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THE LOGIC INDUCED BY A SYSTEM OF HOMOMORPHISMS AND ITS VARIOUS ALGEBRAIC CHARACTERIZATIONS

1. Introduction

The well-known spectral theorem for self-adjoint operators on a Hilbert space can be formulated as follows:

Let H be a complex separable Hilbert space with $\dim H > 2$ and let $L(H)$ denote the orthomodular lattice (shortly, OML) of all orthogonal projections from H onto closed linear subspaces of H. Let $\mathbf{O}$ denote the set of all self-adjoint linear operators on H and $\{m_\alpha \mid \alpha \in \mathrm{S}\}$ the set of all pure probability measures on $L(H)$. Then for every $A \in \mathbf{O}$ there exists a unique $L(H)$-valued measure (spectral measure) μ_A on $\mathbf{B}(\mathbf{R})$ such that for every $\alpha \in \mathrm{S}$ the composed mapping $m_\alpha \circ \mu_A$ is a probability measure on $\mathbf{B}(\mathbf{R})$. (Here and in the following $\mathbf{B}(\mathbf{R})$ denotes the Boolean σ-algebra of all Borel sets of the real line.) Hence, the spectral theorem determines a doubly indexed family $(p_{A,\alpha})_{A\in\mathbf{O},\alpha\in\mathrm{S}}$ of probability measures on $\mathbf{B}(\mathbf{R})$ such that each $p_{A,\alpha}$ can be decomposed in the form $p_{A,\alpha} = m_\alpha \circ \mu_A$ where μ_A is an $L(H)$-valued measure on $\mathbf{B}(\mathbf{R})$ and m_α is a pure probability measure on $L(H)$. The family $(p_{A,\alpha})_{A\in\mathbf{O},\alpha\in\mathrm{S}}$ can be interpreted as the spectral family of probability measures on $\mathbf{B}(\mathbf{R})$ corresponding to $\mathbf{O}$ and S.

Now, by the inverse spectral theorem we may understand the following problem:

Given a doubly indexed family $(p_{A,\alpha})_{A\in\mathbf{O},\alpha\in\mathrm{S}}$ of probability measures on $\mathbf{B}(\mathbf{R})$, what conditions are to be put on $\mathbf{O}$ and S in order that every

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$p_{A,\alpha}$ could be decomposed into an L-valued measure μ_A on $\mathbf{B}(\mathbf{R})$ and a pure probability measure m_α on L for a suitable intermediate system L. This intermediate system L (interpreted as the logic induced by the family $(p_{A,\alpha})_{A\in\mathbf{O},\alpha\in\mathbf{S}}$) should be uniquely determined by this family in the way that L is a subalgebra of $[0,1]^{\mathbf{S}}$.

This problem could be of importance in axiomatic quantum mechanics, where the sets $\mathbf{O}$ and $\mathbf{S}$ are interpreted as the set of all observables and the set of all states of a fixed physical system F, respectively, and where the probability $p_{A,\alpha}(E)$, $E \in \mathbf{B}(\mathbf{R})$, is interpreted as the probability that a measurement of A will lead to a value in E provided that F is in the state α.

The inverse problem formulated above can be interpreted as the problem of determining the logic L of F on basis of the knowledge of all probability measures $p_{A,\alpha}$ induced by measurements performed within F (i. e. on the basis of knowing all results of all measurements which may be performed within F).

We shall first formulate our problem in a very general algebraic setting and then apply it to some concrete algebraic systems.

First observe that not all mappings occurring within the formula $p_{A,\alpha} = m_\alpha \circ \mu_A$ are of the same sort:

$p_{A,\alpha}$ and m_α are probability measures whereas μ_A is an L-valued measure. In order to be able to interprete all mappings occurring within the formula $p_{A,\alpha} = m_\alpha \circ \mu_A$ as homomorphisms, it will be convenient to consider the sets $\mathbf{B}(\mathbf{R})$, $[0,1]$ and L as partial algebras $(A, \oplus, ', 0)$ of type $(2, 1, 0)$ with a partial binary operation $\oplus$ of orthogonal addition and a total unary operation $'$ of orthocomplementation defined as follows:

$$(\mathbf{B}(\mathbf{R}), \oplus, ', \emptyset): \quad E \oplus F := E \cup F \text{ whenever } E \cap F = \emptyset, \qquad E' := \mathbf{R} \setminus E$$
$$([0,1], \oplus, ', 0): \quad a \oplus b := a + b \text{ whenever } a + b \le 1, \qquad a' := 1 - a$$
$$(L, \oplus, ', 0): \quad a \oplus b := a \vee b \text{ whenever } a \perp b \text{ (i. e. } a \le b').$$

Then all mappings occurring within the formula $p_{A,\alpha} = m_\alpha \circ \mu_A$ can be interpreted as (ortho-)homomorphisms within this type of partial algebras.

The inverse problem formulated above can now be described as the problem of decomposing the doubly indexed family $(p_{A,\alpha})_{A\in\mathbf{O},\alpha\in\mathbf{S}}$ of homomorphisms from $\mathbf{B}(\mathbf{R})$ to $[0,1]$ into two independent families $(\mu_A)_{A\in\mathbf{O}}$ and $(m_\alpha)_{\alpha\in\mathbf{S}}$ of homomorphisms from $\mathbf{B}(\mathbf{R})$ to L and from L to $[0,1]$, respectively, such that $p_{A,\alpha} = m_\alpha \circ \mu_A$ holds for all $A \in \mathbf{O}$ and all $\alpha \in \mathbf{S}$. The next section will present a possible solution to this problem.

Section 3 will be devoted to an application of the previous results to quantum logics (orthomodular posets). In Section 4 we will generalize and interpret our concepts from a different point of view: We establish a one-to-one correspondence between ring-like structures and lattices which generalizes the well-known correspondence between Boolean rings and Boolean algebras. In particular, we will characterize orthomodular lattices this way.

2. The generalized inverse spectral theorem for homomorphisms in algebraic systems

Let A_1 and A_2 be two partial algebras of type $(2,1,0)$. For notions concerning the theory of partial algebras we refer the reader to the monograph [1]. For simplicity and also with regard to later generalizations, we shall consider in detail the case where the type of the considered partial algebras is (2), i. e. where there is only one partial binary operation, denoted by $\oplus$. Hence we have $A_1 = (A_1, \oplus)$ and $A_2 = (A_2, \oplus)$. Without loss of generality we may assume that this operation is commutative. By dom $\oplus_{A_1}$ we denote the domain of the partial operation $\oplus$ on A_1. (We do not exclude the case that the operation $\oplus$ on A_1 is total, i. e. that dom $\oplus_{A_1} = A_1^2$.)

In the following let $(p_{A,\alpha})_{A\in \mathbf{O}, \alpha\in \mathbf{S}}$ be a doubly indexed family of homomorphisms from A_1 to A_2. We ask, under which conditions there exists a partial algebra L (of the same type as A_1 and A_2), a family $(\mu_A)_{A\in\mathbf{O}}$ of homomorphisms from A_1 to L and a family $(m_\alpha)_{\alpha\in\mathbf{S}}$ of homomorphisms from L to A_2 such that $p_{A,\alpha} = m_\alpha \circ \mu_A$ for all $A \in \mathbf{O}$ and $\alpha \in \mathbf{S}$. We shall assume that the family $(\mu_A)_{A\in\mathbf{O}}$ is surjective, i. e. that $L = \bigcup_{A\in\mathbf{O}} \mu_A(A_1)$.
We now define the intermediate system L as follows:

Definition 2.1. Let L denote the set $\{[(A,a)] \mid A \in \mathbf{O}, a \in A_1\}$ where for every $A \in \mathbf{O}$ and $a \in A_1$ $[(A,a)]$ denotes the mapping from $\mathbf{S}$ to A_2 defined by

$$[(A,a)](\alpha) := p_{A,\alpha}(a)$$

for all $\alpha \in \mathbf{S}$.

Now we have the following result:

Lemma 2.2. *For all* $A \in \mathbf{O}$ *and all* $(a,b) \in \text{dom } \oplus_{A_1}$ (i) *and* (ii) *hold*:

(i) $([(A,a)],[(A,b)]) \in \text{dom } \oplus_{A_2^{\mathbf{S}}}$.
(ii) $[(A,a)] \oplus [(A,b)] = [(A,a\oplus b)]$.

Proof. This follows from the fact that for all $A \in \mathbf{O}$ and $\alpha \in \mathbf{S}$ $p_{A,\alpha}$ is a homomorphism from A_1 to A_2 and from the definition of the mapping $[(A,a)]$ $(A \in \mathbf{O}, a \in A_1)$. □

Since $A_2^{\mathbf{S}}$ is a cartesian power of A_2, it can be considered as a partial algebra of the same type as A_2 with operations being defined coordinatewise. Obviously, the set L is a subset of the set $A_2^{\mathbf{S}}$, but it may happen that the partial algebra L is not a subalgebra of the partial algebra $A_2^{\mathbf{S}}$ since $f, g \in L$ and $(f,g) \in \operatorname{dom} \oplus_{A_2^{\mathbf{S}}}$ together need not imply $f \oplus g \in L$. In order to ensure that the set L can be regarded as a subalgebra of the partial algebra $A_2^{\mathbf{S}}$ we have to put some conditions on the homomorphisms $p_{A,\alpha}$ from A_1 to A_2. To this aim we introduce the following definitions:

DEFINITION 2.3. Two elements $[(A,a)]$ and $[(B,b)]$ of L are said to be *compatible* with each other if $([(A,a)],[(B,b)]) \in \operatorname{dom} \oplus_{A_2^{\mathbf{S}}}$.

DEFINITION 2.4. Two elements $[(A,a)]$ and $[(B,b)]$ of L are said to be *strongly compatible* with each other if there exist $C \in \mathbf{O}$ and $(c,d) \in \operatorname{dom} \oplus_{A_1}$ such that $([(A,a)],[(B,b)]) = ([(C,c)],[(C,d)])$.

LEMMA 2.5. *Every pair of strongly compatible elements of L is compatible.*

Proof. Let $[(A,a)]$ and $[(B,b)]$ be two elements of L which are strongly compatible with each other. Then there exist $C \in \mathbf{O}$ and $(c,d) \in \operatorname{dom} \oplus_{A_1}$ such that $([(A,a)],[(B,b)]) = ([(C,c)],[(C,d)])$. Because of Lemma 2.2 we have $([(C,c)],[(C,d)]) \in \operatorname{dom} \oplus_{A_2^{\mathbf{S}}}$ and hence $([(A,a)],[(B,b)]) \in \operatorname{dom} \oplus_{A_2^{\mathbf{S}}}$ follows, i. e. the elements $[(A,a)]$ and $[(B,b)]$ are compatible with each other. □

DEFINITION 2.6. The family $(p_{A,\alpha})_{A\in\mathbf{O},\alpha\in\mathbf{S}}$ of homomorphisms from A_1 to A_2 is said to be *compatible* if every pair of mutually compatible elements of L is even strongly compatible.

We are now able to formulate the main theorem of this section:

THEOREM 2.7. *Assume that the family* $(p_{A,\alpha})_{A\in\mathbf{O},\alpha\in\mathbf{S}}$ *of homomorphisms from* A_1 to A_2 *is compatible. Then* (i) - (iv) *hold*:

(i) *L is a subalgebra of* $A_2^{\mathbf{S}}$.
(ii) *For every* $A \in \mathbf{O}$ *the mapping* μ_A *from* A_1 *to L defined by* $\mu_A(a) := [(A,a)]$ *for all* $a \in A_1$ *is a homomorphism from* A_1 *to L.*
(iii) *For every* $\alpha \in \mathbf{S}$ *the mapping* m_α *from L to* A_2 *defined by* $m_\alpha([(A,a)]) := p_{A,\alpha}(a)$ *for all* $A \in \mathbf{O}$ *and* $a \in A_1$ *is a well-defined homomorphism from L to* A_2.
(iv) *For all* $A \in \mathbf{O}$ *and all* $\alpha \in \mathbf{S}$ *we have* $p_{A,\alpha} = m_\alpha \circ \mu_A$.

Proof. (i) Let $A, B \in \mathbf{O}$ and $a, b \in A_1$ and assume that $([(A,a)],[(B,b)]) \in \operatorname{dom} \oplus_{A_2^{\mathbf{S}}}$. Then the two elements $[(A,a)]$ and $[(B,b)]$ of L are compatible with each other and hence by the assumption of the theorem they are even strongly compatible with each other. Hence there exist $C \in \mathbf{O}$

and $(c,d) \in \text{dom} \oplus_{A_1}$ such that $([(A,a)],[(B,b)]) = ([(C,c)],[(C,d)])$. Because of Lemma 2.2 we have $[(C,c)] \oplus [(C,d)] = [(C,c \oplus d)]$. The latter element obviously belongs to the set L. This shows $[(A,a)] \oplus [(B,b)] \in L$.

(ii) Because of Lemma 2.2 we have $(\mu_A(a),\mu_A(b)) \in \text{dom} \oplus_{A_2^{\mathbf{S}}}$ and $\mu_A(a \oplus b) = \mu_A(a) \oplus \mu_A(b)$ for all $A \in \mathbf{O}$ and all $(a,b) \in \text{dom} \oplus_{A_1}$.

(iii) Let α be a fixed element of $\mathbf{S}$. First we have to show that m_α is a well-defined mapping from L to A_2. For this purpose let $A,B \in \mathbf{O}$ and $a,b \in A_1$ and assume $[(A,a)] = [(B,b)]$. Then

$$m_\alpha([(A,a)]) = p_{A,\alpha}(a) = [(A,a)](\alpha) = [(B,b)](\alpha) = p_{B,\alpha}(b) = m_\alpha([(B,b)]).$$

Hence m_α is well-defined. Since $p_{A,\alpha}$ is a homomorphism from A_1 to A_2 for every $A \in \mathbf{O}$, we have $(m_\alpha([(A,a)]), m_\alpha([(A,b)])) \in \text{dom} \oplus_{A_2}$ and $m_\alpha([(A,a \oplus b)]) = m_\alpha([(A,a)]) \oplus m_\alpha([(A,b)])$ for all $A \in \mathbf{O}$ and $(a,b) \in \text{dom} \oplus_{A_1}$. Application of Lemma 2.2 and of the assumption of the theorem completes the proof of (iii).

(iv) For all $A \in \mathbf{O}$, $\alpha \in \mathbf{S}$ and $a \in A_1$ we have

$$(m_\alpha \circ \mu_A)(a) = m_\alpha(\mu_A(a)) = m_\alpha([(A,a)]) = p_{A,\alpha}(a). \qquad \square$$

DEFINITION 2.8. The partial algebra L whose base set was defined in Definition 2.1 will be called the *logic induced by the family* $(p_{A,\alpha})_{A\in\mathbf{O},\alpha\in\mathbf{S}}$ *of homomorphisms from* A_1 *to* A_2.

We can state and prove also an inverted version of Theorem 2.7. But first we introduce two additional definitions.

DEFINITION 2.9. A family $(\mu_A)_{A\in\mathbf{O}}$ of homomorphisms from A_1 to L is said to be *strongly surjective* if it is surjective and for every $(a,b) \in \text{dom} \oplus_L$ there exist $A \in \mathbf{O}$ and $(c,d) \in \text{dom} \oplus_{A_1}$ such that $\mu_A(c) = a$ and $\mu_A(d) = b$.

DEFINITION 2.10. A family $(m_\alpha)_{\alpha\in\mathbf{S}}$ of homomorphisms from L to A_2 is said to be *separating* if $a,b \in L$ and $a \neq b$ together imply the existence of an $\alpha \in \mathbf{S}$ such that $m_\alpha(a) \neq m_\alpha(b)$ and it is said to be *full* if $a,b \in L$ and $(m_\alpha(a), m_\alpha(b)) \in \text{dom} \oplus_{A_2}$ for all $\alpha \in \mathbf{S}$ together imply $(a,b) \in \text{dom} \oplus_L$.

Now we have the following result:

LEMMA 2.11. *Let* $(m_\alpha)_{\alpha\in\mathbf{S}}$ *be a separating and full family of homomorphisms from* L *to* A_2. *Put* $\bar{L} := \{\bar{a} \mid a \in L\}$ *where for every* $a \in L$ $\bar{a}$ *denotes the mapping from* $\mathbf{S}$ *to* A_2 *defined by*

$$\bar{a}(\alpha) := m_\alpha(a)$$

for all $\alpha \in \mathbf{S}$. *Then* $\bar{L}$ *is a subalgebra of* $A_2^{\mathbf{S}}$, $\bar{L} \cong L$ *and the mapping* f *from* L *to* $\bar{L}$ *defined by*

$$f(a) := \bar{a}$$

for all $a \in L$ is an isomorphism from L to $\bar{L}$.

Proof. From the assumptions there follow (i) – (iii):

(i) $(\bar{a}, \bar{b}) \in \operatorname{dom} \oplus_{A_2^{\mathbf{S}}}$ and $\overline{a \oplus b} = \bar{a} \oplus \bar{b}$ for all $(a, b) \in \operatorname{dom} \oplus_L$.

(ii) f is injective.

(iii) $(a, b) \in \operatorname{dom} \oplus_L$ for every $a, b \in L$ with $(\bar{a}, \bar{b}) \in \operatorname{dom} \oplus_{A_2^{\mathbf{S}}}$.

□

We are now able to prove a theorem inverse to Theorem 2.7:

THEOREM 2.12. *Let $(\mu_A)_{A \in \mathbf{O}}$ be a strongly surjective family of homomorphisms from A_1 to L and $(m_\alpha)_{\alpha \in \mathbf{S}}$ a separating and full family of homomorphisms from L to A_2. Then the doubly indexed family $(p_{A,\alpha})_{A \in \mathbf{O}, \alpha \in \mathbf{S}}$ of homomorphisms from A_1 to A_2 defined by $p_{A,\alpha} := m_\alpha \circ \mu_A$ for all $A \in \mathbf{O}$ and $\alpha \in \mathbf{S}$ satisfies the assumptions of Theorem 2.7 (i. e., this family is compatible) and the logic induced by this family (which is denoted here by $\tilde{L}$) is isomorphic to L. Moreover, we have $L \cong \bar{L} = \tilde{L} \subseteq A_2^{\mathbf{S}}$.*

Proof. We show that the family $(p_{A,\alpha})_{A \in \mathbf{O}, \alpha \in \mathbf{S}}$ is compatible. Let $[(A, a)]$ and $[(B, b)]$ be two mutually compatible elements of $\tilde{L}$. Then $([(A, a)], [(B, b)]) \in \operatorname{dom} \oplus_{A_2^{\mathbf{S}}}$. This means that for every $\alpha \in \mathbf{S}$ $(m_\alpha(\mu_A(a)), m_\alpha(\mu_B(b))) \in \operatorname{dom} \oplus_{A_2}$. Since $(m_\alpha)_{\alpha \in \mathbf{S}}$ is full this implies that $(\mu_A(a), \mu_B(b)) \in \operatorname{dom} \oplus_L$. Since $(\mu_A)_{A \in \mathbf{O}}$ is strongly surjective, there exist $C \in \mathbf{O}$ and $(c, d) \in \operatorname{dom} \oplus_{A_1}$ such that $\mu_C(c) = \mu_A(a)$ and $\mu_C(d) = \mu_B(b)$. Hence $([(A, a)], [(B, b)]) = ([(C, c)], [(C, d)])$ which shows that every pair of mutually compatible elements of L is also strongly compatible. Hence the family $(p_{A,\alpha})_{A \in \mathbf{O}, \alpha \in \mathbf{S}}$ is compatible and satisfies the assumption of Theorem 2.7. Since $(\mu_A)_{A \in \mathbf{O}}$ is surjective, the sets $\bar{L}$ and $\tilde{L}$ coincide and hence we have $L \cong \bar{L} = \tilde{L} \subseteq A_2^{\mathbf{S}}$. □

We now return to the example of the spectral theorem in a complex separable Hilbert space H mentioned in the introduction to show that the family $(\mu_A)_{A \in \mathbf{O}}$ of all $L(H)$-valued measures or spectral measures on $\mathbf{B}(\mathbf{R})$ and the family $(m_\alpha)_{\alpha \in \mathbf{S}}$ of all pure probability measures on $L(H)$ satisfy the assumptions of Theorem 2.12. First observe that the probability measures m_α, $\alpha \in \mathbf{S}$, can also be indexed by one-dimensional subspaces or by unit vectors determining these subspaces. It is well-known by Gleason's Theorem that in case of $\dim H \geq 3$ every pure probability measure on $L(H)$ can be represented in the form m_u with $u \in S^1$, where $m_u(P) := (Pu, u)$ for all $P \in L(H)$. (Here and in the following S^1 denotes the set of all unit vectors of H.) So we have $\{m_\alpha \mid \alpha \in \mathbf{S}\} = \{m_u \mid u \in S^1\}$. Clearly, we can interpret these probability measures as homomorphisms from $(L(H), \oplus, ', \{0\})$ to $([0, 1], \oplus, ', 0)$.

The family of all such homomorphisms is separating and full. In fact, if $P, Q \in L(H)$ and $P \neq Q$ then there exists a unit vector $u \in S^1$ such that $(Pu, u) \neq (Qu, u)$. (Projections are uniquely determined by the corresponding quadratic forms.) If $P, Q \in L(H)$ and $((Pu, u), (Qu, u)) \in \text{dom} \oplus_{[0,1]}$ for all $u \in S^1$ then $(Pu, u) \leq 1 - (Qu, u) = (Q^{\perp}u, u)$ for all $u \in S^1$. This implies $P \leq Q^{\perp}$. (The partial order of projections coincides with that of the corresponding quadratic forms.) Hence $P \perp Q$ and $(P, Q) \in \text{dom} \oplus_{L(H)}$. Therefore the family $(m_u)_{u \in S^1}$ is also full.

The family $(\mu_A)_{A \in \mathbf{O}}$ of all homomorphisms from $\mathbf{B}(\mathbf{R})$ to $L(H)$ is strongly surjective as can be seen by the following consideration: Let $(P_1, P_2) \in \text{dom} \oplus_{L(H)}$. Then $P_1 \perp P_2$. Hence we can define an $L(H)$-valued measure μ on $\mathbf{B}(\mathbf{R})$ in the following way:

$$\mu(E) := \sum_{i \in E \cap \{1,2,3\}} P_i$$

for all $E \in \mathbf{B}(\mathbf{R})$ where $P_3 := I - P_1 - P_2$ and where I denotes the identital projection on H. This $L(H)$-valued measure belongs to $\{\mu_A \mid A \in \mathbf{O}\}$ and consequently there exists an $A \in \mathbf{O}$ such that $\mu = \mu_A$. Since $\mu_A(\{1\}) = P_1$ and $\mu_A(\{2\}) = P_2$, $(\mu_A)_{A \in \mathbf{O}}$ is strongly surjective. (That this family is also surjective can be seen by taking $P_2 = O$ where O denotes the projection from H onto the subspace $\{0\}$ of H).

Hence the families $(\mu_A)_{A \in \mathbf{O}}$ and $(m_\alpha)_{\alpha \in \mathbf{S}}$ satisfy the assumptions of Theorem 2.12 and consequently the family $(m_\alpha \circ \mu_A)_{A \in \mathbf{O}, \alpha \in \mathbf{S}}$ is compatible. So Theorem 2.7 may be applied. In this way we obtain the inverse of the spectral theorem as suggested in the introduction.

We can give another example of application of Theorem 2.12 based on the theory of Boolean algebras. Let – as before – $(\mathbf{B}(\mathbf{R}), \oplus, ', \emptyset)$ denote the orthomodular algebra of all Borel sets of the real line and let L denote an arbitrary Boolean algebra (considered also as an orthoalgebra). Further, let $(\mu_A)_{A \in \mathbf{O}}$ denote the family of all homomorphisms from $\mathbf{B}(\mathbf{R})$ to L. It is clear that this family is strongly surjective (by an analogous argument as in the Hilbert space case).

Let $(m_\alpha)_{\alpha \in \mathbf{S}}$ be the family of all homomorphisms from L to the two-element Boolean algebra $2^1 = \{0, 1\}$ (they can also be interpreted as two-valued measures on L). The family $(m_\alpha)_{\alpha \in \mathbf{S}}$ is separating and full. This can be seen as follows: By the Stone representation theorem L is isomorphic to a Boolean algebra of subsets of $\mathbf{S}$ (with set-theoretical operations). So we have

$$L \cong \bar{L} \subseteq \{0, 1\}^{\mathbf{S}}.$$

Let us denote the isomorphism from L to $\bar{L}$ by $a \mapsto \bar{a}$. Then, for all $x \in L$

$\bar{x} = \{\alpha \in \mathbf{S} \mid m_\alpha(x) = 1\}$. If $a, b \in L$ and $(m_\alpha(a), m_\alpha(b)) \in \operatorname{dom} \oplus_{\{0,1\}}$ for all $\alpha \in \mathbf{S}$ then $(m_\alpha(a), m_\alpha(b)) \in \{(0,0),(0,1),(1,0)\}$ for all $\alpha \in \mathbf{S}$ and hence $(\bar{a}, \bar{b}) \in \operatorname{dom} \oplus_{\bar{L}}$ from which $(a,b) \in \operatorname{dom} \oplus_L$ follows. Hence $(m_\alpha)_{\alpha \in \mathbf{S}}$ is full. It is also separating because if $a \neq b$ then also $\bar{a} \neq \bar{b}$ and there exists a two-valued measure m_α on $\bar{L}$ such that $m_\alpha(\bar{a}) \neq m_\alpha(\bar{b})$. This follows from the fact that if the subsets $\bar{a}$ and $\bar{b}$ of $\mathbf{S}$ are not equal then without loss of generality there exists some $x_0 \in \bar{a} \setminus \bar{b}$. We can then define a two-valued measure $\bar{m}_{x_0}$ on $\bar{L}$ by putting $\bar{m}_{x_0}(\bar{c}) := 1$ or 0 depending on whether $x_0 \in \bar{c}$ or $x_0 \notin \bar{c}$, respectively ($\bar{c} \in \bar{L}$). Of course, the homomorphism m_{x_0} from L to $\{0,1\}$ corresponding to the homomorphism $\bar{m}_{x_0}$ from $\bar{L}$ to $\{0,1\}$ belongs to the set $\{m_\alpha \mid \alpha \in \mathbf{S}\}$. Hence $(m_\alpha)_{\alpha \in \mathbf{S}}$ is separating.

So we may apply Theorem 2.12 since all the assumptions are satisfied: $(\mu_A)_{A \in \mathbf{O}}$ is strongly surjective and $(m_\alpha)_{\alpha \in \mathbf{S}}$ is full and separating. Hence by Theorem 2.12 the doubly indexed family $(m_\alpha \circ \mu_A)_{A \in \mathbf{O}, \alpha \in \mathbf{S}}$ of homomorphisms from $\mathbf{B}(\mathbf{R})$ to $\{0,1\}$ (which are also (finitely additive) two-valued probability measures on $\mathbf{B}(\mathbf{R})$) is compatible and the logic induced by this family is isomorphic to L.

Hence we see that every Boolean algebra can be interpreted as the logic induced by a family of probability measures on $\mathbf{B}(\mathbf{R})$ (which is the case of classical mechanics).

We also see that the Stone representation theorem plays the same role in classical mechanics as Gleason's theorem does in quantum mechanics: Both theorems allow us to determine concretely all pure states (pure probability measures) on the logic L of the system. Note only that in classical mechanics this applies also to finitely additive probability measures whereas in quantum mechanics only σ-additive probability measures are involved.

3. An application to orthomodular posets (quantum logics)

We shall apply Theorem 2.7 to the special case where A_1 is the algebra $\mathbf{B}(\mathbf{R})$ of all Borel sets of the real line and A_2 is the interval $[0,1]$. In this case $(\mathbf{B}(\mathbf{R}), \oplus, ', \emptyset)$ is an orthomodular algebra (shortly, an OMA). Also $([0,1], \oplus, ', 0)$ can be considered as a partial algebra with the operations being defined as it was done at the end of Section 1. Let us recall that by an *orthoalgebra* we understand an algebraic system $(A, \oplus, ', 0)$ of type $(2,1,0)$ with a partial binary operation $\oplus$ and a total unary operation $'$ satisfying the following axioms (cf. e. g. [5]):

(OA1) If one side of the commutativity law is defined then so is the other and both are equal (commutativity law).

(OA2) If one side of the associativity law is defined then so is the other and both are equal (associativity law).

(OA3) For each $a \in A$ the element a' is the unique element x of A such that both $a \oplus x$ is defined and $a \oplus x = 0'$ (orthocomplementation law).

(OA4) If $a \in L$ and $a \oplus a$ is defined then $a = 0$ (consistency law).

We shall assume that A is non-degenerate, i. e. that $0' \neq 0$. It is easy to see that any orthoalgebra becomes an orthoposet by defining $a \leq b$ iff there exists an element c of L with $a \oplus c = b$ $(a, b \in A)$. The elements a, b of A are said to be orthogonal to each other, in signs $a \perp b$, if $a \oplus b$ is defined.

From (OA1) to (OA4) it follows that an orthoalgebra $(A, \oplus, ', 0)$ is *orthomodular*. This means that the following law holds:

(OA5) If $a, b \in A$ and $a \oplus b'$ is defined then $a \oplus (a \oplus b')'$ is defined and $a \oplus (a \oplus b')' = b$ (orthomodularity law).

For a theory of orthoalgebras see e. g. [5], for a theory of orthomodular algebras see e. g. [2]. For orthoalgebras we also have

$$a_1 \oplus \ldots \oplus a_n = a_1 \vee \ldots \vee a_n,$$

where $\vee$ denotes the supremum with respect to the partial order $\leq$ on A which was defined above. If this property holds also for an infinite sequence of mutually orthogonal elements then we shall say that $(A, \oplus, ', 0)$ is a *σ-orthomodular algebra* (shortly, a σ-OMA). This property is necessary if we want to define a probability measure on an OMA, since probability measures are assumed to be σ-additive.

In order to apply Theorem 2.7 to probability measures, we have to strengthen the assumptions of this theorem. Namely, we will say that a sequence $[(A_1, a_1)], [(A_2, a_2)], [(A_3, a_3)], \ldots$ of elements of L is *σ-compatible* if for every pair (i, j) of distinct positive integers $([(A_i, a_i)], [(A_j, a_j)]) \in \operatorname{dom} \oplus_{A_2^{\mathbf{S}}}$. Similarly, we will call a sequence $[(A_1, a_1)], [(A_2, a_2)], [(A_3, a_3)], \ldots$ of elements of L *strongly σ-compatible* if there exist $B \in \mathbf{O}$ and $b_1, b_2, b_3, \ldots \in A_1$ such that both $[(A_i, a_i)] = [(B, b_i)]$ for all positive integers i and $(b_i, b_j) \in \operatorname{dom} \oplus_{A_1}$ for all pairs (i, j) of distinct positive integers. In an analogous way as before, it can be proved that every strongly σ-compatible sequence of elements of L is σ-compatible. The family $(p_{A,\alpha})_{A \in \mathbf{O}, \alpha \in \mathbf{S}}$ of homomorphisms from A_1 to A_2 is said to be *σ-compatible* if every σ-compatible sequence of elements of L is even strongly σ-compatible.

We can now reformulate Theorem 2.7 as follows:

THEOREM 3.1. *Let* $(p_{A,\alpha})_{A\in \mathbf{O},\alpha\in \mathbf{S}}$ *be a doubly indexed* σ*-compatible family of probability measures on the* σ*-orthomodular algebra* $(\mathbf{B}(\mathbf{R}),\oplus,',\emptyset)$ *of all Borel sets of the real line. Then in addition to* (i) – (iv) *of Theorem* 2.7 (v) *and* (vi) *hold*:

(v) $(L,\oplus,',0)$ *is a* σ*-orthomodular subalgebra of* $[0,1]^{\mathbf{S}}$.

(vi) *For every* $A\in \mathbf{O}$ μ_A *is a* σ*-homomorphism from* $\mathbf{B}(\mathbf{R})$ *to* L *and for every* $\alpha\in \mathbf{S}$ m_α *is a probability measure on* L.

Proof. In order to prove (v) we only have to show that L is orthomodular. To this aim we may use Theorem 3 of [2] (based on [8]) which characterizes orthomodularity within orthoalgebras $L\subseteq [0,1]^{\mathbf{S}}$ of so-called numerical functions. This theorem states that $(L,\oplus,',0)$ is an OMA iff the following conditions hold:

1° $0\in L$

2° If $f\in L$ then $f':=1-f\in L$.

3° If $f_1,f_2,f_3\in L$ and $f_i+f_j\leq 1$ for $i\neq j$ then $f_1+f_2+f_3\in L$.

In our case, the conditions 1° and 2° clearly hold. In order to show that 3° holds, assume that $f_i=[(A_i,E_i)]$ for $i=1,2,3$. Then the assumption $f_i+f_j\leq 1$ for $i\neq j$ means that $[(A_i,E_i)]\oplus[(A_j,E_j)]\in L$ for $i\neq j$, i. e., the sequence $f_1,f_2,f_3,0,0,0,\ldots$ is σ-compatible. Hence, by the assumptions of the theorem, this sequence is even strongly σ-compatible, i. e., there exist $B\in \mathbf{O}$ and $F_1,F_2,F_3\in \mathbf{B}(\mathbf{R})$ such that $F_i\cap F_j=\emptyset$ for $i\neq j$ and $[(A_i,E_i)]=[(B,F_i)]$ for $i=1,2,3$. (For $i>3$ one may take $F_i=\emptyset$.) But then we obtain:

$$\begin{aligned} f_1+f_2+f_3 &= [(A_1,E_1)]\oplus[(A_2,E_2)]\oplus[(A_3,E_3)]\\ &= [(B,F_1)]\oplus[(B,F_2)]\oplus[(B,F_3)]\\ &= [(B,F_1\cup F_2\cup F_3)]\leq 1 \end{aligned}$$

which shows that condition 3° holds. This means that L is orthomodular. It is now obvious that L is even a σ-OMA and that (vi) holds. □

DEFINITION 3.2. The σ-orthomodular algebra $(L,\oplus,',0)$ (or the σ-orthocomplemented poset $(L,\leq,',0)$) described in Theorem 3.1 will be called the *logic induced by the family* $(p_{A,\alpha})_{A\in \mathbf{O},\alpha\in \mathbf{S}}$ *of probability measures on* $\mathbf{B}(\mathbf{R})$ and will be denoted by $\mathbf{L}(\mathbf{O},\mathbf{S})$.

If we interpret $\mathbf{O}$ and $\mathbf{S}$ as the sets of all observables and all states of a physical system, respectively, then L may be interpreted as the logic of this system. This agrees with the interpretations of the logic of quantum mechanics suggested by G. W. Mackey in [7].

In our case L is a set of mappings from $\mathbf{S}$ to $[0,1]$ and hence L may also be called a numerical logic.

We may interpret L also in another way:

Put $\mathbf{E} := \mathbf{O} \times \mathbf{B}(\mathbf{R})$. The elements of $\mathbf{E}$ can be called experimental propositions. We define an equivalence relation $\sim$ on $\mathbf{E}$ as follows:

$$(A,E) \sim (B,F) \text{ iff for all } \alpha \in \mathbf{S}\ p_{A,\alpha}(E) = p_{B,\alpha}(F)$$

$((A,E),(B,F) \in \mathbf{E})$. Put $L_0 := \mathbf{E}/\sim$ and for every $(A,E) \in \mathbf{E}$ let $|(A,E)|$ denote the equivalence class of (A,E) with respect to $\sim$. Then the correspondence

$$|(A,E)| \longleftrightarrow [(A,E)]$$

is a bijection between L_0 and L. It induces the structure of an OMA on L_0 and by this procedure $(L_0, \leq, ', 0)$ becomes an OMA (or a σ-orthoposet).

Since (via this correspondence) L is isomorphic to L_0, we may now interpret the elements of L as equivalence classes of experimental propositions which gives L an immediate experimental (i. e. physical) meaning.

We see that in our construction the logic of a physical system is determined by the results of all measurements of all observables in all states. (We obtain then the doubly indexed family $(p_{A,\alpha})_{A\in\mathbf{O},\alpha\in\mathbf{S}}$ of probability measures on $\mathbf{B}(\mathbf{R})$.) This means that the logic of a physical system can be experimentically determined. In quantum mechanics, the mathematical model for L_0 is the OML of all orthogonal projections from a complex separable Hilbert space H onto closed linear subspaces of H which is isomorphic to the OML $L(H)$ of all closed linear subspaces of H.

We see from the above considerations that the most important structure in axiomatic foundations of quantum mechanics is the logic L_0 which is a σ-orthocomplemented poset. The structure of L_0 can be equivalently defined as an orthomodular partial algebra. There are several possibilities to give L_0 a concrete interpretation:

L_0 may be assumed to be a Boolean algebra (classical mechanics) or the OML $L(H)$ of all closed linear subspaces of a complex separable Hilbert space H (quantum mechanics). So the structure of L_0 can be generally interpreted as a generalization of a Boolean algebra. There arises the question how to define L_0 in terms of a possibly minimal number of fundamental operations which have a more evident physical interpretation than the ones used usually. The lattice operations $\vee$ and $\wedge$ are not suitable since only the operation $\wedge$ has a physical interpretation (the intersection of subspaces) whereas the interpretation of the operation $\vee$ requires the use of some topological notions. (For $M, N \in L(H)$ we have $M \vee N = \overline{M+N}$ where $\overline{M+N}$

denotes the topological closure of the linear subspace $M + N$ of H.) Hence we may look for more suitable operations. It turns out that the symmetric difference $\triangle$ in a Boolean algebra B is a good starting point for this task since by means of this operation B can be made into a pseudometric space by defining

$$d(a, b) := p(a \triangle b)$$

for all $a, b \in B$ where p is a (subadditive) finitely additive measure on B. So the next section will be devoted to the axiomatization of OMLs and – more general – of bounded lattices with an involutory antiautomorphism by means of the operations $\triangle$ (instead of $\vee$), $\wedge$, 0 and 1 where the symmetric difference $\triangle$ – which has a well-defined meaning in Boolean algebras – has first to be generalized in a suitable way. It can be proved (cf. [3]) that for this generalized symmetric difference $\triangle$ on an arbitrary OML L the following are equivalent:

(i) $\triangle$ is associative.
(ii) $\wedge$ is distributive with respect to $\triangle$.
(iii) L is a Boolean algebra.

This shows that in an arbitrary OML L, if we replace $\triangle$ by $+$ and $\wedge$ by $\cdot$, we obtain a ring only in the case where L is a Boolean algebra. In rings, the operation $+$ is usually assumed to be commutative and associative whereas the operation $\cdot$ may also be non-associative (such structures are usually called non-associative rings). In our case the operation $\cdot$ (corresponding to the intersection $\wedge$) is commutative and associative whereas the operation $+$ (corresponding to the symmetric difference $\triangle$) is commutative but in general not associative. So it seems to be useful to develop the axiomatic foundations for the theory of such generalizations of rings.

4. Boolean quasirings

In order to illustrate the position of Boolean quasirings within a wider class of algebras we start by giving the following definition:

DEFINITION 4.1. An algebra $(R, +, \cdot)$ of type $(2, 2)$ is called a *generalized Boolean quasiring* (GBQR) iff there exist $0, 1 \in R$ such that for all $x, y, z \in R$ the following laws hold:

(1) $x + y = y + x$
(2) $x + 0 = x$
(3) $(xy)z = x(yz)$
(4) $xy = yx$
(5) $xx = x$

(6) $x0 = 0$
(7) $x1 = x$
(8) $1 + (1 + xy)(1 + y) = y$.

DEFINITION 4.2. For a GBQR $R = (R, +, \cdot)$ we define

$$x \vee y := 1 + (1 + x)(1 + y)$$
$$x \wedge y := xy$$
$$x' := 1 + x$$
$$\mathbf{L}(R) := (R, \vee, \wedge, ').$$

LEMMA 4.3. *$\mathbf{L}(R)$ is a bounded lattice with an involutory antiautomorphism.*

Proof. $(R, \wedge)$ is a semilattice with least element 0 and greatest element 1. If in equation (8) we put $y = x$ we obtain $1 + (1 + x) = x$ for all $x \in R$, hence $x'' = x$.

$x \leq y$, i. e. $xy = x$, implies $x' \geq y'$ for $x, y \in R$, because again by (8) $1 + x'y' = 1 + (xy)'y' = y$, wherefrom we conclude $x'y' = y'$. Therefore $y' \leq x'$.

Because of the properties $x \leq y \Leftrightarrow x' \geq y'$ and $x'' = x$ the element $(x' \wedge y')' = 1 + (1 + x)(1 + y)$ is the least upper bound of x and y, therefore $\mathbf{L}(R)$ is a lattice. □

DEFINITION 4.4. Let $L = (L, \vee, \wedge, ')$ be a bounded lattice with an involutory antiautomorphism $'$. (The least and greatest element of L will be denoted by 0 and 1, respectively.) We define

$$x + y := (x \vee y) \wedge (x \wedge y)'$$
$$xy := x \wedge y$$
$$\mathbf{R}(L) := (L, +, \cdot).$$

LEMMA 4.5. *$\mathbf{R}(L)$ is a GBQR in which the following equation holds:*

(9) $$(1 + (1 + x)(1 + y))(1 + xy) = x + y$$

for all $x, y \in L$.

Proof. It is obvious that the defining laws (1) – (7) of GBQRs are satisfied. $1 + x = 1 \wedge x' = x'$. Condition (8) we obtain in the following way:

$$1 + (1 + xy)(1 + y) = [(xy)'y']' = (x \wedge y) \vee y = y$$

and (9) we get by

$$(1 + (1 + x)(1 + y))(1 + xy) = [x'y']'(xy)' = (x \vee y) \wedge (x \wedge y)' = x + y. \quad □$$

Examples of GBQRs

1) $R_{[0,1]} := ([0,1], (x,y) \mapsto |x-y|, \min)$, where $[0,1]$ is the real unit interval and min means the binary operation of forming the minimum. Obviously, equations (1) - (7) hold. Equation (8) is satisfied because

$$\begin{aligned} 1 - \min(1 - \min(x,y), 1 - y) &= \max(1 - (1 - \min(x,y)), 1 - (1 - y)) \\ &= \max(\min(x,y), y) = y \end{aligned}$$

for all $x, y \in [0,1]$. (9) does not hold in $R_{[0,1]}$ because equation (9) with $y = x$ implies $xx' = x + x$, whereas in $R_{[0,1]}$ we have $x(x+1) = \min(x, 1-x)$ and $x + x = |x - x| = 0$.

2) As in Section 1 let $\mathbf{S}$ denote a set (of states) and let $R^{\mathbf{S}}_{[0,1]}$ be the algebra of all functions from $\mathbf{S}$ to $R_{[0,1]}$ endowed with the operations $+$ and $\cdot$ defined coordinatewise (direct product in the sense of universal algebra). Then every substructure of $R^{\mathbf{S}}_{[0,1]}$ is a GBQR. (In the case of these substructures the neutral elements with regard to $+$ and $\cdot$ have not to be the same as in $R^{\mathbf{S}}_{[0,1]}$.)

Homomorphisms of the algebra $(\mathbf{B}(\mathbf{R}), \triangle, \cap)$ of Borel sets to $R^{\mathbf{S}}_{[0,1]}$ can be interpreted in the following way: Considering Borel sets as events, for two events $E, F \in \mathbf{B}(\mathbf{R})$ and a homomorphism h the equation $h(E \triangle F) = |h(E) - h(F)|$ can be thought of as a kind of measure for the distance between E and F. Because of $h(\emptyset) \leq h(E) \leq h(\mathbf{R})$

$$h(E) = h(E \triangle \emptyset) = h(E) - h(\emptyset) \text{ and } h(E') = h(E \triangle \mathbf{R}) = h(\mathbf{R}) - h(E)$$

(where the binary operation $-$ denotes the difference of functions from $\mathbf{S}$ to $[0,1]$). Therefore $h(E)$ and $h(E')$ tell "how far E and E' are away" from the impossible and certain event, respectively. (Of course, the respective distances have to be related to the valuations of $h(\emptyset)$ and $h(\mathbf{R})$.)

3) According to Lemma 4.5 every algebra $\mathbf{R}(L)$ associated with a bounded lattice L having an involutory antiautomorphism gives rise to a GBQR. The underlying lattice L can be interpreted as a certain kind of quantum logic in which the laws $x \vee x' = 1$ and $x \wedge x' = 0$ do not hold in general. If the elements of L are interpreted as questions and x', $x \vee y$ and $x \wedge y$ stand for the negation of x, for "x or y" and for "x and y", respectively, this means that the answer to "x or not x" is not always yes.

On the other hand, because of $'$ not necessarily being an orthocomplementation, L can be regarded as the set-theoretic union of Boolean sublattices of L (with respect to the operations $\vee$, $\wedge$ and $'$; the various subalgebras may have different zeros and ones. (Cf. Fig. 4.1.)) This shows that $\mathbf{R}(L)$ can be the union of homomorphic images of a GBQR that corresponds to a Boolean algebra. This has been required at the beginning of Section 2 ($L = \bigcup_{A \in O} \mu_A(A_1)$).

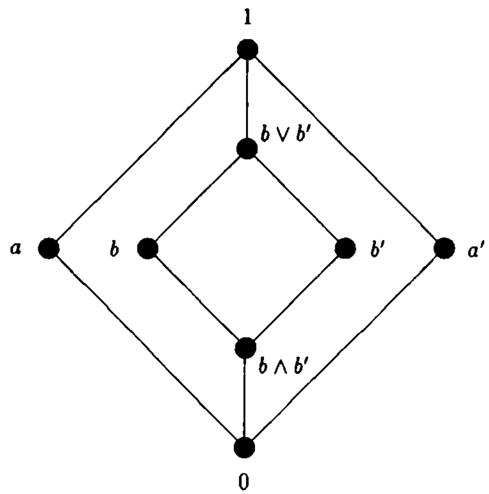

Fig. 4.1

LEMMA 4.6. *For GBQRs R which satisfy equation* (9) *and bounded lattices L with an involutory antiautomorphism we have* $\mathbf{R}(\mathbf{L}(R)) = R$ *and* $\mathbf{L}(\mathbf{R}(L)) = L$.

Proof. Let $\oplus$, $\odot$ be the operations of $\mathbf{R}(\mathbf{L}(R))$.

$$x \oplus y = (x \vee y) \wedge (x \wedge y)' = (1 + (1 + x)(1 + y))(1 + xy) = x + y \quad \text{by (9)}$$
$$x \odot y = x \wedge y = xy.$$

On the other hand, let $\bigvee$, $\bigwedge$, * be the operations of $\mathbf{L}(\mathbf{R}(L))$.

$$x \bigvee y = 1 + (1 + x)(1 + y) = (x' \wedge y')' = x \vee y$$
$$x \bigwedge y = xy = x \wedge y$$
$$x^* = 1 + x = (1 \vee x) \wedge (1 \wedge x)' = x'. \qquad \square$$

DEFINITION 4.7. A GBQR R is said to have *characteristic 2*, if $x + x = 0$ for all $x \in R$.

DEFINITION 4.8. A GBQR of characteristic 2 satisfying equation (9) is called a *Boolean quasiring* (BQR).

THEOREM 4.9. *There is a one-to-one correspondence between Boolean quasirings and ortholattices.*

Proof. For a GBQR R of characteristic 2 from equation (9) and $y = x$ we infer

$$xx' = (1 + (1 + x))(1 + x) = (1 + (1 + x)(1 + x))(1 + xx) = x + x = 0.$$

Again by equation (9), if we put $y = x'$ we obtain $(1 + x'x)(1 + xx') = x + x'$, wherefrom we conclude $x + x' = 1$. Therefore $\mathbf{L}(R)$ has the property $x \wedge x' = xx' = 0$ and

$$x \vee x' = 1 + (1 + x)(1 + x') = 1 + x'x = 1 + 0 = 1,$$

which implies that $\mathbf{L}(R)$ is an ortholattice.

Conversely, if L is an ortholattice, $\mathbf{R}(L)$ is a GBQR which satisfies (9) by Lemma 4.5. Because of $x + x = (x \vee x) \wedge (x \wedge x)' = x \wedge x' = 0$ $\mathbf{R}(L)$ is of characteristic 2. □

Remark. The identities $x + x' = 1$ and $xx' = 0$ suggest to consider BQRs as algebras $(R, +, \cdot, ', 0, 1)$ of type $(2, 2, 1, 0, 0)$.

THEOREM 4.10. *There is a one-to-one correspondence between Boolean quasirings which satisfy the equation*

$$(1 + xy) + (x + xy) = 1 + x \tag{10}$$

and orthomodular lattices.

Proof. Equation (9) with $y = xy$ implies $x(1 + xy) = x + xy$. Therefore, if L is an OML, we deduce in $\mathbf{R}(L)$ using that $(x \wedge y) \vee (x \wedge (x \wedge y)') = x$ in OMLs:

$$\begin{aligned}(1 + xy) + (x + xy) &= (1 + xy) + x(1 + xy) = (1 + xy)(1 + x(1 + xy)) \\ &= [(x \wedge y) \vee (x \wedge (x \wedge y)')]' = x' = 1 + x.\end{aligned}$$

Conversely, if R is a BQR satisfying equation (10) and $x \leq y$ in $\mathbf{L}(R)$, then

$$\begin{aligned}x \vee (y \wedge x') &= 1 + (1 + x)(1 + y(1 + x)) = 1 + ((1 + x) + y(1 + x)) \\ &= 1 + ((1 + xy) + y(1 + xy)) = 1 + ((1 + xy) + (y + xy)) \\ &= 1 + (1 + y) = y.\end{aligned}$$

Therefore in $\mathbf{L}(R)$ we obtain: $x \leq y$ implies $x \vee (y \wedge x') = y$. □

THEOREM 4.11. *A Boolean quasiring is a Boolean ring iff it satisfies*

$$x(1 + y) = x + xy. \tag{11}$$

Proof. Since equation (11) is valid for every Boolean ring because of the distributive law, we have only to show that a BQRR which fulfils equation (11) is a Boolean ring.

As usual we call two elements x, y of R orthogonal to each other – in symbols $x \perp y$ – if $x \leq y'$, i. e. if $xy' = x$. Because of $xy = xy'y = xyy' = x0 = 0$ in case of $x \perp y$ the relation $x \perp y$ implies $xy = 0$, but the converse is not true for every BQR. However, if equation (11) holds, $xy = 0$ yields $xy' = x$, hence $x \perp y$. From this it follows that $\mathbf{L}(R)$ is pseudocomplemented with regard to $'$. (For the definition of a pseudocomplemented lattice see [4].) It is well-known (cf. [4]) that for a pseudocomplemented lattice L the set $S := \{x' \mid x \in L\}$ is a Boolean algebra with respect to the operations $\wedge$ (from L) and $x \bigvee y := (x' \wedge y')'$ and that for such a lattice $x \in S$ iff $x'' = x$. In case of $L = \mathbf{L}(R)$ we have $x \bigvee y = x \vee y$ and $S = \mathbf{L}(R)$. Therefore we obtain that $R = \mathbf{R}(\mathbf{L}(R))$ is a Boolean ring. □

Remark. In our proof we have made use of the theory of pseudocomplemented meet-semilattices. One could also proof Theorem 4.11 by taking into account that for a BQR equation (11) is equivalent to $x \wedge y' = x \wedge (x \wedge y)'$ within the corresponding OML. It is well-known (cf. e. g. [6]) that the latter equation is valid if and only if x and y commute. If this is the case for all elements x and y of the OML this means that the OML is a Boolean algebra.

As far as applications to quantum mechanics are concerned, if we take into account BQRs or GBQRs instead of lattices when dealing with observables, the mapping μ_A occurring in the formula $p_{A,\alpha} = m_\alpha \circ \mu_A$ (see Section 1) can be interpreted as a homomorphism within the variety of BQRs and GBQRs, respectively, whereas the mapping m_α can only be considered as a (full) homomorphism with respect to operations of type 1 and also, if they do occur, of type 0. If we consider only the 2-place operation $+$, m_α is only a homomorphism with respect to orthogonal elements, the very assumption we have made in Section 1.

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