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Krzysztof Czarnowski

ON THE STRUCTURE OF FIXED POINT SETS OF "k-SET-CONTRACTIONS" IN B_0 SPACES

A theorem of Krasnosielski-Perov-Rabinowitz type on fixed point sets properties is given in B_0 spaces for a class of maps broader then compact maps. This is applied to the Darboux problem for a hyperbolic equation.

This paper deals with characterization of sets of solutions of equations in locally convex linear topological spaces, or, to be more specific, in B_0 spaces. We use topological degree methods to obtain our main Theorem (16). Theorem (16) is a generalization of Theorem [4;(2.2)], which applies to fixed point sets of compact maps in B_0 spaces, to a broader class of "k-set-contractive" maps. It goes parallel to a theorem of W. V. Petryshyn [11] on fixed point sets properties of some k-set-contractions in Banach spaces. The required extension of the Banach space notion of measure of noncompactness and k-set-contraction to the case of a B_0 space is done in the first part of the paper. Next we use the ideas of R. D. Nussbaum [10] to define a topological degree for some k-set-contractions in B_0 spaces. The theory of k-set-contractive and more general condensing maps in locally convex linear topological spaces is also given in the paper of B. N. Sadovskii [12].

It should be stated that the proof of theorem (16) makes use, via Lemma (15) ([4;(3.1)], for a corresponding Banach space result see [8]), of theorem of N. Aronszajn [2]: *the intersection A of a sequence of subsets $\{A_n\}$ of a metric space X is an $\mathcal{R}_\delta$-set in X, provided A_n are compact absolute retracts and $\{A_n\}$ converges to A in the sense of Hausdorff metric.* Recall that $A \subset X$ is an $\mathcal{R}_\delta$-set in X iff it is homeomorphic to an intersection of a *decreasing* sequence of compact absolute retracts. An $\mathcal{R}_\delta$-set is acyclic—in particular compact and connected.

At the end, as an application, the structure of the Darboux problem for a

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hyperbolic partial differential equation is studied in an unbounded domain. The right side is assumed to satisfy Carathéodory conditions and allowed to depend on derivatives of the unknown function. An existence theorem for a similar problem (with continuous right-hand side and in a bounded domain) was proved by K. Goebel [5].

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1. Semimeasures of noncompactness

Let E denote a metrizable complete locally convex space (i.e. a B_0-space or in other words a Frechet space) with topology induced by a sequence of seminorms $\{q_n \mid n \in \mathbb{N}\}$. If $U_n = \{x \in E \mid q_n(x) < 1\}$, then $\mathcal{U} = \{\varepsilon U_n \mid n \in \mathbb{N}, \varepsilon > 0\}$ is a basis of absolutely convex neighbourhoods of zero. We say that a set $A \subset E$ is q_n-bounded if $q_n(A)$ is a bounded subset of $\mathbb{R}$, and denote by $\mathcal{B}_n$ the family of all q_n-bounded subsets of E. Then $\mathcal{B} = \bigcap\{\mathcal{B}_n \mid n \in \mathbb{N}\}$ is the family of all bounded subsets of E.

(1) DEFINITION. For $n \in \mathbb{N}$ let $\gamma_n : \mathcal{B}_n \to \mathbb{R}_+$,

$$\gamma_n(A) = \inf\{\delta > 0 \mid \text{there exists a finite set } X \subset E \text{ such that } A \subset X + \delta U_n\}.$$

(2) Remark. *If $E_n = E/q_n^{-1}(0)$ is a quotient space and $\pi_n : E \to E_n$ the projection, then the formula $\mu_n(A) = \gamma_n(\pi_n^{-1}(A))$ for any $\|\cdot\|_n$-bounded set $A \subset E_n$, where $\|[x]\|_n = q_n(x)$, defines a ball measure of noncompactness μ_n in the Banach space $\langle E_n, \|\cdot\|_n\rangle$.*

Justified by the above remark we shall call the functions $\{\gamma_n \mid n \in \mathbb{N}\}$ semimeasures of noncompactness induced by seminorms $\{q_n \mid n \in \mathbb{N}\}$. They have the known properties of a Banach space ball measure of noncompactness (c.f. [5]).

(3) PROPERTIES.

(3.1) *if $A \in \mathcal{B}_n$, $B \subset A$, then $B \in \mathcal{B}_n$ and $\gamma_n(B) \le \gamma_n(A)$,*

(3.2) *if $A, B \in \mathcal{B}_n$, then $A \cup B \in \mathcal{B}_n$ and $\gamma_n(A \cup B) \le \max(\gamma_n(A), \gamma_n(B))$,*

(3.3) *if $A \in \mathcal{B}_n$, then $\overline{A} \in \mathcal{B}_n$ and $\gamma_n(\overline{A}) = \gamma_n(A)$,*

(3.4) *if $A \in \mathcal{B}_n$ and $\alpha \in \mathbb{R}$, then $\alpha A \in \mathcal{B}_n$ and $\gamma_n(\alpha A) = |\alpha| \cdot \gamma_n(A)$,*

(3.5) *if $A, B \in \mathcal{B}_n$, then $A + B \in \mathcal{B}_n$ and $\gamma_n(A + B) \le \gamma_n(A) + \gamma_n(B)$,*

(3.6) *if $A \in \mathcal{B}_n$, then $\operatorname{co} A \in \mathcal{B}_n$ and $\gamma_n(\operatorname{co} A) \le \gamma_n(A)$, where $\operatorname{co} A$ denotes the smallest convex set containing A.*

Since relatively compact sets in a complete space are totally bounded sets we have

(4) PROPOSITION. *$\overline{A}$ is a compact subset of E iff $A \in \mathcal{B}$ and $\gamma_n(A) = 0$ for each $n \in \mathbb{N}$.*

The next point is a version of a well known theorem of Kuratowski for measures of noncompactness in metric spaces [7].

(5) PROPOSITION. *Let $\{A_m \mid m \in \mathbb{N}\}$ be a decreasing sequence of nonempty bounded closed sets such that $\lim_{m\to\infty} \gamma_n(A_m) = 0$ for each $n \in \mathbb{N}$. Then $A_\infty = \bigcap\{A_m \mid m \in \mathbb{N}\}$ is a nonempty compact set.*

Proof. We first prove that $A_\infty \neq \emptyset$. We shall use the double diagonal method. For every $n, m \in \mathbb{N}$ can be chosen a positive δ_m^n and a finite set X_m^n such that $A_m \subset X_m^n + \delta_m^n U_n$ and $\lim_{m\to\infty} \delta_m^n = 0$ for each $n \in \mathbb{N}$. Take any sequence $\{a_m \mid m \in \mathbb{N}\}$ such that $a_m \in A_m$.

Fix $n = 1$. We choose a subsequence $\{a_{1,m}\}$ of $\{a_m\}$ such that $\{a_{1,m}\} \subset x_1^1 + \delta_1^1 U_1$ for some $x_1^1 \in X_1^1$, then we choose a subsequence $\{a_{2,m}\}$ of $\{a_{1,m}\}$ such that $\{a_{2,m}\} \subset x_2^1 + \delta_2^1 U_1$ for some $x_2^1 \in X_2^1$, and so on—when $\{a_{k-1,m}\}$ is already defined, we choose $\{a_{k,m}\}$ to be its subsequence such that $\{a_{k,m}\} \subset x_k^1 + \delta_k^1 U_1$ for some $x_k^1 \in X_k^1$. Puting $a_m^1 = a_{m,m}$ we get a subsequence $\{a_m^1\}$ of $\{a_m\}$ which is a Cauchy sequence with respect to q_1.

Using the same method for $n = 2$ we define a subsequence $\{a_m^2\}$ of $\{a_m^1\}$ which is a Cauchy sequence with respect to both q_1 and q_2, and generally when $\{a_m^{n-1}\}$ is already defined and is a Cauchy sequence with respect to $q_1, q_2, \ldots, q_{n-1}$ we define its subsequence $\{a_m^n\}$ which is a Cauchy sequence with respect to $q_1, q_2, \ldots, q_n$. We put $\tilde{a}_m = a_m^m$ for $m \in \mathbb{N}$. Then $\{\tilde{a}_m\}$ is a subsequence of $\{a_m\}$ and is a Cauchy sequence (with respect to each q_n). Hence $\tilde{a}_m \to a$ for some $a \in E$ and obviously $a \in A_\infty$.

The compactness of A_∞ follows from (4) and the fact that $\gamma_n(A_\infty) = 0$ for each $n \in \mathbb{N}$ since by (3.1), $\gamma_n(A_\infty) \le \gamma_n(A_m)$ for every $n, m \in \mathbb{N}$. ∎

Let $F : D \to E$, where D is a closed subset of E, be a continuous map. We shall introduce the following:

(6) DEFINITION. Let $\{\gamma_n\}$ denote a sequence of semimeasures of noncompactness and let $\{k_n\}$ be a sequence of non negative reals.

(6.1) F is a $\{k_n\}$-$\{\gamma_n\}$-contraction, if $\gamma_n(F(A)) \le k_n\gamma_n(A)$ for every bounded $A \subset D$ and $n \in \mathbb{N}$;

(6.2) F is a strict $\{\gamma_n\}$-contraction, if its image $F(D)$ is bounded and it is a $\{k_n\}$-$\{\gamma_n\}$-contraction with $\{k_n\}$ such that $k_n < 1$ for each $n \in \mathbb{N}$;

(6.3) F is $\{\gamma_n\}$-condensing, if its image $F(D)$ is bounded and $\gamma_n(F(A)) < \gamma_n(A)$ for each $n \in \mathbb{N}$ and bounded $A \subset D$ with $\gamma_n(A) > 0$;

(6.4) F is a strict set-contraction (is condensing), if it is a strict $\{\tilde{\gamma}_n\}$-contraction (is $\{\tilde{\gamma}_n\}$-condensing) for some sequence of seminorms $\{\tilde{q}_n\}$

equivalent to $\{q_n\}$ and an induced sequence of semimeasures of noncompactness $\{\tilde{\gamma}_n\}$.

(7) PROPOSITION. *If $F : D \to E$ is a condensing map, then the vector field $I - F : D \to E$, where I denotes the identity, is proper and hence also closed.*

Proof. For an arbitrary compact $C \subset E$ let $A = (I - F)^{-1}(C)$. Then $A \subset F(A) + C$ and $\gamma_n(A) \le \gamma_n(F(A) + C) \le \gamma_n(F(A))$ for each $n \in \mathbb{N}$ because of (3) and (4). But this implies that $\gamma_n(A) = 0$ for each $n \in \mathbb{N}$ since F is condensing. ∎

An example of a $\{k_n\}$-$\{\gamma_n\}$-contraction is given by

(8) PROPOSITION. *Let us assume that a continuous map $V : D \times D \to E$ satisfies the following conditions*

(8.1) *$V(A, y)$ is relatively compact for every bounded $A \subset D$ and $y \in D$ (i.e. the map $V(\cdot, y) : D \to E$ is completely continuous),*

(8.2) *for each $n \in \mathbb{N}$ there exists $k_n \ge 0$ such that*

$$q_n(V(x, y_1) - V(x, y_2)) \le k_n q_n(y_1 - y_2)$$

for all $x \in D$ and $y_1, y_2 \in D$ (i.e. the section $V(x, \cdot) : D \to E$ is a $\{k_n\}$-$\{q_n\}$-contraction).

Then the map $F : D \to E$, $F(x) = V(x, x)$ is a $\{k_n\}$-$\{\gamma_n\}$-contraction.

Proof. Let $A \in D$ be a bounded set and let $\delta_n = \gamma_n(A)$, where $n \in \mathbb{N}$ is arbitrarily fixed. For any $\varepsilon > 0$ there exists $Y = \{y_1, y_2, \ldots, y_l\}$ such that $A \subset Y + (\delta_n + \varepsilon)U_n$. Thus we get

$$F(A) \subset V(A, A) \subset V(A, Y + (\delta_n + \varepsilon)U_n) \subset \bigcup_{i=1}^{l} V(A, y_i + (\delta_n + \varepsilon)U_n)$$

$$\subset \bigcup_{i=1}^{l} (V(A, y_i) + k_n(\delta_n + \varepsilon)U_n) \subset \bigcup_{i=1}^{l} \Big(X_i + \big(k_n(\delta_n + \varepsilon) + \varepsilon\big)U_n\Big),$$

where a finite set X_i was taken so that $V(A, y_i) \subset X_i + \varepsilon U_n$, since $V(A, y_i)$ is totally bounded. Hence $\gamma_n(F(A)) \le k_n(\gamma_n(A) + \varepsilon) + \varepsilon$, and therefore $\gamma_n(F(A)) \le k_n\gamma_n(A)$, since $\varepsilon > 0$ was arbitrarily chosen. ∎

2. Topological degree. Admissible maps and homotopies

We follow the method of R. D. Nussbaum [10] to define a degree for a class of maps including strict set-contractive vector fields in B_0 spaces. We start with a general construction for a continuous map $F : \overline{G} \to E$, $G \subset E$ open. We define

$$\begin{aligned} K_1 &\equiv K_1(F,\overline{G}) = \overline{\mathrm{co}}\, F(\overline{G}), \\ K_m &\equiv K_m(F,\overline{G}) = \overline{\mathrm{co}}\, F(\overline{G} \cap K_{m-1}), \quad m > 1, \\ K_\infty &\equiv K_\infty(F,\overline{G}) = \bigcap \{K_m \mid m \in \mathbb{N}\}. \end{aligned}$$

Notice that $\{K_m\}$ is a decreasing sequence of closed and convex sets and $F(G \cap K_m) \subset K_{m+1}$ for $m \in \mathbb{N}$. Hence K_∞ is a closed, convex and invariant set (i.e. $F(\overline{G} \cap K_\infty) \subset K_\infty$). Moreover the set of fixed points $\mathrm{Fix}\, F$ of the map F is contained in K_∞. We denote by $r : E \to K_\infty$ a retraction.

(9) LEMMA. *Let us assume that $C \subset E$ is a closed, convex and invariant set (i.e. $F(\overline{G} \cap C) \subset C$) such that $K_\infty \subset C$, and $R : E \to C$ is a retraction. Consider the map*

$$H : \overline{G} \times [0,1] \to E, \quad \textit{given by} \quad H(x,t) = (1-t)\, r\, F(x) + t\, R\, F(x).$$

Then $\mathrm{Fix}\, H = \mathrm{Fix}\, F$ *(where* $\mathrm{Fix}\, H = \{x \in \overline{G} \mid x = H(x,t)$ *for some* $t \in [0,1]\}$*).*

Proof. Since $\mathrm{Fix}\, F \subset K_\infty \subset C$, the inclusion $\mathrm{Fix}\, F \subset \mathrm{Fix}\, H$ follows. In order to prove the inclusion $\mathrm{Fix}\, H \subset \mathrm{Fix}\, F$ let us suppose that $x = (1-t)\, rF(x) + t\, RF(x)$ for some $t \in [0,1]$. Then $x \in C$ and by the invariance of C, $RF(x) = F(x)$, hence $x = (1-t)\, rF(x) + t\, F(x)$. We get $x \in K_1$, since $F(x) \in F(\overline{G}) \subset K_1$, and if $x \in K_m$, then $F(x) \in K_{m+1}$. Thus $x \in K_\infty$ and $x \in \mathrm{Fix}\, F$. ■

Let us note that a topological degree is defined for the class of compact vector fields (c.f. [9]). The following corollary is an immediate consequence of the above lemma.

(10) COROLLARY. *Let us assume that* $\mathrm{Fix}\, F \subset G$, $C \subset E$ *satisfies the assumptions of lemma (9) and* $rF, RF : \overline{G} \to E$ *are compact maps. Then*

$$\deg(I - rF, G, 0) = \deg(I - RF, G, 0).$$

For a given open $G \subset E$ we shall call a continuous map $F : \overline{G} \to E$ to be admissible (notation: $F \in \mathcal{A}(G)$) if $\mathrm{Fix}\, F \subset G$ and $K_\infty \equiv K_\infty(F,\overline{G})$ is a compact set. We define

$$\deg(I - F, G, 0) = \deg(I - rF, G, 0) \qquad (= 0, \text{ when } K_\infty = \emptyset),$$

for $F \in \mathcal{A}(G)$. Corollary (10) shows that this definition is independent of the choice of the retraction $r : E \to K_\infty$ (take $C = K_\infty$ and $R : E \to K_\infty$—any other retraction), and that the degree thus defined agrees with the degree for compact vector fields whenever $F : \overline{G} \to E$ is a compact map (take $C = E$ and $R = I$).

Let $H : \overline{G} \times [0,1] \to E$ be a continuous map and put

$$\begin{aligned} Q_1 &\equiv Q_1(H,\overline{G}) = \overline{\mathrm{co}}\, H(\overline{G} \times [0,1]),\\ Q_m &\equiv Q_m(H,\overline{G}) = \overline{\mathrm{co}}\, H\big((\overline{G} \cap Q_{m-1}) \times [0,1]\big), \quad m > 1,\\ Q_\infty &\equiv Q_\infty(H,\overline{G}) = \bigcap\{Q_m \mid m \in \mathbb{N}\}. \end{aligned}$$

The map H is an admissible homotopy (notation: $H \in \mathcal{AH}(G)$) if $\operatorname{Fix} H \subset G$ and $Q_\infty(H,\overline{G})$ is a compact set. The degree defined above has standard properties of normalization, additivity and homotopy invariance (c.f. [1]).

(11) PROPOSITION.

(11.1) *if* $0 \in G$, *then* $\deg(I,G,0) = 1$,

(11.2) *if* $F \in \mathcal{A}(G)$ *and* $G_1, G_2 \subset G$ *are open and disjoint sets such that* $\operatorname{Fix} F \subset G_1 \cup G_2$, *then* $F \in \mathcal{A}(G_1) \cap \mathcal{A}(G_2)$ *and*

$$\deg(I - F, G, 0) = \deg(I - F, G_1, 0) + \deg(I - F, G_2, 0),$$

(11.3) *if* $H \in \mathcal{AH}(G)$, *then* $H(\cdot,t) \in \mathcal{A}(G)$ *for each* $t \in [0,1]$ *and*

$$\deg(I - H(\cdot,0), G, 0) = \deg(I - H(\cdot,1), G, 0).$$

Proof. (11.1) is obvious. To prove (11.2) let $K_\infty = K_\infty(F,\overline{G})$, $r : E \to K_\infty$ and $K_{\infty,i} = K_\infty(F,\overline{G}_i)$, $r_i : E \to K_{\infty,i}$ for $i = 1,2$. Since $K_{\infty,i} \subset K_\infty$, then $F \in \mathcal{A}(G_i)$. Since $F(\overline{G}_i \cap K_\infty) \subset K_\infty$ and $K_{\infty,i} \subset K_\infty$, we get

$$\deg(I - r_i F, G_i, 0) = \deg(I - rF, G_i, 0)$$

by means of (10). Hence

$$\begin{aligned} \deg(I - F, G, 0) &= \deg(I - rF, G, 0) = \deg(I - rF, G_1, 0) + \deg(I - rF, G_2, 0)\\ &= \deg(I - r_1 F, G_1, 0) + \deg(I - r_2 F, G_2, 0)\\ &= \deg(I - F, G_1, 0) + \deg(I - F, G_2, 0). \end{aligned}$$

To prove (11.3) let $R : E \to Q_\infty$ and $K_{\infty,t} = K_\infty(H(\cdot,t),\overline{G})$, $r_t : E \to K_{\infty,t}$ for $t \in [0,1]$. Since, for any $t \in [0,1]$, $K_{\infty,t} \subset Q_\infty$, then $H(\cdot,t) \in \mathcal{A}(G)$. Moreover, from the inclusion $H(\overline{G} \cap Q_\infty, t) \subset Q_\infty$ for $t \in [0,1]$, we have

$$\deg(I - r_t H(\cdot,t), G, 0) = \deg(I - RH(\cdot,t), G, 0)$$

by means of (10). The compact homotopy RH is admissible and the proof is complete. ∎

The following propositions give examples of admissible maps and homotopies.

(12) PROPOSITION. *If $F : \overline{G} \to E$ is a strict set-contraction, then the set $K_\infty \equiv K_\infty(F, \overline{G})$ is compact (possibly empty). Hence $F \in \mathcal{A}(G)$, provided* $\operatorname{Fix} F \subset G$.

Proof. The set $K_1 = \overline{\text{co}}\, F(\overline{G})$ is a bounded set, hence $\{K_n\}$ is a decreasing sequence of closed bounded sets (possibly empty) and for each $n \in \mathbb{N}$ we have

$$\tilde{\gamma}_n(K_{m+1}) = \tilde{\gamma}_n(\overline{\text{co}}\, F(\overline{G} \cap K_m)) \le k_n \tilde{\gamma}_n(\overline{G} \cap K_m) \le k_n \tilde{\gamma}_n(K_m),$$

and hence $\lim_{m\to\infty} \tilde{\gamma}_n(K_m) = 0$. Thus $\tilde{\gamma}_n(K_\infty) = 0$ and the assertion follows. ∎

(13) PROPOSITION. *Let $H : \overline{G} \times [0,1] \to E$ be a strict set-contractive homotopy (i.e. we assume that $H(\overline{G} \times [0,1])$ is a bounded set and $\tilde{\gamma}_n(H(A \times [0,1])) \le k_n \tilde{\gamma}_n(A)$ with $k_n < 1$, $n \in \mathbb{N}$, for some $\{\tilde{\gamma}_n\}$ and an arbitrary bounded $A \subset \overline{G}$). Then $Q_\infty \equiv Q_\infty(H, G)$ is a compact set. Moreover $H \in \mathcal{AH}(G)$, provided* $\operatorname{Fix} H \subset G$.

Proof. The set $Q_1 = \overline{\text{co}}\, H(G \times [0,1])$ is a bounded set, hence $\{Q_m\}$ is a decreasing sequence of closed bounded sets (possibly empty) and for each $n \in \mathbb{N}$ we have

$$\tilde{\gamma}_n(Q_{m+1}) = \tilde{\gamma}_n(\overline{\text{co}}\, H((\overline{G} \cap Q_m) \times [0,1])) \le k_n \tilde{\gamma}_n(Q_m),$$

and so $\lim_{m\to\infty} \tilde{\gamma}_n(Q_m) = 0$. Thus $\tilde{\gamma}_n(Q_\infty) = 0$. ∎

(14) PROPOSITION. *Let $F_0, F_1 : \overline{G} \to E$ be strict $\{\tilde{\gamma}_n\}$-contractions for some sequence $\{\tilde{\gamma}_n\}$. Then the linear homotopy*

$$H : \overline{G} \times [0,1] \to E, \quad H(x,t) = (1-t)\, F_0(x) + t\, F_1(x)$$

satisfies assumptions of (13) and hence $H \in \mathcal{AH}(G)$ provided $\operatorname{Fix} H \subset G$.

Proof. The set $H(\overline{G} \times [0,1]) \subset \text{co}(F_0(\overline{G}) \cup F_1(\overline{G}))$ is bounded and for an arbitrary $n \in \mathbb{N}$ and a bounded $A \subset \overline{G}$

$$\begin{aligned}\tilde{\gamma}_n(H(A \times [0,1])) &\le \tilde{\gamma}_n(\text{co}(F_0(A) \cup F_1(A))) = \max(\tilde{\gamma}_n(F_0(A)), \tilde{\gamma}_n(F_1(A))) \\ &\le \max(k_{0,n}\tilde{\gamma}_n(A), k_{1,n}\tilde{\gamma}_n(A)) \le k_n \tilde{\gamma}_n(A),\end{aligned}$$

where $k_n = \max(k_{0,n}, k_{1,n}) < 1$. ∎

3. Structure of fixed point sets of strict set-contractions

The sequence of seminorms $\{q_n\}$ generating the topology of E is now assumed to be *non decreasing*. Let $f : \overline{G} \to E$ be a continuous map and $\{\varepsilon_n\}$ a sequence of positive reals tending to zero. A sequence of continuous maps $\{f_n : \overline{G} \to E \mid n \in \mathbb{N}\}$ is an $\{\varepsilon_n\}$-$\{q_n\}$-approximation of the map f, if $q_n(f_n(x) - f(x)) \leq \varepsilon_n$ for all $n \in \mathbb{N}$ and $x \in \overline{G}$.

The following lemma was proved in [4]:

(15) LEMMA. *Suppose that a continuous map $f : \overline{G} \to E$ satisfies the following conditions*

(15.1) *every sequence $\{x_n\} \subset \overline{G}$ such that $f(x_n) \to 0$ contains a convergent subsequence;*

(15.2) *there exists an $\{\varepsilon_n\}$-$\{q_n\}$-approximation $\{f_n : \overline{G} \to E \mid n \in \mathbb{N}\}$ of f such that the map*

$$\tilde{f}_n : f_n^{-1}(\varepsilon_n \overline{U}_n) \to \varepsilon_n \overline{U}_n, \quad \tilde{f}_n(x) = f_n(x)$$

is a homeomorphism for each $n \in \mathbb{N}$.

Then $f^{-1}(0)$ is an $\mathcal{R}_\delta$-set in the space E.

Let $\{\tilde{q}_n\}$ denote a sequence of seminorms equivalent to $\{q_n\}$ (not necessarily non decreasing) and let $\{\tilde{\gamma}_n\}$ be a corresponding sequence of semimeasures of noncompactness. Now we prove our main result.

(16) THEOREM. *Suppose that $F : \overline{G} \to E$ is a strict $\{\tilde{\gamma}_m\}$-contraction satisfying the following conditions*

(16.1) $\operatorname{Fix} F \subset G$ *and* $\deg(I - F, G, 0) \neq 0$;

(16.2) *there exists an $\{\varepsilon_n\}$-$\{q_n\}$-approximation $\{F_n : \overline{G} \to E \mid n \in \mathbb{N}\}$ of F such that the equation $x - F_n(x) = y$ has at most one solution for every $y \in \varepsilon_n \overline{U}_n$ and $n \in \mathbb{N}$;*

Then the set of fixed points $\operatorname{Fix} F$ *is an $\mathcal{R}_\delta$-set in the space E.*

Proof. It suffices to check that the maps $f = I - F$ and $f_n = I - F_n$, $n \in \mathbb{N}$, satisfy the assumptions of lemma (15). By virtue of (7) the map f is proper, hence it satisfies the condition (15.1). Moreover f is a closed map, hence $(2\varepsilon_n \overline{U}_n) \cap f(\partial G) = \emptyset$ for sufficiently large $n \in \mathbb{N}$, since $0 \notin f(\partial G)$. Then for every $y \in \varepsilon_n \overline{U}_n$ the linear homotopy

$$H_{n,y} : G \times [0,1] \to E, \quad H_{n,y}(x,t) = (1-t)\,F(x) + t\,\big(F_n(x) - y\big)$$

has no fixed points on ∂G and, by (14), $H_{n,y} \in \mathcal{AH}(G)$. Thus

$$\deg(f_n - y, G, 0) = \deg(f, G, 0) \neq 0$$

for all $y \in \varepsilon_n \overline{U}_n$.

We have shown that $\varepsilon_n \overline{U}_n \subset f_n(G)$ for sufficiently large $n \in \mathbb{N}$. By virtue of (16.2) and (7) $\tilde{f}_n : f_n^{-1}(\varepsilon_n \overline{U}_n) \to \varepsilon_n \overline{U}_n$ is a one-to-one closed continuous map, hence it is a homeomorphism. ∎

(17) Corollary. *If $F : E \to E$ is a strict $\{\tilde{\gamma}_m\}$-contraction satisfying condition* (16.2), *then* $\operatorname{Fix} F$ *is an $\mathcal{R}_\delta$-set in the space E*

Proof. The homotopy

$$H : E \times [0,1] \to E, \quad H(t,x) = t\,F(x)$$

is admissible by (14). Hence $\deg(I - F, E, 0) = \deg(I, E, 0) = 1$ and the assumptions of theorem (16) are fullfilled. ■

Now we are going to apply the above theorem to the Darboux problem for a hyperbolic equation. Let $\mathbb{R}_+ = [0,+\infty)$, $\Delta = \mathbb{R}_+ \times \mathbb{R}_+$ and $\Delta_n = [0,n] \times [0,n]$, $n \in \mathbb{N}$. Let $f : \Delta \times \mathbb{R}^{4\nu} \to \mathbb{R}^\nu$ be a Carathéodory map, i. e. we assume that all its sections

$$f(x,y;\cdot,\cdot,\cdot,\cdot) : \mathbb{R}^{4\nu} \to \mathbb{R}^\nu, \qquad (x,y) \in \Delta$$

are continuous and all sections

$$f(\cdot,\cdot;u,r,s,t) : \Delta \to \mathbb{R}^\nu, \qquad (u,r,s,t) \in \mathbb{R}^{4\nu}$$

are measurable in the Lebesgue sense.

The Darboux problem is stated as follows:

$$\text{(D)} \qquad \begin{cases} u_{xy} = f(x,y;u,u_x,u_y,u_{xy}) & \text{in } \Delta, \\ u(0,y) = g(y),\ u(x,0) = h(x) & \text{on } \partial\Delta, \end{cases}$$

where $g, h : \mathbb{R}_+ \to \mathbb{R}^\nu$ are given absolutely continuous maps which satisfy condition $g(0) = h(0)$. A solution of this problem is any absolutely continuous map $u : \Delta \to \mathbb{R}^\nu$ which satisfies the differential equation almost everywhere in Δ and the boundary condition for all $x, y \in \mathbb{R}_+$.

In the sequel we say that a measurable function $v : \Delta \to \mathbb{R}_+$ is *locally bounded (locally less then a, $a > 0$)*, if

$$\operatorname{ess\,sup}_{(x,y)\in\Delta_n} \big|v(x,y)\big| < +\infty \qquad (\text{respectively: } \ldots < a)$$

for each $n \in \mathbb{N}$.

In the following theorem we study the set of all solutions of the problem (D) as a subset of the B_0 space $\mathcal{C}$ of continuous maps from Δ into $\mathbb{R}^\nu$:

$$\mathcal{C} = \Big\langle C(\Delta, \mathbb{R}^\nu), \{q_n | n \in \mathbb{N}\} \Big\rangle, \qquad q_n(u) = \sup\Big\{ |u(x,y)| \,\Big|\, (x,y) \in \Delta_n \Big\}.$$

The sequence $\{q_n\}$ of seminorms in $\mathcal{C}$ is nondecreasing.

(18) THEOREM. *Let $b, c, K, M, N : \Delta \to \mathbb{R}_+$ be measurable locally bounded functions and let N be locally less then 1. We assume that the following two conditions are satisfied:*

(18.1) *for all* $(x, y; u, r, s, t) \in \Delta \times \mathbb{R}^{4\nu}$,

$$|f(x, y; u, r, s, t)| \leq b(x, y)|u| + c(x, y),$$

(18.2) *for all* $(x, y; u, r_1, s_1, t_1), (x, y; u, r_2, s_2, t_2) \in \Delta \times \mathbb{R}^{4\nu}$,

$$|f(x, y; u, r_1, s_1, t_1) - f(x, y; u, r_2, s_2, t_2)| \\ \leq K(x, y)|r_1 - r_2| + M(x, y)|s_1 - s_2| + N(x, y)|t_1 - t_2|.$$

Then the set of all solutions of the Darboux problem (D) is an $\mathcal{R}_\delta$-set in the space C.

P r o o f. In the proof we consider the B_0 space of locally integrable maps

$$\mathcal{L} = \Big\langle \mathcal{L}(\Delta, \mathbb{R}^\nu), \{p_n | n \in \mathbb{N}\} \Big\rangle, \qquad p_n(u) = \int\limits_{\Delta_n} |u(x, y)|\, dx\, dy$$

($\{p_n\}$ is nondecreasing). With a Carathéodory map $f : \Delta \times \mathbb{R}^{4\nu} \to \mathbb{R}^\nu$ satisfying condition (18.1) we associate a continuous map $V_f : \mathcal{L} \times \mathcal{L} \to \mathcal{L}$ (c.f. [6]) given by the formula

$$V_f(v, w)(x, y) \\ = f\Big(x, y;\ \int\limits_0^x \int\limits_0^y v(\xi, \eta)\, d\xi\, d\eta,\ \int\limits_0^y w(x, \eta)\, d\eta,\ \int\limits_0^x w(\xi, y)\, d\xi, w(x, y)\Big)$$

and a corresponding map $F_f : \mathcal{L} \to \mathcal{L}$, $F_f(v) = V_f(v, v)$.

A map $u : \Delta \to \mathbb{R}^\nu$ is a solution of the problem (D) if and only if $v = u_{xy}$ is a fixed point of the map $F_{\tilde{f}} : \mathcal{L} \to \mathcal{L}$, where

$$\tilde{f} : \Delta \times \mathbb{R}^{4\nu} \to \mathbb{R}^\nu, \\ \tilde{f}(x, y; u, r, s, t) = f\big(x, y; u + h(x) + g(y) - g(0), r + h'(x), s + g'(y), t\big)$$

and $\tilde{f}$ satisfies the same assumptions as f.

If $v \in \operatorname{Fix} F_{\tilde{f}}$, then, by (18.1),

$$|v(x, y)| \leq c(x, y) + b(x, y) \int\limits_0^x \int\limits_0^y |v(\xi, \eta)|\, d\xi\, d\eta$$

and, by an inequality of Wendroff ([3], chapter 4, §30),

$$|v(x, y)| \leq c(x, y)\, e^{xy\, b(x, y)} \equiv \alpha(x, y), \quad (x, y) \in \Delta,$$

where $\alpha : \Delta \to \mathbf{R}_+$ is measurable, locally bounded and we can also assume that $\alpha(x,y) > 0$ for each $(x,y) \in \Delta$. Hence the set of all fixed points of $F_{\bar{f}}$ coincides with the fixed point set of the map $\widehat{F}_{\bar{f}}$ given by the formula $\widehat{F}_{\bar{f}}(v) = V_{\bar{f}}(R(v), v)$ where $R : \mathcal{L} \to \mathcal{L}$ is a retraction onto the set

$$\mathcal{D}_\alpha = \{\, v \in \mathcal{L} : \ |v(x,y)| \le \alpha(x,y) \text{ almost everywhere in } \Delta \,\}$$

defined by

$$(Rv)(x,y) = \begin{cases} v(x,y), & |v(x,y)| \le \alpha(x,y), \\ \dfrac{v(x,y)}{|v(x,y)|}\alpha(x,y), & |v(x,y)| \ge \alpha(x,y). \end{cases}$$

With a positive locally bounded measurable function α on Δ, we also associate the following subset of $\Delta \times \mathbf{R}^{4\nu}$:

$$\Omega_\alpha = \Big\{ (x,y;u,r,s,t) \in \Delta \times \mathbf{R}^{4\nu} : \quad |u| \le \int_0^x \int_0^y \alpha(\xi,\eta)\, d\xi\, d\eta,$$
$$|r| \le \int_0^y \alpha(x,\eta)\, d\eta, \quad |s| \le \int_0^x \alpha(\xi,y)\, d\xi, \quad |t| \le \alpha(x,y) \Big\}.$$

In order to complete the proof of Theorem (18) we need the following two lemmas. The first of them is stated without proof, since it is very similar to the approximation lemma in [4] (Lemma (3.2)). The proof in [4] requires only minor changes.

(19) LEMMA. *Suppose $\alpha : \Delta \to \mathbf{R}_+$ is a positive locally bounded measurable function and $f : \Delta \times \mathbf{R}^{4\nu} \to \mathbf{R}^\nu$ is a Carathéodory map which satisfies the hypothesis of Theorem* (18).

For each $\varepsilon > 0$ and $n > 0$ there exists a Carathéodory map

$$\bar{f} : \Delta \times \mathbf{R}^{4\nu} \to \mathbf{R}^\nu,$$

measurable locally bounded functions $\bar{b}, \bar{c}, \bar{L} : \Delta \to \mathbf{R}_+$ and an integrable function $\varphi : \Delta_n \to \mathbf{R}_+$ such that

$$\int_{\Delta_n} \varphi(x,y)\, dx\, dy < \varepsilon,$$

and the map $\bar{f}$ satisfies conditions (18.1), (18.2) (*with b and c replaced by $\bar{b}$ and $\bar{c}$*), *and moreover*

(19.1) *for all $(x,y;u_1,r_1,s_1,t_1), (x,y;u_2,r_2,s_2,t_2) \in \Omega_\alpha$,*

$$|\bar{f}(x,y;u_1,r_1,s_1,t_1) - \bar{f}(x,y;u_2,r_2,s_2,t_2)| \le \bar{L}(x,y)|u_1 - u_2|$$

$$+ K(x,y)|r_1 - r_2| + M(x,y)|s_1 - s_2| + N(x,y)|t_1 - t_2|$$

and

(19.2) *for all* $(x,y;u,r,s,t) \in \Omega_\alpha \cap (\Delta_n \times \mathbb{R}^{4\nu})$,

$$|\overline{f}(x,y;u,r,s,t) - f(x,y;u,r,s,t)| \leq \varphi(x,y).$$

(20) LEMMA. *Suppose* $\alpha : \Delta \to \mathbb{R}_+$ *is a positive locally bounded measurable function and* $f : \Delta \times \mathbb{R}^{4\nu} \to \mathbb{R}^\nu$ *is a Carathéodory map which satisfies the hypothesis of Theorem* (18). *Then the map*

$$\widehat{F}_f : \mathcal{L} \to \mathcal{L}, \qquad \widehat{F}_f(v) = V_f(R(v), v)$$

is a strong $\{\tilde{\gamma}_n\}$*-contraction, for some sequence of seminorms* $\{\tilde{q}_n\}$ (*and implied* $\{\tilde{\gamma}_n\}$) *equivalent to* $\{q_n\}$, *where the choice of* $\{\tilde{q}_n\}$ *depends only on the functions* K, M *and* N.

Proof. It follows from the definition of retraction R and (18.1) that the image $V(R(\mathcal{L}), w)$ is a bounded subset of the space $\mathcal{L}$ for each $w \in \mathcal{L}$. Moreover, since the image of the set $R(\mathcal{L}) = \mathcal{D}_\alpha$ under the map $J : \mathcal{L} \to \mathcal{L}$,

$$J(v)(x,y) = \int_0^x \int_0^y v(\xi,\eta)\, d\xi\, d\eta$$

is relatively compact (it is a set of functions equicontinuous on each Δ_n), the section

$$v \mapsto V(R(v), w) : \mathcal{L} \to \mathcal{L}$$

is a compact map.

Now we are going to show that for arbitrary $v \in \mathcal{L}$ the section $V(v, \cdot) : \mathcal{L} \to \mathcal{L}$ is a $\{k_n\}$-$\{\tilde{p}_n\}$-contraction with $k_n < 1$, $n \in \mathbb{N}$, for a suitable sequence of seminorms $\{\tilde{p}_n\}$ in $\mathcal{L}$, equivalent to $\{p_n\}$. The assertion will then follow from (8) applied to the map $(v, w) \mapsto V_f(R(v), w)$.

We use the method of Bielecki considering the family of seminorms

$$p_{n,\kappa}(v) = \int_{\Delta_n} e^{-\kappa(x+y)}\, |v(x,y)|\, dx\, dy, \quad n \in \mathbb{N},\ \kappa > 0.$$

We introduce the following denotations

$$K_n = \operatorname{ess\,sup}_{(x,y)\in\Delta_n} |K(x,y)| < \infty,$$
$$M_n = \operatorname{ess\,sup}_{(x,y)\in\Delta_n} |M(x,y)| < \infty,$$
$$N_n = \operatorname{ess\,sup}_{(x,y)\in\Delta_n} |N(x,y)| < 1, \qquad n \in \mathbb{N}$$

and note that $N_n < 1$ for each $n \in \mathbb{N}$. By virtue of (18.2) and integration by parts we have

$$\begin{aligned}
&p_{n,\kappa}\big(V(v,w_1) - V(v,w_2)\big) \\
&\le \int_{\Delta_n} dx\,dy\, e^{-\kappa(x+y)}\Big[K(x,y)\int_0^y d\eta\,|w_1(x,y) - w_2(x,\eta)| \\
&\quad + M(x,y)\int_0^x d\xi\,|w_1(\xi,y) - w_2(\xi,y)| + N(x,y)|w_1(x,y) - w_2(x,y)|\Big] \\
&\le K_n \int_0^n dx\, e^{-\kappa x}\int_0^n dy\, e^{-\kappa y}\int_0^y d\eta\,|w_1(x,\eta) - w_2(x,\eta)| \\
&\quad + M_n \int_0^n dy\, e^{-\kappa y}\int_0^n dx\, e^{-\kappa x}\int_0^x d\xi\,|w_1(\xi,y) - w_2(\xi,y)| + N_n p_{n,\kappa}(w_1 - w_2) \\
&\le \Big(\frac{K_n + M_n}{\kappa} + N_n\Big) p_{n,\kappa}(w_1 - w_2).
\end{aligned}$$

It is sufficient now to take $\tilde{p}_n = p_{n,\kappa}$ with $\kappa > (K_n - M_n)/(1 - N_n)$, to get $k_n = (K_n + M_n)/\kappa + N_n < 1$ and the proof is complete. ∎

Now we complete the **proof of Theorem (18).** Let $f_n : \Delta \times \mathbf{R}^{4\nu} \to \mathbf{R}^\nu$ denote the approximation of $\tilde{f}$ given by lemma (19) for $n \in \mathbb{N}$ and $\varepsilon = \frac{1}{n}$, and let $F = \widehat{F}_{\tilde{f}}$ and $F_n = \widehat{F}_{f_n}$. Then $\{F_n\}$ is a $\{1/n\}$-$\{p_n\}$-approximation of F and, by Lemma (20), F and F_n $(n \in \mathbb{N})$ are strict $\{\tilde{\gamma}_m\}$-contractions for some common sequence of semimeasures of noncompactness $\{\tilde{\gamma}_m\}$.

Suppose now $v_1 - F_n(v_1) = v_2 - F_n(v_2)$ for some $n \in \mathbb{N}$ and $v_1, v_2 \in \mathcal{L}$. Then $v_1 - v_2 = F_n(v_1) - F_n(v_2)$ and, since F_n satisfies (19.1) with some $\overline{L} \equiv L_n$, then

$$\begin{aligned}
|v_1(x,y) - v_2(x,y)| &\le \frac{L_n(x,y)}{1 - N(x,y)}\int_0^x\int_0^y |v_1(\xi,\eta) - v_2(\xi,\eta)|\,d\xi\,d\eta \\
&+ \frac{K(x,y)}{1 - N(x,y)}\int_0^y |v_1(x,\eta) - v_2(x,\eta)|\,d\eta + \frac{M(x,y)}{1 - N(x,y)}\int_0^x |v_1(\xi,y) - v_2(\xi,y)|\,d\xi
\end{aligned}$$

and so we get $v_1 = v_2$ (see inequalities [3], chapter 4, §30).

Thus $F : \mathcal{L} \to \mathcal{L}$ satisfies assumptions of corollary (17) and hence Fix F is an $\mathcal{R}_\delta$-set in the space $\mathcal{L}$. Let us notice that the set of all solutions of (D) coincides with the set $S(\operatorname{Fix} F)$, where

$$S : \mathcal{L} \to \mathcal{C}, \qquad S(u)(x,y) = h(x) + g(y) - g(0) + \int_0^x \int_0^y v(\xi,\eta)\, d\xi\, d\eta.$$

Since S is continuous and one-to-one, it is homeomorphic on compact sets, and $S(\operatorname{Fix} F)$ is an $\mathcal{R}_\delta$-set in $\mathcal{C}$. ■

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INSTITUTE OF MATHEMATICS
UNIVERSITY OF GDAŃSK
ul. Wita Stwosza 57
89-952 GDAŃSK, POLAND
E-mail: kczarn@ksinet.univ.gda.pl

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