stemed from his reflection on information systems. Generally speaking, rough sets are mathematical models of approximate classification. Classification concerns objects which are describable by means of a list of attributes, i.e., certain features of objects.

In turn, the theory of non-classical logics has provided a motivation for the notion of a Heyting algebra valued set. The latter notion was introduced by Scott in 1972 in his work on the intuitionistic set theory.

Investigations into the relationship between the theory of rough sets and the theory of Heyting algebra valued sets were initiated by A. Obtułowicz [5]. He discovered a representation of Pawlak's rough sets by means of Heyting algebra valued sets which are four-element chains. The above result provided the inspiration for examining some properties of the set of all rough subsets of a rough set. This paper presents some results of a research in this area. The main results are two theorems placed in Chapter 3 of this paper: the first theorem says that the lattice of all rough subsets of a rough set is a complete Heyting algebra while the second theorem provides a necessary and sufficient condition for the above lattice to be a Boolean algebra.

The relationship between rough sets and Heyting algebra valued sets can be also conveniently analysed in terms of category theory. This problem is discussed in Chapters 4 and 5. We mention here a noticeable result which states that an object which is isomorphic with a rough set need not be a rough set itself.

1. Preliminary notions

The relations of a partial order on a set are denoted by the symbol $\leq$. Let $\langle L, \leq\rangle$ be a partially ordered set and X be a non-empty subset of L. The least upper bound of X in L and the greatest lower bound of X in L are denoted by $\vee X$ and $\wedge X$, respectively. If X is a two-element set, $X = \{a, b\}$, the respective bounds are denoted by $a \vee b$ and $a \wedge b$. A *lattice* is a partially ordered set $\langle L, \leq\rangle$ with the property that for every pair a, b of elements of L, the supremum $a \vee b$ and the infimum $a \wedge b$ exist. A lattice $\mathcal{L} = \langle L, \leq\rangle$ is called *complete*, if for each non-empty set $X \subseteq L$, the least upper bound $\bigvee X$ and the greatest lower bound $\bigwedge X$ exist.

Let a and b be elements of a lattice $\langle L, \leq\rangle$. An element $x \in L$ is called the *pseudocomplement* of a *relative to* b, if x is the largest element of L with the property that $a \wedge x \leq b$. This element is denoted by $a \rightarrow b$.

If the lattice $\mathcal{L}$ possesses the least element (which is denoted by $\mathbf{0}$), then the element $a \rightarrow \mathbf{0}$ is called the *pseudocomplement* of a and is denoted by $\neg a$.

Any lattice with the least element $\mathbf{0}$ such that the operation of relative pseudocomplementation $\rightarrow$ is defined for every pair a, b, i.e., $a \rightarrow b$ exists for all a, b, is called a *Heyting algebra*. (Instead of "Heyting algebra" the term "pseudo-Boolean algebra" is also often used in the literature.)

If a lattice with the above properties is complete, it is called a *complete Heyting algebra.*

It is a well-known fact (see e.g. [9]) that a complete lattice $\mathcal{L}$ is a Heyting algebra if and only if, for every indexed subset $\{a_t\}_{t\in T}$ of this lattice and for every $a \in L$, the following equality holds in the lattice

$$a \wedge \bigvee_{t\in T} a_t = \bigvee_{t\in T}(a \wedge a_t).$$

(The condition is referred to as infinite distributivity). It follows from this result that not every complete and distributive lattice is a Heyting algebra. For example, in the lattice of all closed subsets of a stright line there does not exists the pseudocomplement of the element p relative to the empty set $\emptyset$, where p is any point of the line.

Clearly, every finite distributive lattice is a Heyting algebra. Also every Boolean algebra is a Heyting algebra. It is also known (cf. [9]) that a Heyting algebra is a Boolean algebra iff, for every element a of this algebra, $a \vee \neg a = \mathbf{1}$, where $\mathbf{1}$ stands for the greatest element of the lattice.

2. A representation of rough sets by Heyting algebra valued sets

We begin with recalling some basic facts contained in the papers [4], [5], [6].

Let R be an equivalence relation on a set U. A subset T of U is called *R-open*, if the following condition is satisfied

$$\forall x, y \in U[x \in T \ \&\ xRy \Rightarrow y \in T].$$

Given an element $x \in U$, we let $[x]_R$ denote the set $\{y \in U : xRy\}$. Then, for every $Y \subseteq U$, the sets

$$\underline{A}_R(X) := \{x \in U : [x]_R \subseteq X\},$$
$$\overline{A}_R(X) := \{x \in U : [x]_R \cap X \neq \emptyset\},$$
$$Fr_R(X) := \overline{A}_R(X) - \underline{A}_R(X)$$

are called, respectively, the *lower approximation*, the *upper approximation*, and the *boundary* of the set X in U.

The equivalence relation R on U determines the equivalence relations S on the set $\mathcal{P}(U)$ of all subsets of U:

$$XSY \Leftrightarrow [\underline{A}_R(X) = \underline{A}_R(Y) \ \&\ \overline{A}_R(X) = \overline{A}_R(Y)],$$

or equivalently,

$$XSY \Leftrightarrow [\underline{A}_R(X) = \underline{A}_R(Y) \ \&\ Fr_R(X) = Fr_R(Y)].$$

In the paper [6] a rough set is understood to be the equivalence class $[X]_S$ (in the power set $\mathcal{P}(U)$) of any set $X \subseteq U$ modulo S, i.e., the set

$$[X]_S = \{Y \subseteq U : XSY\}, \quad \text{for } X \subseteq U.$$

The objects of the form $[X]_S$ are interchangeably referred to as rough sets in the sense of Pawlak or *abstract rough sets.* As opposed to the above by a *rough set* we will mean any quadruple of the form (U, R, I) such that U is a set, R is an equivalence relation on U, and I, B are subsets of U which satisfy the following conditions:

(a_1) $I \cap B = \emptyset$,
(a_2) I and B are R-open sets,
(a_3) $\forall x \in B.\ \exists y \in B.\ [x \neq y \ \&\ xRy]$.

It is a known fact (cf. [5]) that if R is an equivalence relation on a set U, then the assignement

$$[X]_S \rightarrow (\underline{A}_R(X), Fr_R(X))$$

is a bijection from the quotient set $\mathcal{P}(U)/S$ onto the set of all ordered pairs (I, B) of subsets of the set U satisfying the conditions (a_1)–(a_2). Thus there exists a one-to-one correspondence between abstract rough sets and rough sets in the above sense.

Let A be a complete Heyting algebra. By a *set valued by the algebra* A, shortly: an A-*set* (cf. [4]), we will mean any pair (U, δ) such that U is a set and $\delta : U \times U \to A$ is a mapping satisfying the following conditions:

(i) $\forall x, y \in U.\ \delta(x, y) = \delta(y, x)$,
(ii) $\forall x, y, z \in U.\ \delta(x, y) \wedge \delta(y, z) \leq \delta(x, z)$.

The intuitive sense of the definition of an A-set is this: for any two given elements x, y of the set U, the element $\delta(x, y)$ of the Heyting algebra A define the extent with respect to which the element x is equal to y.

Let us denote by $\mathcal{L}_4$ the chain with the underlying set $\mathcal{L}_4 = \{0, 1, 2, 3\}$. We admit the following definition:

DEFINITION 2.1. An $R\mathcal{L}_4$-*set* is any $\mathcal{L}_4$-set $\underline{U} = (U, \delta)$ satisfying the following conditions:

(R_1) $\forall x \in U.\ 1 \leq \delta(x, x)$,
(R_2) $\forall x, y \in U.\ [2 \leq \delta(x, y) \Rightarrow x = y]$,
(R_3) $\forall x, y \in U.\ [\delta(x, y) = 1 \Rightarrow \delta(x, x) = \delta(y, y)]$,
(R_4) $\forall x \in U.\ [\delta(x, x) = 2 \Rightarrow \exists y \in U.\ \delta(x, y) = 1]$.

According to the presented above intuitions connected with the function δ, the conditions (R_1)–(R_4) can be formulated in a less formal way as follows:

(R_1) every element x is equal to itself to a degree at least 1,

(R_2) any two elements whose degree of equality is not less than 2 are identical,

(R_3) if x is equal to itself to the degree 2, then there exists an element y which is equal to x to the degree 1.

The following two theorems of Obtułowicz determine a representation of rough sets by means of $R\mathcal{L}_4$-sets.

THEOREM 2.2. *For an arbitrary* $R\mathcal{L}_4$-*set* $\underline{U} = (U, \delta)$ *define the relation* $R_{\underline{U}}$ *on* U *and the sets* $I_{\underline{U}}, B_{\underline{U}}$ *as follows:*

$$x R_{\underline{U}} y \Leftrightarrow 1 \leq \delta(x, y),$$
$$I_{\underline{U}} = \{x \in U : \delta(x, x) = 3\}, \quad B_{\underline{U}} = \{x \in U : \delta(x, x) = 2\}.$$

Then the quadruple $\underline{Z}_{\underline{U}} = (U, R_{\underline{U}}, I_{\underline{U}}, B_{\underline{U}})$ *is a rough set.*

THEOREM 2.3. *If $\underline{Z} = (U, R, I, B)$ is a rough set, then the pair $\underline{U}_{\underline{Z}} = (U, \delta_{\underline{Z}})$, where $\delta_{\underline{Z}}$ is the mapping defined as follows:*

$$\delta_{\underline{Z}}(x,y) = \begin{cases} 3 & \textit{if } x = y \,\&\, x \in I, \\ 2 & \textit{if } x = y \,\&\, x \in B, \\ 1 & \textit{if } (xRy \,\&\, x \neq y) \textit{ or } (x = y \,\&\, x \in U - (I \cup B)), \\ 0 & \textit{if it is not the case that } xRy, \end{cases}$$

is an $R\mathcal{L}_4$-set.

To each complete Heyting algebra A the category of all A-sets is assigned (cf. [4]). We will speak of these categories in Chapter 4. Following the familiar category-theoretic terminology, instead of the term "A-set" we will also use the word "object" in this and in further parts of the paper.

DEFINITION 2.4. Let A be a complete Heyting algebra and let $\underline{U} = (U, \delta)$ be an A-set. By a *description* of the subobject of the object $\underline{U}$ we shall understand aby mapping $\alpha : U \to A$ satisfying the following conditions:

(o_1) $\forall x \in U.\ \alpha(x) \leq \delta(x,x)$,
(o_2) $\forall x, y \in U.\ \alpha(x) \wedge \delta(x,y) \leq \alpha(y)$.

Let $\underline{U}$ be any A-set. We let $P(U,\delta)$ denote the set of all subobjects of $\underline{U}$. We then define the subset $S(U,\delta)$ of $P(U,\delta)$ in the following way:

(1) $$S(U,\delta) = \{\alpha \in P(U,\delta) : (\forall x, y \in U.) \alpha(x) \wedge \alpha(y) \leq \delta(x,y)\}.$$

The elements of the set $S(U,\delta)$ are called *singletons*. For each A-set $\underline{U} = (U,\delta)$, we define the mapping $\Gamma_\delta : S(U,\delta) \times S(U,\delta) \to A$ according to the formula:

(2) $$\Gamma_\delta(\alpha,\beta) = \bigvee\{\alpha(x) \wedge \beta(x) : x \in U\}.$$

Then the pair $(S(U,\delta), \Gamma_\delta)$ is also an A-set.

Let us also notice that for each $x \in U$, the mapping α_x given by the formula:

$$\alpha_x(y) = \delta(x,y), \quad \text{for every } y \in U,$$

is an element of the set $S(U,\delta)$.

Let $\underline{U} = (U,\delta)$ be an $R\mathcal{L}_4$-set. Then for every element x of U, the function α_x defined as above "describes" one of the equivalence classes of the relation $R_{\underline{U}}$ (cf. Theorem 2.2), viz. the equivalence class of the element x.

The set $P(U,\delta)$ of all descriptions of the subobjects of the object $\underline{U} = (U,\delta)$ is known to be partially ordered by the relation $\leq$ defined as follows (cf. [4]):

$$\alpha \leq \beta \Leftrightarrow \forall x \in U.\ \alpha(x) \leq \beta(x).$$

Furthermore $\mathcal{P}(\underline{U}) := \langle P(U,\delta), \leq \rangle$ is a lattice isomorphic with the lattice of all subobjects of the object $\underline{U}$.

DEFINITION 2.5. Let $\underline{U} = (U,\delta)$ be an $R\mathcal{L}_4$-set. An *r-description* of the subobject of the object $\underline{U}$ is any mapping $\alpha : U \to \mathcal{L}_4$ which satisfies the following conditions:

(r_1) $\forall x \in U. [\alpha(x) \leq \delta(x,x)]$,
(r_2) $\forall x \in U. [1 \leq \alpha(x)]$,
(r_3) $\forall x, y \in U. [\delta(x,y) = 1 \Rightarrow \alpha(x) = \alpha(y)]$,
(r_4) $\forall x \in U. [\alpha(x) = 2 \Rightarrow (\exists y \in U.)\ \delta(x,y) = 1]$.

The conditions $(r_1), (r_2), (r_3)$ above and the condition (R_2) of Definition 2.1 imply that if $\underline{U} = (U,\delta)$ is an $R\mathcal{L}_4$-set, then any r-description of a subobject of the object $\underline{U}$ is at the same time a description of this subobject.

If (U,δ) is a non-trivial $R\mathcal{L}_4$-set, i.e., $\delta(x,y) = 0$ for some $x, y \in U$, then the set of all r-descriptions of the subobjects of the object (U,δ) is disjoint with the set $S(U,\delta)$ of all singletons of (U,δ). Indeed, if x and y are elements of U such that $\delta(x,y) = 0$, then for every $\alpha \in S(U,\delta)$ we would have that $\alpha(x) = 0$ or $\alpha(y) = 0$. This would contradict the condition (r_2) of Definition 2.5.

It is easy to notice that if $\underline{U} = (U,\delta)$ is an $\mathcal{L}_4$-set, $\alpha : U \to \mathcal{L}_4$ is a description of a subobject of the object $\underline{U}$, and the pair $\underline{U}' = (U,\delta_\alpha)$ with δ_α defined as follows:

$$\delta_\alpha(x,y) = \delta(x,y) \wedge \alpha(x) \quad \text{for all } x, y \in U, \tag{3}$$

is an $R\mathcal{L}$-set, then α is an r-description of a subobject of U. In turn, if $\underline{U} = (U,\delta)$ is an $R\mathcal{L}_4$-set, $\alpha : U \to \mathcal{L}_4$ is an r-description of a subobject of the object $\underline{U}$, then the pair $\underline{U}' = (U,\delta_\alpha)$ with δ_α defined as in (3), is an $R\mathcal{L}_4$-set.

3. Rough subsets of a rough set

We admit the following definition:

DEFINITION 3.1. Let $\underline{Z} = (U,R,I,B)$ be a rough set. A *rough subset of* $\underline{Z}$ is any rough set $\underline{Z}' = (U,R',I',B')$ such that $R' = R, I' \subseteq I$ and $B' \subseteq I \cup B$.

Let $\underline{Z}' = (U,R',I',B')$ be a rough subset of a rough set $\underline{Z} = (U,R,I,B)$. Let $\underline{U}' = (U,\delta_{\underline{Z}'})$ and $\underline{U} = (U,\delta_{\underline{Z}})$ be the $R\mathcal{L}_4$-sets resulting from $\underline{Z}'$ and $\underline{Z}$, respectively, by applying to them the assignement described in Theorem 2.3. Then, for every pair $x, y \in U$:

$$1 \leq \delta_{\underline{Z}'}(x,y) \quad \text{iff} \quad 1 \leq \delta_{\underline{Z}}(x,y).$$

Furthermore, $\delta_{\underline{Z}'}(x,x) \leq \delta_{\underline{Z}}(x,x)$ for every $x \in U$. On the basis of this one easily proves that the mapping $\alpha : U \to \mathcal{L}_4$, defined by the formula: $\alpha(x) = \delta_{\underline{Z}'}(x,x)$, is an r-description of a subobject of the object $\underline{U}$.

Now, let $\underline{U} = (U, \delta)$ be an $R\mathcal{L}_4$-set and α an r-description of a subobject of the object $\underline{U}$. Let furthermore $\underline{U}' = (U, \delta_\alpha)$ be the $R\mathcal{L}_4$-set with α given by the formula (1). Then the rough set $\underline{Z}' = (U, R_{\underline{U}'}, I_{\underline{U}'}, B_{\underline{U}'})$, which results from $\underline{U}'$ by applying to it the assignement described in Theorem 2.2, is a rough subset of the rough set $\underline{Z} = (U, R_{\underline{U}}, I_{\underline{U}}, B_{\underline{U}})$. According to this interpretation, r-descriptions of subobjects of the object $\underline{U}$ can be regarded as the characteristic functions of rough subsets of the rough set $\underline{U}$.

The function α, occurring in Definition 2.4, can be interpreted in the following way: for each element x of U, the element $\alpha(x)$ of the Heyting algebra $\mathcal{L}_4$ defines the extent with respect to which the element x belongs to the rough subset $\underline{Z}'$.

The conditions (r_1)–(r_4) thus say that

— an element x belongs to a subset at most to the degree to which x is equal to itself,

— an element belongs to a set at most to the degree 1,

— if x is equal to y at the degree 1, then the degrees of membership of each of these two elements to a set are the same,

— if x belongs to a rough set at the degree 2, then there exists an element y which is equal to x at the degree 1.

Let $\underline{U} = (U, \delta)$ be an $R\mathcal{L}_4$-set and let $P_r(U, \delta)$ denote the set of all r-descriptions of subobjects of the object $\underline{U}$. It is easy to notice that the relation $\leq$ defined as follows:

$$\alpha \leq \beta \Leftrightarrow \forall x \in U.\ \alpha(x) \leq \beta(x),$$

for all $\alpha, \beta \in P_r(U, \delta)$, is a partial order on $P_r(U, \delta)$. Moreover, for every pair $\alpha, \beta \in P_r(U, \delta)$, the functions $\alpha \vee \beta$ and $\alpha \wedge \beta$ defined by:

$$(\alpha \vee \beta)(x) = \alpha(x) \vee \beta(x),$$
$$(\alpha \wedge \beta)(x) = \alpha(x) \wedge \beta(x),$$

for all $x \in U$, are also elements of the set $P_r(U, \delta)$. They define the supremum and the infimum of the elements α and β, respectively, in the partially ordered set $\mathcal{P}_r(\underline{U}) = \langle P_r(U, \delta), \leq \rangle$. $\mathcal{P}_r(\underline{U})$ is therefore a lattice $\underline{U}$ is a sublattice of the lattice $\mathcal{P}(\underline{U})$ of all descriptions of subobjects of the object $\underline{U}$.

There also holds the following theorem:

THEOREM 3.2. *If $\underline{U} = (U, \delta)$ is an $R\mathcal{L}_4$-set, then the lattice $\mathcal{P}_r(\underline{U})$ of all r-descriptions of subobjects of the object of the object $\underline{U}$ is distributive and complete.*

Proof. The distributivity of the lattice $\mathcal{P}_r(\underline{U}) = \langle P_r(U,\delta), \leq \rangle$ follows from the distributivity of the lattice $\mathcal{L}_4$. For each non-empty set $X \subseteq P_r(U,\delta)$ denote $\bigvee X = \gamma$ and $\bigwedge X = \sigma$. Then clearly $\gamma(x) = \bigvee\{\alpha(x) : \alpha \in X\}$ and $\sigma(x) = \bigwedge\{\alpha(x) : \alpha \in X\}$. It is not difficult to see that γ and σ are r-descriptions. This means that $\mathcal{P}_r(\underline{U})$ is a complete lattice. ∎

Let us notice that for each $R\mathcal{L}_4$-set $\underline{U} = (U, \delta)$, the mappings $\mathbf{0}$ and $\mathbf{1}$ defined as follows:

$$\mathbf{0}(x) = 1, \quad \mathbf{1}(x) = \delta(x,x) \quad \text{for all } x \in U,$$

are r-descriptions of subobjects of the object $\underline{U}$. On the strength of the conditions (r_2) and (r_1), $\mathbf{0}$ is the smallest element and $\mathbf{1}$ is the largest element of the lattice $\mathcal{P}_r(\underline{U})$, respectively.

Let α and β be arbitrary elements of the lattice $\mathcal{P}_r(\underline{U})$. The set $X = \{\gamma : \alpha \wedge \gamma \leq \beta\}$ possesses the greatest element $\gamma_{\max}$, namely

$$\gamma_{\max}(x) = \begin{cases} \mathbf{1}(x) & \text{if } \alpha(x) \leq \beta(x), \\ \beta(x) & \text{if } \beta(x) \leq \alpha(x) \ \&\ \beta(x) \neq \alpha(x). \end{cases}$$

This means that in the lattice $\mathcal{P}_r(\underline{U})$ for every pair α, β of elements there exists the pseudocomplement $\alpha \rightarrow \beta$ of α relative to β (this is the element $\gamma_{\max}$). This remark and Theorem 3.2 yield

THEOREM 3.3. *For every $R\mathcal{L}_4$-set $\underline{U} = (U, \delta)$ the lattice $\mathcal{P}_r(\underline{U})$ of r-descriptions of subobjects of the object $\underline{U}$ is a complete Heyting algebra.* ∎

On account of the considerations carried out thus far, the operations $\vee$ and $\wedge$ can be interpreted as the join and the meet of rough subsets of the $R\mathcal{L}_4$-set $\underline{U}$. Therefore $\mathcal{P}_r(\underline{U})$ can be called the algebra of rough subsets of $\underline{U}$.

THEOREM 3.4. *Let $\underline{U} = (U, \delta)$ be an $R\mathcal{L}_4$-set. The lattice $\mathcal{P}_r(\underline{U})$ of all r-descriptions of subobjects of the object $\underline{U}$ is a Boolean algebra iff $\underline{U}$ satisfies the following condition*:

(4) $$\forall x, y \in U.\ [\delta(x,x) = \delta(y,y) = 3 \ \&\ \delta(x,y) = 1 \Rightarrow x = y].$$

Proof. Suppose $\underline{U} = (U, \delta)$ is an $R\mathcal{L}_4$-set which satisfies the condition (4). Letting $P_r(U,\delta)$ be the set of all r-descriptions of the object $\underline{U}$, we notice that for any $\alpha \in P_r(U,\delta)$ there hold the conditions:

$$(\neg\alpha)(x) = \begin{cases} \delta(x,x) & \text{if } \alpha(x) = 1, \\ 1 & \text{if } \alpha(x) = 2 \text{ or } \alpha(x) = 3 \end{cases}$$

and

$$(\alpha \vee \neg\alpha)(x) = \begin{cases} \delta(x,x) & \text{if } \alpha(x) = 1 \text{ or } \alpha(x) = 3, \\ 2 & \text{if } \alpha(x) = 2. \end{cases}$$

The above conditions readily imply that $\alpha \vee \neg\alpha = \mathbf{1}$ for every $\alpha \in P_r(U, \delta)$. Thus $\mathcal{P}_r(\underline{U})$ is a Boolean algebra.

To prove the reverse implication, suppose $\underline{U} = (U, \delta)$ is an $R\mathcal{L}_4$-set which does not satisfy the condition (4). Consequently, there exist elements $x_0, y_0 \in U$ such that $\delta(x_0, y_0) = \delta(y_0, y_0) = 3, \delta(x_0, y_0) = 1$ and $x_0 \neq y_0$. Let us consider the mapping $\alpha : U \to \mathcal{L}_4$ defined by means of the formula:

$$\alpha(x) = \begin{cases} 2 & \text{if } x \in [x_0]_{R_{\underline{U}}}, \\ 1 & \text{otherwise,} \end{cases}$$

where $R_{\underline{U}}$ is the relation defined as in Theorem 2.2. α is r-description of a subobject of the object $\underline{U}$. Moreover $(\alpha \vee \neg\alpha)(x_0) = 2 \neq \delta(x_0, x_0)$, i.e., $\alpha \vee \neg\alpha \neq \mathbf{1}$. This shows that $\mathcal{P}_r(\underline{U})$ is not a Boolean algebra. ∎

The condition (4) says that the restriction of the relation $R_{\underline{U}}$ to the interior $I_{\underline{U}}$, (i.e., the set $R_{\underline{U}} \cap (I_{\underline{U}} \times I_{\underline{U}})$), is the identity relation.

4. Morphisms of A-sets. The category of $R\mathcal{L}_4$-sets

Our nearest goal is to present a thorough explanation of our understanding of a mapping of one rough set to another. Instead of the word "mapping" we will use in this context the term "morphism". The definitions of a morphism of rough sets and of the category of rough sets will be presented in Chapter 5. In this chapter we will focus our attention on the notion of a morphism of A-sets (cf. Higgs [4]). We shall also discuss some properties of the category of $R\mathcal{L}_4$-sets.

Let A be a complete Heyting algebra and let $\underline{U} = (U, \delta)$ and $\underline{W} = (W, \sigma)$ be arbitrary A-sets. A *morphism* from $\underline{U}$ to $\underline{W}$ is any triple of the form $(\underline{U}, f, \underline{W})$, where f is a function from $U \times W$ to A (A is the underlying set of the algebra A) which satisfies the following conditions:

(m_1) $\forall x, x' \in U.\ \forall y \in W.\ f(x,y) \wedge \delta(x,x') \leq f(x',y)$,
(m_2) $\forall x \in U.\ \forall y, y' \in W.\ f(x,y) \wedge \sigma(y,y') \leq f(x,y')$,
(m_3) $\forall x \in U.\ \forall y, y' \in W.\ f(x,y) \wedge f(x,y') \leq \sigma(y,y')$,
(m_4) $\forall x \in U.\ \bigvee\{f(x,y) : y \in W\} = \delta(x,x)$.

Any such a function f is called an *A-function.*

Each A-function $f : U \times W \to A$ can be treated as a "characteristic function" of a "subset" of the set $U \times W$. For each pair (x, y) belonging to $U \times W, f(x,y)$ is interpreted as the element of the algebra A which defines the degree of relatedeness of the element y to x through f.

EXAMPLE 4.1. Let $\underline{U} = (U, \delta)$ be an arbitrary A-set. According to considerations presented in Chapter 2, the pair $(S(U,\delta), \Gamma_\delta)$ is also an A-set. We remind that $S(U,\delta)$ is the set of all singletons (cf. (1)) and Γ_δ is defined as in (2). The A-set $(S(U,\delta), \Gamma_\delta)$ is denoted by C_δ. We define the function $f : U \times S(U,\delta) \to A$ in the following way:

$$f(x,\beta) = \delta(x,x) \wedge \Gamma_\delta(\alpha_x, \beta), \tag{5}$$

for all $x \in U$ and $\beta \in S(U,\delta)$, where $\alpha_x(y) = \delta(x,y)$ for all $y \in U$.

From the properties of the supremum and the infimum in Heyting algebras and from the conditions (o_1) and (o_2) of Definition 2.3 and the conditions (i)–(ii) of Chapter 2 it follows that the function f defined by means of (5) satisfies the conditions (m_1)–(m_4). Hence the triple $(\underline{U}, f, C_\delta)$ is a morphism from $\underline{U}$ to C_δ.

In turn, if $(\underline{U}, f, \underline{W})$ and $(\underline{W}, f', \underline{V})$ are morphisms from $\underline{U} = (U,\delta)$ to $\underline{W} = (W,\zeta)$ and from $\underline{W}$ to $\underline{V} = (V,\gamma)$, respectively, then the *composition* of these morphisms is the triple $(\underline{U}, f' \circ f, \underline{V})$, where

$$(f' \circ f)(x,z) = \bigvee \{ f(x,y) \wedge f'(y,z) : y \in W \}, \tag{6}$$

for all $(x,z) \in U \times V$.

The *identity* morphism is any triple of the form $(\underline{U}, \delta, \underline{U})$, where $\underline{U} = (U,\delta)$ is any A-set. The identity morphism is denoted by the symbol $i_{\underline{U}}$.

Let $A = \mathcal{L}_4$ and let $\underline{U} = (U,\delta), \underline{W} = (W,\sigma)$ be $R\mathcal{L}_4$-sets. Let $(\underline{U}, f, \underline{W})$ be any morphism from $\underline{U}$ to $\underline{W}$. The function f is then an $\mathcal{L}_4$-function. The equality $f(x,y) = a$, where $a \in \mathcal{L}_4$, can be then interpreted as follows:

"y is not an f-image of x" in case when $a = 0$.
"y is an f-image of x" when $1 \leq a$.

If $1 \leq f(x,y)$ and the knowledge of the exact value of $f(x,y)$ is irrelevant for us, we will shortly say that y is an f-image of x.

In order to interpret the conditions (m_1)–(m_2) in the case when $A = \mathcal{L}_4$, we will say that elements x_1, x_2 of an arbitrary $R\mathcal{L}_4$-set (U,σ) are *entirely different* if $\delta(x_1, x_2) = 0$.

It follows from the conditions (m_1)–(m_2) that:

if y is an f-image of x at the degree a_1 and x is equal to x' at the degree a_2, then y is an f-image of x' at the degree at least $min\{a_1, a_2\}$.

The condition (m_2) expresses a certain kind of the injectivity property of the morphism f:

if both y and y' are f-images of x, then the elements y and y' cannot be entirely different.

The condition (m_4) says that each element x of U has an f-image y which belongs to W at the same degree at which x is equal to itself.

The axiom (m_3)–(m_4) imply that

$$f(x,y) \leq \delta(x,x) \wedge \sigma(y,y), \tag{7}$$

i.e., the element y is an f-image of x at the degree not greater than that at which x is equal to itself and not greater than the degree at which y is equal to itself.

We infer from the conditions (m_1)–(m_3) that if elements x_1, x_2 are not entirely different, then their f-images y_1, y_2 are not entirely different either.

A-Set denotes the category whose objects are all A-sets and the morphisms from one object $\underline{U} = (U, \delta)$ to another $\underline{W} = (W, \sigma)$ are all the triples $(\underline{U}, f, \underline{W})$ in which f is an A-function.

The following theorems are true for the category A-set:

4.2. A morphism $(\underline{U}, f, \underline{W})$ from an object $\underline{U} = (U, \delta)$ to an object $\underline{W} = (W, \sigma)$ is a monomorphism iff

$$f(x,y) \wedge f(x',y) \leq \delta(x,x'),$$

for all $x, x' \in U$ and $y \in W$.

4.3. A morphism $(\underline{U}, f, \underline{W})$ is an epimorphism iff

$$\bigvee\{f(x,y) : x \in U\} = \sigma(y,y),$$

for every $y \in W$.

4.4. If $(\underline{U}, f, \underline{W})$ is both a monomorphism and an epimorphism, then it is an isomorphism.

The proofs of these results can be found in Higgs [4].

The following corollary readily follows from Theorem 4.2–4.4 and Example 4.1:

COROLLARY 4.5. *For every A-set $\underline{U} = (U, \delta)$, the triple $(\underline{U}, f, C_\delta)$, where $C_\delta = (S(U,\delta), \Gamma_\delta)$ and $f : U \times S(U,\delta) \to A$ is defined as in (5), is an isomorphism in the category A-Set.*

Let $R\mathcal{L}_4$-Set denote the category whose objects are all $R\mathcal{L}_4$-sets and the morphisms from an object $\underline{U}$ to an object $\underline{W}$ are all triples of the form $(\underline{U}, f, \underline{W})$, where f is an $\mathcal{L}_4$-function. The composition of morphisms is defined according to the equality (6). The identity morphism in the category $R\mathcal{L}_4$-Set is any triple of the form $(\underline{U}, \delta, \underline{U})$, where $\underline{U} = (U, \delta)$ is any $R\mathcal{L}_4$-set.

The category $R\mathcal{L}_4$-Set is a full subcategory of the category $\mathcal{L}_4$-Set; moreover this category has products and a terminal object. The product $\underline{U} \times \underline{W}$

of two objects $\underline{U} = (U,\delta), \underline{W} = (W,\sigma)$ is defined as $\underline{U} \times \underline{W} = (U \times W, \xi)$, where

$$\xi((x,y),(x',y')) = \delta(x,x') \wedge \sigma(y,y'), \tag{8}$$

for all $x, x' \in U$ and all $y, y' \in W$.

A terminal object $\underline{T}$ in the category $R\mathcal{L}_4$-Set is any pair (U,τ) such that U is a one-element set, i.e., $U = \{y\}$ and $\tau(y,y) = 3$.

A certain fact connected with the notion of isomorphic closedness of a subcategory relative to its supercategory deserves attention while investigating the category $R\mathcal{L}_4$-Set as a subcategory of $\mathcal{L}_4$-Set. This fact is expressed in Theorem 4.6. But first we shall recall the definition of isomorphic closedness of a subcategory.

A subcategory B of a category C is *isomorphically closed* (cf. [2]) if any C-object (i.e., an object of the category C) which is isomorphic with a B-object is actually a B-object.

THEOREM 4.6. *The category $R\mathcal{L}_4$-Set is not an isomorphically closed subcategory of the category $\mathcal{L}_4$-Set.*

Proof. Let $\underline{U} = (U,\delta)$ be an $R\mathcal{L}_4$-set such that $\delta(x_0,x_0) = 3$ for some $x_0 \in U$. Let $S(U,\delta)$ be the set of all singletons for $\underline{U}$. We define the functions $\alpha_{x0} : U \to \mathcal{L}_4, \beta_{x0} : U \to \mathcal{L}_4$ in the following way:

$$\alpha_{x0}(x) = \delta(x_0,x),$$
$$\beta_{x0}(x) = \begin{cases} 2 & \text{if } x = x_0, \\ \delta(x_0,x) & \text{otherwise.} \end{cases}$$

Clearly both α_{x0} and β_{x0} belong to the set $S(U,\delta)$. On account of Corollary 4.5, the $R\mathcal{L}_4$-set $\underline{U}$, which is clearly also an $\mathcal{L}_4$-set, is isomorphic (in the category $\mathcal{L}_4$-Set) with the $\mathcal{L}_4$-set $C_\delta = (S(U,\delta), \Gamma_\delta)$, where Γ_δ is given by the formula (2). However $\Gamma_\delta(\alpha_{x0},\beta_{x0}) = 2$ and $\alpha_{x_0} \neq \beta_{x_0}$, so the function Γ_δ does not satisfy the condition (R_2) of Definition 2.1. This means that C_δ is not an $R\mathcal{L}_4$-set.

The proof of the theorem is complete. ∎

5. The category of rough sets

The representation of rough sets through $R\mathcal{L}_4$-sets enables us to expect that the notion of a morphism of rough sets is strictly related to the notion of a morphism in the sense of Higgs. The relationship between the two notions is expressed in the theorem formulated below. This theorem concludes our considerations.

We admit the following definition:

DEFINITION 5.1. Let $\underline{Z} = (U, R, I, B)$ and $\underline{Z}' = (U', R', I', B')$ be arbitrary rough sets. A *morphism* from $\underline{Z}$ to $\underline{Z}'$ is any quadruple $(\underline{Z}, F, \varphi, \underline{Z}')$ such that

(s_1) F is a function defined on the quotient set U/R with values in the set U'/R',
(s_2) φ is a function defined on $I \cup B$ with values in $I' \cup B'$,
(s_3) $\forall x, x' \in I \cup B.\ [xRx' \Rightarrow \varphi(x)R'\varphi(x')]$,
(s_4) $x \in I \Rightarrow \varphi(x) \in I'$,
(s_5) $\forall x \in I \cup B.\ [\varphi(x) \in F([x]_R)$.

The category RS of rough sets is defined to be the category whose objects are all rough sets and morphisms are all quadruples $(\underline{Z}, F, \varphi, \underline{Z}')$ defined as above. The composition of morphisms in the category RS is defined in the natural way by means of the compositions of functions. An identity morphism is any quadruple of the form $(\underline{Z}, Id, id, \underline{Z})$, where $\underline{Z} = (U, R, I, B)$ is a rough set, Id is the identity relation on U/R and id is the identity on $I \cup B$. A terminal object in the category RS is any quadruple $\underline{Z} = (U, R, I, B)$ such that U is a one-element set, i.e., $U = \{a\}$ for some a, and $R = \{(a, a)\}, I = \{a\}, B = \emptyset$.

The facts and lemmas we will present are easy consequences of the considerations carried out thus far. They will be employed in the proof of a theorem which establishes the relationship between the categories RS and $R\mathcal{L}_4$-Set.

For any two rough sets $\underline{Z} = (U, R, I, B), \underline{Z}' = (U', R', I', B')$ and a morphism $(\underline{Z}, F, \varphi, \underline{Z}')$ between them we define the function $f^{\sim} : U \times U' \to \mathcal{L}_4$ in the following way:

$$f^{\sim}(x, y) = \begin{cases} 3 & \text{if } x \in I \ \&\ y = \varphi(x), \\ 2 & \text{if } x \in B \ \&\ y = \varphi(x), \\ 1 & \text{if } y \in F([x]_R) \ \&\ y \neq \varphi(x), \\ 0 & \text{if } y \in F([x]_R). \end{cases} \tag{9}$$

(The symbol $y \neq \varphi(x)$ in the above formula means that either x does not belong to the domain of φ or the element $\varphi(x)$ is different from y.)

LEMMA 5.2. *Suppose $(\underline{Z}, F, \varphi, \underline{Z}')$ is a morphism of the category RS from a rough set $\underline{Z} = (U, R, I, B)$ to a rough set $\underline{Z}' = (U', R', I', B')$. Let $\underline{U}_{\underline{Z}} = (U, \delta_{\underline{Z}}), \underline{U}_{\underline{Z}'} = (U, \delta_{\underline{Z}'})$ (cf. 2.3) and let $f^{\sim} : U \times U' \to \mathcal{L}_4$ be the function defined as in (9). Then the triple $(\underline{U}_{\underline{Z}}, f^{\sim}, \underline{U}_{\underline{Z}'})$ is a morphism of the category $R\mathcal{L}_4$-Set.*

5.3. *Let $\underline{U} = (U, \delta)$ and $\underline{W} = (W, \sigma)$ be any $R\mathcal{L}_4$-sets. Let furthermore $R_{\underline{U}}, R_{\underline{W}}$ be the relations and $I_{\underline{U}}, B_{\underline{U}}, I_{\underline{W}}, B_{\underline{W}}$ the sets defined as in Theorem*

2.2. Then $(\underline{U}, f, \underline{W})$ is a morphism from $\underline{U}$ to $\underline{W}$ in the category $R\mathcal{L}_4$-Set.

We define two functions F_f and φ_f. F_f is the function from $U/R_{\underline{U}}$ to $W/R_{\underline{W}}$ defined as follows: F_f assigns to any equivalence class $[x]_{R_{\underline{U}}}$ belonging to $U/R_{\underline{U}}$ the equivalence class $[y]_{R_{\underline{W}}}$ in $W/R_{\underline{W}}$, where y is any element which satisfies the condition $1 \leq f(x, y)$. (In virtue of (m_4) and (R_1) such an element y exists.) In turn, the function $\varphi_f : I_{\underline{U}} \cup B_{\underline{U}} \rightarrow I_{\underline{W}} \cup B_{\underline{W}}$ is defined according to the following rule: to each element x belonging to $I_{\underline{U}} \cup B_{\underline{U}}$ is assigned an element y such that $f(x, y) = \delta(x, x)$. (Such an element y exists on account of (m_4); moreover y belongs to $I_{\underline{W}} \cup B_{\underline{W}}$ by (7).)

Both the functions F_f and φ_f are well-defined — for F_f this fact is a consequence of the condition (m_3) while for φ_f this follows from the conditions (m_3) and (R_2).

LEMMA 5.4. *Suppose $(\underline{U}, f, \underline{W})$ is a morphism from an object $\underline{U} = (U, \delta)$ to an object $\underline{W} = (W, \sigma)$ in the category $R\mathcal{L}_4$-Set. Let $\underline{Z}_{\underline{U}} = (U, R_{\underline{U}}, I_{\underline{U}}, B_{\underline{U}})$, $\underline{Z}_{\underline{W}} = (W, R_{\underline{W}}, I_{\underline{W}}, B_{\underline{W}})$ (cf. 2.2) and let the functions F_f and φ_f be defined as in 5.3. Then the quadruple $(\underline{Z}_{\underline{U}}, F_f, \varphi_f, \underline{Z}_{\underline{W}})$ is a morphism of the category RS.*

THEOREM 5.5. *The category RS of rough sets is isomorphic with the category $R\mathcal{L}_4$-Set.*

Proof. We shall prove that there exist covariant functors Φ, Ψ such that $\Phi : RS \rightarrow R\mathcal{L}_4$-Set, $\Psi : R\mathcal{L}_4$-Set $\rightarrow RS$ and

$$(10) \qquad \Phi\Psi = I_{RS}, \Psi\Phi = I_{R\mathcal{L}_4-Set},$$

where I_K stands for the identity function in the category K.

We define the functor Φ. To each object $\underline{Z} = (U, R, I, B)$ of the category RS is assigned the object $\Phi(\underline{Z}) = (U, \delta_{\underline{Z}})$ (cf. 2.3) of the category $R\mathcal{L}_4$-Set. In turn, to each morphism $h = (\underline{Z}, F, \varphi, \underline{Z}')$ of the category RS is assigned the morphism $\Phi(h) = (\underline{U}_{\underline{Z}}, f, \underline{U}_{\underline{Z}'})$ in $R\mathcal{L}_4$-Set, defined as in Lemma 5.2. Let us notice that if h is a morphism from an object as in Lemma 5.2. Let us notice that if h is a morphism from an object $\underline{Z}$ to an object $\underline{Z}'$ in RS, then $\Phi(h)$ is a morphism from $\Phi(\underline{Z})$ to $\Phi(\underline{Z}')$ in $R\mathcal{L}_4$-Set.

We see that if $\underline{Z}$ is an arbitrary object of the category RS, then Φ assigns to the identity $\iota_{\underline{Z}}$ a certain identity $\iota_{\underline{U}}$ in the category $R\mathcal{L}_4$-Set.

It is not difficult to show that Φ preserves compositions, i.e., if $h_1 = (\underline{Z}, F_1, \varphi_1, \underline{Z}')$ and $h_2 = (\underline{Z}', F_2, \varphi_2, \underline{Z}'')$ are morphisms in RS, then

$$(11) \qquad \Phi(h_2 \circ h_1) = \Phi(h_2) \circ \Phi(h_1).$$

For we have: $\Phi(h_1) = (\underline{U}_{\underline{Z}}, f_1^{\sim}, \underline{U}_{\underline{Z}'}), \Phi(h_2) = (\underline{U}_{\underline{Z}'}, f_2^{\sim}, \underline{U}_{\underline{Z}''})$ and $\Phi(h_2) \circ \Phi(h_1) = (\underline{U}_{\underline{Z}}, f_2^{\sim} \circ f_1^{\sim}, \underline{U}_{\underline{Z}''})$, where $f_2^{\sim} \circ f_1^{\sim}$ is defined according to (6). In turn, $\Phi(h_2 \circ h_1) = (\underline{U}_{\underline{Z}}, g^{\sim}, \underline{U}_{\underline{Z}''})$, where $g^{\sim} : U \times U'' \to \mathcal{L}_4$ is defined as follows:

$$g^{\sim}(x,z) = \begin{cases} 3 & \text{if } x \in I \ \& \ (\varphi_2 \circ \varphi_1)(x) = z, \\ 2 & \text{if } x \in B \ \& \ (\varphi_2 \circ \varphi_1)(x) = z, \\ 1 & \text{if } z \in (F_2 \circ F_1)([x]_R) \ \& \ (\varphi_2 \circ \varphi_1)(x) \neq z, \\ 0 & \text{if } z \notin (F_2 \circ F_1)([x]_R). \end{cases}$$

(11) then follows from the equality $(f_2^{\sim} \circ f_1^{\sim})(x,z) = g^{\sim}(x,z)$, for all $x \in U, z \in U''$.

This proves that Φ is a covariant functor from the category RS to the category $R\mathcal{L}_4$-Set.

We now define the functor Ψ. Each object $\underline{U} = (U, \delta)$ of the category $R\mathcal{L}_4$-Set is assigned the object $\underline{Z}_{\underline{U}} = (U, R_{\underline{U}}, I_{\underline{U}}, B_{\underline{U}})$ (cf. 2.2) of the category RS. Any morphism $(\underline{U}, f, \underline{W})$ from an object $\underline{U} = (U, \delta)$ to an object $\underline{W} = (U, \sigma)$ of $R\mathcal{L}_4$-Set is assigned the morphism $(\underline{Z}_{\underline{U}}, F_f, \varphi_f, \underline{Z}_{\underline{W}})$ in RS (cf. Lemma 5.4).

We then notice that for any identity morphism $\iota_{\underline{U}}$ of the category $R\mathcal{L}_4$-Set its image $\Psi(\iota_{\underline{U}})$ is an identity morphism in RS. Furthermore, if $(\underline{U}, f_1, \underline{W})$ and $(\underline{W}, f_2, \underline{V})$ are arbitrary morphisms in $R\mathcal{L}_4$-Set, then $F_{f2} \circ F_{f1} = F_{(f2 \circ f1)}$ and $\varphi_{f2} \circ \varphi_{f1} = \varphi_{(f2 \circ f1)}$. This ultimately proves that Ψ is covariant functor from the category $R\mathcal{L}_4$-Set to the category RS.

We also see that the functors Φ, Ψ satisfy the equations (10). The functor Φ is thus a covariant bijector from RS to $R\mathcal{L}_3$-Set (and Ψ is a covariant bijector from $R\mathcal{L}_4$-Set to RS). So the category RS is isomorphic with the category $R\mathcal{L}_4$-Set. ∎

In the light of the above result, Theorem 4.6 can be formulated as follows: an $\mathcal{L}_4$-set which is isomorphic with a rough set need not be a rough set itself.

References

[1] G. Grätzer, *General Lattice Theory*, Akademie-Verlag, Berlin, 1978.

[2] H. Herrlich and G. E. Strecker, *Category Theory*, Boston 1973, Printed in the United States of America.

[3] D. Higgs, *A category approach to Boolean valued models*. Preprint, University of Waterloo, 1973.

[4] D. Higgs, *Injectivity in the topos of complete Heyting algebra valued sets*, Canad. J. Math., 36 (1984), 550–568.

[5] A. Obtułowicz, *Rough sets and Heyting algebra valued sets*, Bull. Polish Acad. Sci. Math., 35 (9–10), (1987) 667–673.

[6] Z. Pawlak, *Rough sets, Basic notions*, ICS PAS Reports, 436 (1981).
[7] Z. Pawlak, *Rough sets, Algebraic and topological approach*, ICS PAS Reports, 482 (1982).
[8] Z. Pawlak, *Rough sets*, Internat. J. Inform. Comput. Sci., 11 (5), (1982). 341–356.
[9] H. Rasiowa, R. Sikorski, *The Mathematics of Metamathematics*, Vol. 41 Warszawa (1970), PWN.
[10] A. Wiweger, *Knowledge representation systems, logical kits, and contexts*, Bull. Polish Acad. Sci. Math., 36 (3–4), 1988.

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