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ON THE ASYMPTOTIC BEHAVIOUR OF SOLUTIONS OF DIFFERENCE EQUATIONS

For the difference equation

$$\Delta y_n = \sum_{i=0}^{\infty} a_n^i y_{n+i}$$

sufficient conditions for the existence of an asymptotically constant solution are presented.

We denote: N-the set of natural numbers, R-the set of real numbers, C-the set of complex numbers and we write $y(n) = y_n$. For each function $y : N \to R$ or $y : N \to C$ the difference operator

$$\Delta y_n = y_{n+1} - y_n, \quad n \in N, \quad \Delta^i y_n = \Delta(\Delta^{i-1} y_n), \quad \text{for } i > 1$$

has been defined.

To simplify formulae we use the conventional assumption that the void sum is equal to zero and the void product is equal to one, that is

$$\sum_{j=n}^{k} y_j := 0, \quad \prod_{j=n}^{k} y_j := 1 \quad \text{for } k < n.$$

Instead of $\lim_{n\to\infty} y_n = K$ we shall write $y_n = K + o(1)$.

THEOREM. *Let $a^i : N \to R$, $a_n^0 \neq -1$ for $n \in N$, $\sup_{n\geq m}[\max_i |a_n^i|] > 0$ for each $m \in N$ and*

$$\sum_{i=0}^{\infty} \sum_{j=1}^{\infty} |a_j^i| < \infty. \tag{1}$$

Then for each arbitrary constant $K \in R$, there exists a solution $y : N \to R$ of the equation

$$\Delta y_n = \sum_{i=0}^{\infty} a_n^i y_{n+i} \tag{2}$$

such that

$$y_n = K + o(1). \tag{3}$$

Proof. If $K = 0$ then the zero sequence y (i.e. $y_n = 0$ for $n \in N$) is the solution of equation (2) and the theorem holds. From now on let $K > 0$ (for $K < 0$ the proof runs analogically). It means that there exists a positive constant ε such that $K - \varepsilon > 0$. Let $K_1 = K + \varepsilon, I = [K - \varepsilon, K + \varepsilon]$, and

$$\alpha_n = K_1 \sum_{i=0}^{\infty} \sum_{j=n}^{\infty} |a_j^i|, \qquad n \in N. \tag{4}$$

From assumption (1) it follows that there exists $n_2 \in N$ such that $\alpha_n \leq \varepsilon$ for any $n > n_2$. Let $n_1 = \min\{n_2 \in N : \alpha_n \geq \varepsilon\}$. Further let l_∞ be a Banach space of bounded sequences $x = \{\xi_n\}_{n=1}^{\infty}$ with the norm $\|x\| = \sup_{n \geq 1} |\xi_n|$. Denote by $T = T(K, n_1)$ the set of sequences $x = \{\xi_n\}_{n=1}^{\infty} \in l_\infty$ such that

$$\begin{cases} \xi_n = K & \text{for } n = 1, 2, \ldots, n_1 - 1 \\ \xi_n \in I_n & \text{for } n \geq n_1 \end{cases}$$

where $I_n = [K - \alpha_n, K + \alpha_n]$ and observe that $I_n \subseteq I$ for each $n \geq n_1$.

The closedness of the set T will be shown by the examination that the set $l_\infty \setminus T$ is open. Let's take a sequence $h = \{h_i\}_{i=1}^{\infty} \notin T$. Then $h_k \neq K$ for an index $k < n_1$ or $h_k \notin I_k$ for an index $k \geq n_1$. Thus one can find $\varepsilon_1 > 0$ such that $K \notin (h_k - \varepsilon_1, h_k + \varepsilon_1)$ or $I_k \cap (h_k - \varepsilon_1, h_k + \varepsilon_1) = \emptyset$, respectively. Let $B(x, \varepsilon_1)$ denote ε_1-neighbourhood of arbitrarily chosen point x in the space l_∞. We will show that $B(h, \varepsilon_1) \cap T = \emptyset$. Let's take an arbitrary element $h' = \{h_i\}_{i=1}^{\infty} \in B(h, \varepsilon_1)$. Then $h_i' \in (h_i - \varepsilon_1,\ h_i + \varepsilon_1)$ for every $i \in N$. Thus either $h_k' \neq K$ for an index $k < n_1$ or $h_k' \notin I_k$ for an index $k \geq n_1$. It implies that the sequence h' together with its ε_1-neighbourhood is contained in $l_\infty \setminus T$. This proves that the set $l_\infty \setminus T$ is open.

It is easy to check that T is a convex subset of l_∞.

We will prove that T is a compact subset in l_∞. Let's take any $\varepsilon_2 > 0$. If $\operatorname{diam} I_{n_1} < \varepsilon_2$, then $v = \{K, K, \ldots\} \in T$ is the ε_2-net on the set T. Suppose that there exists $n_3 \geq n_1$ such that $\operatorname{diam} I_{n_3} \geq \varepsilon_2$ and $\operatorname{diam} I_{n_3+1} < \varepsilon_2$. We denote $r_i = [\frac{\operatorname{diam} I_{n_1+i-1}}{\varepsilon_2}] + 1$. Then the set $\{K, K, \ldots, K, K - s_1\varepsilon_2, \ldots, K - s_{n_3-n_1+1}\varepsilon_2, K, \ldots\} \in l_\infty$, where s_i takes all values from the set $\{1, 2, \ldots, r_i\}$ for each $i = 1, 2, \ldots, n_3 - n_1 + 1$, is the ε_2-net on the set T. This net is finite, i.e. it consists of finite number of sequences such that their ε_2-neighbourhood covers the considered set.

Next we define the operator $\mathsf{A} = \mathsf{A}(K, n_1) : l_\infty \to l_\infty$ as follows: for an arbitrary $x = \{\xi_n\}_{n=1}^{\infty} \in I_\infty$ put $\mathsf{A}x = y = \{\eta_n\}_{n=1}^{\infty}$, where

$$\eta_n = \begin{cases} K, & \text{for } n = 1, 2, \ldots, n_1 - 1 \\ K - \sum_{i=0}^{\infty} \sum_{j=n}^{\infty} a_j^i \xi_{j+i}, & \text{for } n \geq n_1, \end{cases}$$

We show that operator A tranforms the set T into T. Let us take any $x = \{\xi_n\}_{n=1}^{\infty} \in T$. Using (4) and since $\xi_{j+i} \in I_{j+i} \subset I$ for each $j \geq n_1$, $i = 0, 1, 2, \ldots$, we get the following estimation

$$|\eta_n - K| \leq \sum_{i=0}^{\infty} \sum_{j=n}^{\infty} |a_j^i||\xi_{j+i}| \leq K_1 \sum_{i=0}^{\infty} \sum_{j=n}^{\infty} |a_j^i|.$$

Hence $\eta_n \in I_n$ for each $n \in N$ and therefore the operator A transforms T into T.

Now we show that the operator A is continuous on the set T. Fix arbitrary $\varepsilon_3 > 0$ and let $\delta_3 = \frac{\varepsilon_3}{\alpha}$, where

$$\alpha = \sum_{i=0}^{\infty} \sum_{j=n_1}^{\infty} |a_j^i|.$$

Take two arbitrary elements $x = \{\xi_n\}_{n=1}^{\infty}$ and $z = \{\zeta_n\}_{n=1}^{\infty}$ of the set T such that $\|x - z\| < \delta_3$. By the assumption (1) we see that the series

$$\sum_{i=0}^{\infty} \sum_{j=n_1}^{\infty} a_j^i \xi_{j+i} \quad \text{and} \quad \sum_{i=0}^{\infty} \sum_{j=n_1}^{\infty} a_j^i \zeta_{j+i}$$

are absolutely convergent. So, we have

$$\begin{aligned} \|\mathsf{A}x - \mathsf{A}z\| &= \sup_{n \geq n_1} \left| \left\{ K - \sum_{i=0}^{\infty} \sum_{j=n}^{\infty} a_j^i \xi_{j+i} \right\} - \left\{ K - \sum_{i=0}^{\infty} \sum_{j=n}^{\infty} a_j^i \zeta_{j+i} \right\} \right| \\ &\leq \sup_{n \geq n_1} \sum_{i=0}^{\infty} \sum_{j=n}^{\infty} |a_j^i||\xi_{j+i} - \zeta_{j+i}| \leq \|x - z\| \sum_{i=0}^{\infty} \sum_{j=n}^{\infty} |a_j^i| < \varepsilon_3. \end{aligned}$$

Since ε_3 is arbitrary the latter proves that the operator A is continuous on the set T. Hence by virtue of the Schauder's Fixed Point Theorem the equation $x = \mathsf{A}x$ has a solution in T. Let $w = \{\omega_n\}_{n=1}^{\infty}$ be fixed point of $\mathsf{A}x = x$. Then $w \in T$ can be written in the form

$$w = \{K, K, \ldots, K, \omega_{n_1}, \omega_{n_1+1}, \ldots, \omega_n, \ldots\}$$

and

$$\mathsf{A}w = \left\{ K, K, \ldots, K, K - \sum_{i=0}^{\infty} \sum_{j=n_1}^{\infty} a_j^i \omega_{j+i}, \ldots, K - \sum_{j=0}^{\infty} \sum_{j=n}^{\infty} a_j^i \omega_{j+i}, \ldots \right\}.$$

Hence

(5) $$\omega_n = K - \sum_{i=0}^{\infty} \sum_{j=n}^{\infty} a_j^i \omega_{j+i}, \quad \text{for } n \geq n_1$$

and therefore

$$\Delta\omega_n = \sum_{i=0}^{\infty} a_n^i \omega_{n+i} \quad \text{for } n \ge n_1.$$

So the sequence $w = \{\omega_n\}_{n=1}^{\infty}$ satisfies equation (2) for $n \ge n_1$ only, i.e. $y_n = \omega_n$ for $n \ge n_1$. We can derive the others y_j's, $j = n_1 - 1, \ldots, 2, 1$, directly from the equation (2) which can be written down in the form

$$y_n = -(1 + a_n^0)^{-1}\Big[(a_n^1 - 1)y_{n+1} + \sum_{i=2}^{\infty} a_n^i y_{n+i}\Big]. \tag{6}$$

Finally, we get a sequence satisfying equation (2) for any $n \in N$. Moreover, this sequence has for $n \ge n_1$ identical elements as sequence w and satisfies the condition (3), because $\omega_n \in I_n$ and $\operatorname{diam} I_n \to 0$ and $n \to \infty$. This completes the proof of the theorem.

The similary prove methods are presented in papers [1] and [2]. The equation

$$\Delta x_n = \sum_{i=0}^{r} a_n^i x_{n+1}$$

is investigated in the paper [2].

EXAMPLE. Let us consider the equation (2), where

$$a_n^i = \frac{3}{8}\frac{1}{2^i(2^{n+1} - 1)}.$$

By the theorem, for an arbitrary K we can find the solution x converging to K. For example for $K = 1$ this is $y_n = 1 + \frac{1}{2^n}$.

Remark. Let $a^i : N \to C$, $a_n^0 \ne -1$ for $n \in N$, and $\sup_{n \ge m}[\max_i |a_n^i|] > 0$ for each $m \in N$, and

$$\sum_{i=0}^{\infty}\sum_{j=1}^{\infty} |a_j^i| < \infty.$$

Then for each arbitrary constant $K \in C$ there exists a solution $z : N \to C$ of the equation

$$\Delta z_n = \sum_{i=0}^{\infty} |a_n^i| z_{n+1}$$

such that

$$z_n = K + o(1).$$

References

[1] A. Drozdowicz, J. Popenda, *Asymptotic behaviour of solutions of the second order difference equation*, Proc. Amer. Math. Soc. 99, (1987), 135–140.

[2] J. Popenda, E. Schmeidel, *On the asymptotic behaviour of solutions of linear difference equations*, Publ. Mat. 38, (1994), 3–9.

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