DEMONSTRATIO MATHEMATICA
Vol. XXX No 1 1997

Mumtaz Ahmad Khan

ON AN ALGEBRAIC STUDY OF A CLASS OF DISCRETE ANALYTIC FUNCTIONS

1. Introduction

A theory of discrete analytic functions, called q-analytic functions, was developed by Harman [2], [3] on the geometric lattice

$$(1.1)\qquad H=\{(q^m x_0, q^n y_0);\ m,n\in Z,\ 0<q<1,(x_0,y_0)\text{ fixed},\ x_0>0,\ y_0>0\}.$$

The symbol $[\alpha]$ will denote the q-number defined as

$$(1.2)\qquad [\alpha]=\frac{1-q^\alpha}{1-q},\quad 0<q<1,$$

where α is any real or complex number.

Also $[n]!$ will denote the q-factorial function given by

$$(1.3)\qquad [n]!=[1][2][3]\dots[n]\equiv\frac{(1-q)_n}{(1-q)^n}.$$

Further, the q-difference operators $D_{q,x}$ and $D_{q,y}$ are defined as

$$(1.4)\qquad D_{q,x}[f(z)]=\frac{f(z)-f(qx,y)}{(1-q)x}$$

and

$$(1.5)\qquad D_{q,y}[f(z)]=\frac{f(z)-f(x,qy)}{(1-q)iy},$$

respectively, where f is a discrete function. The two operators involve a *basic triad* of points denoted by

$$(1.6)\qquad T(z)=\{(x,y),(qx,y),(x,qy)\}.$$

1991 *Mathematics Subject Classification*: Primary 30G20; Secondary 30H05.

Let D be a discrete domain. Then a discrete function f is said to be *q-analytic* at $z \in D$, if

(1.7) $$D_{q,x}[f(z)] = D_{q,y}[f(z)].$$

(1.8) If in addition (1.7) holds for every $z \in D$ such that $T(z) \subseteq D$, then f is said to be *q-analytic* in D.

For simplicity, if (1.7) or (1.8) holds, the common operator D_q is used, where

(1.9) $$D_q = D_{q,x} = D_{q,y}.$$

Harman also introduced the operator C_y defined by

(1.10) $$C_y \equiv \sum_{j=0}^{\infty} \frac{(1-q)^j}{(1-q)_j} (iy)^j D_{q,x}^j$$

and called the function

(1.11) $$f(z) = C_y[f(x,0)] = \sum_{j=0}^{\infty} \frac{(1-q)^j}{(1-q)_j} (iy)^j D_{q,x}^j [f(x,0)]$$

the *q-analytic continuation* of $f(x,0)$, into a q-analytic function f defined at the point $(x,y) \in H$ (cf. (1.1)). Similarly

(1.12) $$C_x[f(0,y)] = \sum_{j=0}^{\infty} \frac{(1-q)^j}{(1-q)_j} x^j D_{q,y}^j [f(0,y)]$$

represents the q-analytic continuation from the y-axis.

As a discrete analogue of the classical function z^n, Harman defined a function $z^{(n)}$ satisfying the following conditions:

(1.13) $$\begin{cases} \text{(i)} & D_q[z^{(n)}] = \frac{(1-q^n)}{(1-q)} z^{(n-1)}, \\ \text{(ii)} & z^{(0)} = 1, \\ \text{(iii)} & 0^{(n)} = 0, \ n > 0, \end{cases}$$

where n is a non-negative integer. Such a function is obtained by applying the operator C_y, given by (1.10), to the real function x^n.

In fact $z^{(n)}$, for a non-negative integer n, is given by

$$z^{(n)} \equiv C_y(x^n) = \sum_{j=0}^{\infty} \frac{(1-q)^j}{(1-q)_j} (iy)^j D_{q,x}^j (x^n)$$

or

(1.14) $$z^{(n)} = \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_q x^{n-j} (iy)^j,$$

where $\begin{bmatrix} n \\ j \end{bmatrix}_q = \frac{[n]!}{[j]![n-j]!}$.

It can also be written as

$$(1.15)\qquad z^{(n)} = \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_q x^j (iy)^{n-j}.$$

Harman also defined the function $\mathrm{e}(z)$ by the equality

$$(1.16)\qquad \mathrm{e}(z) = C_y[\mathrm{e}_q((1-q)x)].$$

Thus

$$(1.17)\qquad \mathrm{e}(z) = \sum_{j=0}^{\infty} \frac{(1-q)^j}{(1-q)_j} z^{(j)}.$$

In 1969, Sen [4] studied the topological and algebraic structure of the set of all complex valued functions $f(z) = \sum_{n=0}^{\infty} a_n z^n$, where $n!|a_n|$ is bounded.

Let R denote the set of all complex valued functions of the type

$$(1.18)\qquad f(z) = \sum_{n=0}^{\infty} a_n z^{(n)}, \quad \text{where } [n]!|a_n| \text{ is bounded.}$$

It can be verified easily that the elements (1.18) of the set R are all discrete functions. In R we define addition and multiplication as

$$(1.19)\qquad f(z) + g(z) = \sum_{n=0}^{\infty} (a_n + b_n) z^{(n)}$$

and

$$(1.20)\qquad f(z) \circ g(z) = \sum_{n=0}^{\infty} [n]! a_n b_n z^{(n)},$$

respectively, where $f(z)$, given by (1.18), and $g(z) = \sum_{n=0}^{\infty} b_n z^{(n)}$ are two elements of R.

The aim of the present paper is to study the topological and algebraic structure of R.

2. Algebraic and topological structure of R

We are going to prove that R is a commutative ring with identity element.

LEMMA 1. *R is closed with respect to the two operators '$+$' and '$\circ$' given by* (1.19) *and* (1.20), *respectively.*

Proof. Since $[n]!|(a_n + b_n)| \le [n]!|a_n| + [n]!|b_n|$, therefore $f(z) + g(z)$ is an element of R, when $f(z), g(z) \in R$.

Again $[n]!|a_n b_n| \le [n]!|a_n|[n]!|b_n|$. This implies that, if $f(z), g(z) \in R$, then $f(z) \circ g(z) \in R$. Hence, Lemma 1 is proved.

LEMMA 2. $\mathrm{e}(z) = \sum_{n=0}^{\infty} \frac{(1-q)^n}{(1-q)_n} z^{(n)}$ *is the identity element of R.*

Proof. It is obvious that e(z) is an element of R. Now for $f(z) \in R$ we have, by (1.3),

$$\mathrm{e}(z) \circ f(z) = \sum_{n=0}^{\infty} \frac{z^{(n)}}{[n]!} \circ \sum_{n=0}^{\infty} a_n z^{(n)} = \sum_{n=0}^{\infty} [n]! \frac{1}{[n]!} a_n z^{(n)} = f(z).$$

Similarly, it can be proved that $f(z) \circ \mathrm{e}(z) = f(z)$. Hence e$(z)$ is an identity element of R.

THEOREM 1. *R is a commutative ring with identity.*

Proof. From Lemma 1 we find that R is closed with respect to the operators '+' and '$\circ$'. It can be proved easily that

$$[f(z) + g(z)] + h(z) = f(z) + [g(z) + h(z)],$$
$$[f(z) \circ g(z)] \circ h(z) = f(z) \circ [g(z) \circ h(z)],$$

where $f(z), g(z), h(z) \in R$. Again $f(z) = 0$ is the element of R. In fact, it is true that for (1.18) from R the element $-f(z) = \sum_{n=0}^{\infty}(-a_n)z^{(n)}$ is the additive inverse of (1.18), because $[n]!|-a_n| = [n]!|a_n|$ is bounded so that $-f(z) \in R$ and

$$f(z) + (-f(z)) = \sum_{n=0}^{\infty}(a_n - a_n)z^{(n)} = 0.$$

Now, for $f(z), g(z)$ as above and $h(z) = \sum_{n=0}^{\infty} c_n z^{(n)}$, we verify the distributive property

$$[f(z) + g(z)] \circ h(z) = \sum_{n=0}^{\infty}(a_n + b_n)z^{(n)} \circ \sum_{n=0}^{\infty} c_n z^{(n)}$$
$$= \sum_{n=0}^{\infty}[n]! a_n c_n z^{(n)} + \sum_{n=0}^{\infty}[n]! b_n c_n z^{(n)} = f(z) \circ h(z) + g(z) \circ h(z).$$

Similarly, it can be proved that

$$f(z) \circ [g(z) + h(z)] = f(z) \circ g(z) + f(z) \circ h(z).$$

Commutative property for addition and multiplication follows easily. Also from Lemma 2 we find that e(z) is the identity element of R. Hence the Theorem 1 is proved.

3. Further results

We are going to show that R is a Banach algebra.

Let C denote the set of complex numbers. For $\alpha \in C$ and $f(z) \in R$ scalar multiplication is defined by $\alpha f(z) = \sum_{n=0}^{\infty} \alpha a_n z^{(n)}$.

The following axioms are satisfied:

(1) if $\alpha \in C$, $f(z) \in R$, then $\alpha f(z) \in R$,
(2) $(\lambda + \mu)f(z) = \lambda f(z) + \mu f(z)$, $\lambda, \mu \in C$,
(3) $\alpha(\beta f(z)) = (\alpha\beta)f(z)$, $\alpha, \beta \in C$,
(4) $\lambda(f(z) + g(z)) = \lambda f(z) + \lambda g(z)$, $\lambda \in C$,
(5) $1f(z) = f(z)$.

This proves the following theorem.

THEOREM 2. *The set R is a linear space over the set of complex numbers.*

THEOREM 3. *The set R is a commutative algebra with the identity element.*

Proof. We have

$$\begin{aligned}\lambda(f(z) \circ g(z)) &= \lambda \sum_{n=0}^{\infty}[n]!a_n b_n z^{(n)} = \sum_{n=0}^{\infty}[n]!(\lambda a_n)b_n z^{(n)} \\ &= \sum_{n=0}^{\infty}(\lambda a_n)z^{(n)} \circ \sum_{n=0}^{\infty} b_n z^{(n)} = \lambda \sum_{n=0}^{\infty} a_n z^{(n)} \circ \sum_{n=0}^{\infty} b_n z^{(n)} \\ &= (\lambda f(z)) \circ g(z).\end{aligned}$$

Hence the theorem follows from the results of Theorems 1 and 2.

We now define the norm of (1.18) from R by

$$\|f(z)\| = \sup_n [n]!|a_n|;$$

since $[n]!|a_n|$ is bounded, $\sup_n[n]!|a_n|$ exists. Now

(i) $\|f(z)\| = \sup_n[n]!|a_n| \geq 0$ and $\|f(z)\| = 0$ iff $\sup_n[n]!|a_n| = 0$, i.e., $a_n = 0$ for all n or $f(z) = \sum_{n=0}^{\infty} a_n z^{(n)} = 0$;

(ii) $\|f(z)+g(z)\| = \sup_n[n]!|a_n+b_n| \leq \sup_n[n]!(|a_n|+|b_n|) = \sup_n[n]!|a_n| + \sup_n[n]!|b_n| = \|f(z)\| + \|g(z)\|$,

(iii) $\|\alpha f(z)\| = \|\alpha \sum_{n=0}^{\infty} a_n z^{(n)}\| = \|\sum_{n=0}^{\infty}(\alpha a_n)z^{(n)}\| = \sup_n[n]!|\alpha a_n| = |\alpha| \sup_n[n]!|a_n| = |\alpha|\|f(z)\|$.

Hence we have the following result.

THEOREM 4. *R is a normed linear space.*

Our next aim is to prove the following theorem.

THEOREM 5. *R is a Banach space.*

Proof. Consider the sequence $\{f_p(z)\}$, where $f_p(z) = \sum_{n=0}^{\infty} a_{pn} z^{(n)}$ is an element of R. Let $\{f_p(z)\}$ be a Cauchy sequence. Hence for every $\varepsilon > 0$ there exists a positive integer p_0 such that $\|f_p(z) - f_q(z)\| < \varepsilon$ for $p, q \geq p_0$, i.e.,

$$\sup_n [n]!|a_{pn} - a_{qn}| < \varepsilon \quad \text{for } p, q \geq p_0.$$

This implies that

$$(3.1) \qquad [n]!|a_{pn} - a_{qn}| < \varepsilon$$

for $p, q \geq p_0$ and for every n. We regard n fixed and consider the sequence $a_{1n}, a_{2n}, a_{3n}, \ldots, a_{rn}, \ldots$ On account of (3.1) this sequence would converge to a limit a_n (say) according to Cauchy test. From (3.1) we have

$$(3.2) \qquad [n]!|a_n - a_{pn}| < \varepsilon$$

for $p \geq p_0$ and for all n. Hence $[n]!|a_n|$ is bounded, for

$$[n]!|a_n| = [n]!|a_n - a_{pn} + a_{pn}| \leq [n]!|a_n - a_{pn}| + [n]!|a_{pn}|.$$

Then (1.18) is an element of R.

Now we find from (3.2) that $\sup_n [n]!|a_n - a_{pn}| < \varepsilon$ for $p \geq p_0$, that is $\|f_p(z) - f(z)\| < \varepsilon$ for $p \geq p_0$. Therefore $f_p(z) \to f(z) \in R$, when $p \to \infty$.

This implies that R is complete. So the proof of the theorem follows from Theorem 4.

THEOREM 6. *R is a commutative Banach algebra with identity element.*

Proof. Let $f(z), g(z) \in R$. Since

$$\|f(z) \circ g(z)\| = \Big\| \sum_{n=0}^{\infty} [n]! a_n b_n z^{(n)} \Big\| = \sup_n ([n]!)^2 |a_n b_n| \leq \sup_n [n]!|a_n| \sup_n [n]!|b_n|,$$

i.e.,

$$\|f(z) \circ g(z)\| \leq \|f(z)\| \|g(z)\|.$$

Again $\mathrm{e}(z)$ is the identity element of R, since we have

$$\|\mathrm{e}(z)\| = \sup_n [n]! \left| \frac{1}{[n]!} \right| = 1.$$

Hence, Theorem 6 is proved.

References

[1] W. Hahn, *Beiträge zür Theorie der Heineschen Reihen*, Math. Nachr. 2 (1949), 340–379.

[2] C. J. Harman, *Discrete geometric function theory I*, Appl. Anal. 7 (1977), 315–336.

[3] C. J. Harman, *Discrete geometric function theory II*, Appl. Anal. 9 (1979), 191–203.

[4] M. K. Sen, *On a class of integral functions*, Bull. Calcutta Math. Soc. 61 (1969), 67–74.

DEPARTMENT OF APPLIED MATHEMATICS
FACULTY OF ENGINEERING
A.M.U., ALIGARH-202002, U.P., INDIA

Received November 29, 1995.