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Jiling Cao, I. L. Reilly, M. K. Vamanamurthy

COMPARISON OF CONVERGENCES FOR MULTIFUNCTIONS

In this paper, we extend the concept of continuous convergence for single-valued functions to multifunctions and compare it with topological convergence in points, topological convergence in graphs, quasiuniform convergence and almost quasiuniform convergence. Relationships among these kinds of convergences are established and some of results from [3], [9], [11] and [14] are generalized.

1. Introduction

The concept of convergence of functions is indispensable in both analysis and topology. The purpose of this paper is to compare several types of convergences for multifunctions which have appeared in recent years.

Let X and Y be two topological spaces. A subset A of X is said to be *α-paracompact* [1] if every open cover of A in X has a locally finite open covering refinement in X. Let $\{A_i : i \in D\}$ be a net of subsets of X. A point $x \in X$ is called a *limit point* [10], [13] of $\{A_i : i \in D\}$, denoted by $x \in LiA_i$, if for every neighbourhood U of x there is $i_0 \in D$ such that $A_i \cap U \neq \emptyset$ for all $i \geq i_0$. Furthermore, $x \in X$ is called a *cluster point* [10], [13] of $\{A_i : i \in D\}$, denoted by $x \in LsA_i$, if for every neighbourhood U of x and every $i \in D$ there is $i_0 \in D$ such that $i_0 \geq i$ and $A_{i_0} \cap U \neq \emptyset$. We say that $\{A_i : i \in D\}$ *topologically converges* to A, denoted by $LtA_i = A$, if $LiA_i = LsA_i = A$. Note that *a* net $\{x_i : i \in D\}$ is convergent to x_0, denoted by $x_i \to x_0$, iff $\{\{x_i\}Li \in D\}$ is topologically convergent to $\{x_0\}$ when X is Hausdorff.

By a multifunction (or multi-valued function) $F: X \to Y$, we mean a point to set correspondence from X into Y such that $F(x) \neq \emptyset$ for each

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point $x \in X$. Recall that $F: X \to Y$ is said to be *upper* (*lower*) *semicontinuous* [10], [13], abbreviated by usc (lsc), at a point $x \in X$ if for each open subset $V \subset Y$ satisfying $F(x) \subset V$ ($F(x) \cap V \neq \emptyset$) there exists an open neighbourhood U of x such that $F(U) \subset V$ ($F(x') \cap V \neq \emptyset$ for all $x' \in U$). The set $Gr(F) = \{(x, y) \in X \times Y : x \in X, y \in F(x)\}$ is called the *graph* of F. Moreover, $F: X \to Y$ is called *point-compact* (*point-connected, point-paracompact*) if for every $x \in X$ the set $F(x)$ is a compact (connected, α-paracompact) subset of Y. Throughout this paper, $C(X,Y)$ ($C_m(X,Y)$) denotes the family of all continuous single-valued functions (continuous and point-compact multifunctions) from a topological space X into a topological space Y.

In [11], Kowalczyk discussed two different types of convergences for multifunctions: topological convergence and graph convergence by means of topological convergence of subsets. We find that the concept of continuous convergence will enable us to compare these convergences. Thus, in Section 2, we extend in a natural way this concept from the case of single-valued functions (see [9]) to the case of multifunctions and establish some characterizations. In the next section, we compare continuous convergence with topological convergence and graph convergence. In the last section, multifunctions whose range is a quasiuniform space are considered, and relationships among quasiuniform convergence, almost quasiuniform convergence as well as all three kinds of convergences mentioned above are established.

2. Continuous convergence for multifunctions

Let X and Y be two topological spaces. Let $\mathcal{P}_0(X)$ stand for the collection of all nonempty subsets of X. For each nonempty open subset $G \subset X$, denote by $G^+ = \mathcal{P}_0(G) = \{A \in \mathcal{P}_0(X) : A \subset G\}$ and by $G^- = \{A \in \mathcal{P}_0(X) : A \cap G \neq \emptyset\}$. The *upper* (*lower*) *topology* on $\mathcal{P}_0(X)$ is generated by $\{G^+ : G \text{ is open in } X\}$ ($\{G^- : G \text{ is open in } X\}$) [10]. Recall that a net $\{F_i : i \in D\}$ of multifunctions from X into Y converges pointwise to F [5], denoted by $F_i \xrightarrow{pc} F$, if for each point $x \in X$ the net $\{F_i(x) : i \in D\}$ converges to $F(x)$ with respect to both upper and lower topologies on $\mathcal{P}_0(Y)$.

DEFINITION 2.1. A net $\{F_i : i \in D\}$ of multifunctions from X into Y is called

(1) *upper continuously convergent* to F, denoted by $F_i \xrightarrow{ucc} F$, if for each point $x \in X$ and each net $\{x_\beta : \beta \in E\}$ on X with $x_\beta \to x$, the net $\{F_i(x_\beta) : (i, \beta) \in D \times E\}$ converges to $F(x)$ with respect to the upper topology on $\mathcal{P}_0(Y)$.

(2) *lower continuously convergent* to F, denoted by $F_i \xrightarrow{lcc} F$, if for each point $x \in X$ and each net $\{x_\beta : \beta \in E\}$ on X with $x_\beta \to x$, the

net $\{F_i(x_\beta) : (i,\beta) \in D \times E\}$ converges to $F(x)$ with respect to the lower topology on $\mathcal{P}_0(Y)$.

(3) *continuously convergent* to F, denoted by $F_i \xrightarrow{cc} F$, if both $F_i \xrightarrow{ucc} F$ and $F_i \xrightarrow{lcc} F$.

Obviously, continuous convergence defined above is a natural extension of the corresponding notion for single-valued functions in Frink [9], and it implies pointwise convergence. The following two fundamental lemmas are very important in the sequel. The proof of the second lemma is similar to that of the first one, so we omit it.

LEMMA 2.2. *Let $\{F_i : i \in D\}$ be a net of multifunctions from X into Y. Then the following statements are equivalent.*

(1) $F_i \xrightarrow{ucc} F$.

(2) *For any point $x \in X$ and any neighbourhood V of $F(x)$ in Y and any net $\{x_\beta : \beta \in E\}$ on X with $x_\beta \to x$, there exist $i_0 \in D$ and $\beta_0 \in E$ such that $F_i(x_\beta) \subset V$ for all $i \geq i_0$ and all $\beta \geq \beta_0$.*

(3) *For any point $x \in X$ and any neighbourhood V of $F(x)$ in Y, there exist a neighbourhood U of x and $i_0 \in D$ such that $F_i(U) \subset V$ for all $i \geq i_0$.*

Proof. The implication of (1) $\Rightarrow$ (2) is trivial.

(2) $\Rightarrow$ (3). If not, then there exist a point $x \in X$ and a neighbourhood V of $F(x)$ in Y such that for any neighbourhood U of x and any $i \in D$ one can find $\lambda \geq i$ and a point $x_{U,\lambda} \in U$ with $F_\lambda(x_{U,\lambda}) \not\subset V$. Thus for any given neighbourhood U of x, there exists a cofinal subset D_U of D such that $F_\lambda(x_{U,\lambda}) \not\subset V$ for all $\lambda \in D_U$. Let $D_0 = \bigcup_{U \in \mathcal{N}(x)} D_U$, where $\mathcal{N}(x)$ is the family of all neighbourhoods of x ordered by inclusion. Obviously, D_0 is a cofinal subset of D and the net $\{x_{U,\lambda} : (U,\lambda) \in \mathcal{N}(x) \times D_0\}$ is convergent to x. From (2), there exists a pair $(U_0,\lambda_0) \in \mathcal{N}(x) \times D_0$ such that $F_\lambda(x_{U,\lambda}) \subset V$ for all $(U,\lambda) \in \mathcal{N}(x) \times D_0$ whenever $U \subset U_0$ and $\lambda \geq \lambda_0$. This is a contradiction.

(3) $\Rightarrow$ (1). Fix $x \in X$ and let $\{x_\beta : \beta \in E\}$ be a net on X such that $x_\beta \to x$. From (3), for any neighbourhood V of $F(x)$ in Y there exist a neighbourhood U of x and $i_0 \in D$ such that $F_i(U) \subset V$ for all $i \geq i_0$. Since $x_\beta \to x$, there exists $\beta_0 \in E$ such that $x_\beta \in U$ for all $\beta \geq \beta_0$. Hence $F_i(x_\beta) \subset V$ for all $i \geq i_0$ and $\beta \geq \beta_0$.

LEMMA 2.3. *Let $\{F_i : i \in D\}$ be a net of multifunctions from X into Y. Then the following statements are equivalent.*

(1) $F_i \xrightarrow{lcc} F$.

(2) *For any point $x \in X$ and any open set V of Y with $F(x) \cap V \neq \emptyset$ and any net $\{x_\beta : \beta \in E\}$ on X with $x_\beta \to x$, there exist $i_0 \in D$ and $\beta_0 \in E$ such that $F_i(x_\beta) \cap V \neq \emptyset$ for all $i \geq i_0$ and $\beta \geq \beta_0$.*

(3) *For any point $x \in X$ and any open set V of Y with $F(x) \cap V \neq \emptyset$, there exist a neighbourhood U of x and $i_0 \in D$ such that $F_i(x') \cap V \neq \emptyset$ whenever $x' \in U$ and $i \geq i_0$.*

3. Topological convergences in points and graphs

Let $\{F_i : i \in D\}$ be a net of multifunctions from X into Y.

DEFINITION 3.1. [11] A net $\{F_i : i \in D\}$ is called

(1) *topologically convergent in points* to F, denoted by $F_i \xrightarrow{ptc} F$, if $LtF_i(x) = F(x)$ for every point $x \in X$.

(2) *topologically convergent in graphs* to F, denoted by $F_i \xrightarrow{gtc} F$, if $LtGr(F_i) = Gr(F)$.

Remark: In [11], topological convergence in points and in graphs are called "topological convergence" and "graph convergence", respectively. In [2] and [3], topological convergence in graphs is also called "topological convergence"; while, in [14] it is called " Hausdorff topological convergence of graphs". To avoid confusion, we rename these two types of convergences as in the above. Obviously, topological convergence in points and pointwise convergence are equivalent for single-valued functions when the range is a Hausdorff space.

It is pointed out in [11] that the notions of topological convergence in points and topological convergence in graphs are for multifunctions independent, even when the domain is a metric space. Now we will establish relationships between continuous convergence and topological convergence in graphs.

THEOREM 3.2. *Let $\{F_i : i \in D\}$ be a net of multifunctions from X into Y.*

(1) *If $F_i \xrightarrow{lcc} F$ then $Gr(F) \subset LiGr(F_i)$.*

(2) *If Y is a Hausdorff space and F is a point-paracompact multifunction such that $F_i \xrightarrow{ucc} F$ then $LsGr(F_i) \subset Gr(F)$.*

Proof. (1) Take an arbitrary pair $(x_0, y_0) \in Gr(F)$ and let U and V be neighbourhoods of x_0 and y_0 respectively. Since $F(x_0) \cap V \neq \emptyset$ then there exist a neighbourhood G of x_0 and $i_0 \in D$ such that $F_i(x) \cap V \neq \emptyset$ for all $x \in G$ and all $i \geq i_0$. Hence $(U \times V) \cap Gr(F_i) \neq \emptyset$ for all $i \geq i_0$ which implies that $(x_0, y_0) \in LiGr(F_i)$.

(2) Suppose that $(x_0, y_0) \in LsGr(F_i) \setminus Gr(F)$. Then in particular $y_0 \notin F(x_0)$. Since Y is a Hausdorff space, then for every point $y \in F(x_0)$ there exist disjoint open subsets V_y and W_y containing y and y_0, respectively. The family $\{V_y : y \in F(x_0)\}$ forms an open cover of $F(x_0)$. By α-paracompactness of $F(x_0)$, there is a locally finite open cover $\mathcal{V} = \{U_\lambda : \lambda \in \Lambda\}$

of $F(x_0)$ which refines $\{V_y : y \in F(x_0)\}$. Therefore there exists an open neighbourhood W_0 of y_0 such that W_0 intersects only finitely many members $U_{\lambda_1}, U_{\lambda_2}, ..., U_{\lambda_n}$ of $\mathcal{V}$. So we may choose finitely many points $y_1, y_2, ..., y_n$ of $F(x_0)$ such that $U_{\lambda_k} \subset V_{y_k}$ for each k, $1 \leq k \leq n$. Put $W = W_0 \cap (\bigcap_{k=1}^{n} W_{y_k})$ and $V = \bigcup_{\lambda \in \Lambda} U_\lambda$. Observe that W is an open neighbourhood of y_0 disjoint with V and $F(x_0) \subset V$. Since $F_i \xrightarrow{ucc} F$ there exist a neighbourhood G of x and $i_0 \in D$ such that $F_i(G) \subset V$ for all $i \geq i_0$. Now $G \times W$ is a neighbourhood of (x_0, y_0) in $X \times Y$ and $(G \times W) \cap Gr(F_i) = \emptyset$ for all $i \geq i_0$. This contradicts the fact that $(x_0, y_0) \in LsGr(F_i)$. Thus, $LsGr(F_i) \subset Gr(F)$.

COROLLARY 3.3. *Let $\{F_i : i \in D\}$ be a net of multifunctions from X into a Hausdorff space Y and let F be a point-paracompact multifunction. If $F_i \xrightarrow{cc} F$, then $F_i \xrightarrow{gtc} F$.*

COROLLARY 3.4. [9] *Let $\{f_i : i \in D\}$ be a net of functions from X into a Hausdorff space Y. If $f_i \xrightarrow{cc} f$, then $f_i \xrightarrow{gtc} f$.*

By a similar proof to that one of Theorem 3.2 we can get the following result.

THEOREM 3.5. *Let $\{F_i : i \in D\}$ be a net of multifunctions from X into a Hausdorff space Y and let F be a point-paracompact multifunction. If $F_i \xrightarrow{cc} F$ then $F_i \xrightarrow{ptc} F$.*

Following [6], a space X is called *rimcompact* if for each point $x \in X$ and each neighbourhood U of x there exists a neighbouhood V of x such that the boundary $Fr(V)$ of V is compact and $V \cup Fr(V) \subset U$. It is well-known that rimcompact Hausdorff spaces are regular.

THEOREM 3.6. *Let X be a locally connected space and Y be a rimcompact space. Let $\{F_i : i \in D\} \subset C_m(X, Y)$ be a net such that all F_i are point-connected. If $F_i \xrightarrow{gtc} F$ and F is point-compact multifunction, then $F_i \xrightarrow{cc} F$.*

Proof. (1) We shall prove first that $F_i \xrightarrow{ucc} F$. If not, then there exist a point $x \in X$ and a neighbourhood V' of $F(x)$ in Y such that for any neighbourhood U of x, there is a cofinal subset D_U of D satisfying $F_i(U) \cap (Y - V') \neq \emptyset$ for all $i \in D_U$. Since Y is rimcompact then for every point $y \in F(x)$ there exists a neighbourhood V_y of y such that $Fr(V_y)$ is compact and $V_y \cup Fr(V_y) \subset V'$. The family $\{Int(V_y) : y \in F(x)\}$ forms an open cover of $F(x)$. Thus there are finitely many points $y_1, y_2, \ldots, y_n \in F(x)$ such that $F(x) \subset \bigcup_{k=1}^{n} V_{y_k}$. Put $V = \bigcup_{k=1}^{n} V_{y_k}$ and observe that $Fr(V) \subset \bigcup_{k=1}^{n} Fr(V_{y_k})$. It is easy to see that $Fr(V)$ is compact and $V \cup Fr(V) \subset V'$. Therefore $F_i(U) \cap (Y - V) \neq \emptyset$ for all $i \in D_U$.

Let $\mathcal{N}_c(x)$ be the family of all connected neighbourhoods of x ordered by inclusion. Since X is locally connected, $\mathcal{N}_c(x)$ is a base of the neighbourhood system $\mathcal{N}(x)$ of x. Let $U \in \mathcal{N}_c(x)$. Since $F_i \xrightarrow{gtc} F$ then there exists $i_1 \in D$ such that $(U \times V) \cap Gr(F_i) \neq \emptyset$ whenever $i \geq i_1$. Let $D'_U = \{i \in D : i \geq i_1\}$. Then D'_U is a cofinal subset of D such that for each $i \in D'_U$, we have both $F_i(U) \cap V \neq \emptyset$ and $F_i(U) \cap (Y - V) \neq \emptyset$. From Theorem 7.4.4 in [10] the sets $F_i(U)$ are connected for all $i \in D'_U$. This implies that $F_i(U) \cap Fr(V) \neq \emptyset$. For each $i \in D'_U$ choose points $x_{i,U} \in U$ and $y_{i,U} \in F_i(x_{i,U}) \cap Fr(V)$. Thus, we get a net $\{y_{i,U} : i \in D'_U\} \subset Fr(V)$. Let $y_U \in Fr(V)$ be a cluster point of $\{y_{i,U} : i \in D'_U\}$. Then $\{y_U : U \in \mathcal{N}_c(x)\}$ is also a net in $Fr(V)$ and hence it has a cluster point $y \in Fr(V)$.

We will show that $(x, y) \in LsGr(F_i)$. To see this, let G and W be any open neighbourhoods of x and y respectively. Since X is locally connected and y is a cluster point of $\{y_U : U \in \mathcal{N}_c(x)\}$ then we can choose $U \in \mathcal{N}_c(x)$ such that $x \in U \subset G$ and $y_U \in W$. Thus, for any given $i \in D$, there exists a $\lambda \in D'_U$ such that $\lambda \geq i$ and $y_{\lambda,U} \in W$. From the definitions of $x_{\lambda,U}$ and $y_{\lambda,U}$, we have $(x_{\lambda,U}, y_{\lambda,U}) \in Gr(F_\lambda) \cap (G \times W)$. This implies that $(x, y) \in LsGr(F_i)$. Since $F_i \xrightarrow{gtc} F$, then $(x, y) \in Gr(F)$, i.e., $y \in F(x) \subset V$. But this contradicts with the fact that $y \in Fr(V)$. Therefore $F_i \xrightarrow{ucc} F$.

(2) $F_i \xrightarrow{lcc} F$. If not, one can find a point $x \in X$ and an open subset V' of Y with $F(x) \cap V' \neq \emptyset$ such that for each neighbourhood U of x there exists a cofinal subset D_U of D satisfying $F_i(x'_{i,U}) \subset (Y - V')$ for some point $x'_{i,U} \in U$, whenever $i \in D_U$. Let $y_0 \in F(x) \cap V'$. Since Y is rimcompact then there exists a neighbourhood V of y_0 such that $Fr(V)$ is compact and $V \cup Fr(V) \subset V'$. Thus, for any given neighbourhood U of x there exists a cofinal subset D_U of D such that $F_i(U) \cap (Y - V) \neq \emptyset$ for all $i \in D_U$. The rest of the proof is similar to that of (1).

CORROLARY 3.7. [14] *Let X be a locally connected space and Y be a rimcompact space. If $\{f_i : i \in D\}$ is a net in $C(X,Y)$ and $f \in C(X,Y)$ such that $f_i \xrightarrow{gtc} f$, then $f_i \xrightarrow{cc} f$.*

As an immediate corollary of Theorems 3.5 and 3.6, we can establish the following relationship between topological convergence in points and topological convergence in graphs.

COROLLARY 3.8. *Assume that X is a locally connected space and Y is a rimcompact space. Let $\{F_i : i \in D\} \subset C_m(X,Y)$ be a net such that each F_i is point-connected. If $F_i \xrightarrow{gtc} F$ and F is a point-compact multifunction then $F_i \xrightarrow{ptc} F$.*

THEOREM 3.9. *Let $\{F_i : i \in D\}$ be a net of multifunctions from X into a regular space Y. If $F_i \xrightarrow{cc} F$ and F is a point-paracompact multifunction then F is continuous.*

Proof. (1) F is usc. Fix $x \in X$ and an open neighbourhood V of $F(x)$ in Y. Since Y is regular than for each point $y \in F(x)$, there exists an open subset U_y such that $y \in U_y \subset \overline{U}_y \subset V$. The sets $\{U_y : y \in F(x)\}$ form an open cover of $F(x)$. By α-paracompactness of $F(x)$ there exists a locally finite open cover $\mathcal{V} = \{V_\lambda : \lambda \in \Lambda\}$ of $F(x)$, which refines $\{U_y : y \in F(x)\}$. Then we have

$$F(x) \subset \bigcup_{\lambda\in\Lambda} V_\lambda \subset \overline{\bigcup_{\lambda\in\Lambda} V_\lambda} \subset \bigcup_{\lambda\in\Lambda} \overline{V_\lambda} \subset \bigcup_{y\in F(x)} \overline{U_y} \subset V.$$

Since $F_i \xrightarrow{cc} F$, by Lemma 2.2., there are $i_0 \in D$ and a neighbourhood G of x such that $F_i(G) \subset \bigcup_{\lambda\in\Lambda} V_\lambda$ for all $i \geq i_0$. By Theorem 3.5 we have $F_i \xrightarrow{ptc} F$ and thus we have

$$F(x') \subset \bigcap_{i\in D} \overline{\bigcup_{j\geq i} F_j(x')} \subset \overline{\bigcup_{\lambda\in\Lambda} V_\lambda} \subset V$$

for all $x' \in G$. Hence F is usc.

(2) F is lsc. Fix $x \in X$ and an open subset V of Y with $F(x) \cap V \neq \emptyset$. Choose a point $y \in F(x) \cap V$ and a closed neighbourhood W of y such that $W \subset V$. Since $F_i \xrightarrow{lcc} F$, by Lemma 2.3, there exist a neighbourhood G of x and $i_0 \in D$ such that $F_i(x') \cap W \neq \emptyset$ for all $x' \in G$, whenever $i \geq i_0$. We claim that $F(x') \cap V \neq \emptyset$ for all $x' \in G$. If not, then there exists a point $x'' \in G$ such that $F(x'') \cap V = \emptyset$. Thus $F(x'') \subset Y - V \subset Y - W$. Using the fact that $F_i \xrightarrow{ucc} F$, there exists $i_1 \in D$ such that $F_i(x'') \subset Y - W$ for all $i \geq i_1$. So $F_i(x'') \cap W = \emptyset$ whenever $i \geq i_0$ and $i \geq i_1$ which is a contradiction. Therefore F is lsc.

CORROLARY 3.10. [9] *Let $\{f_i : i \in D\}$ be a net of functions from X into a regular space Y. If $f_i \xrightarrow{cc} f$, then f is continuous.*

4. Convergence of multifunctions into quasiuniform spaces

Let $(Y,\mathcal{U})$ be a quasiuniform space [7]. The conjugate quasiuniformity of $\mathcal{U}$ is denoted by $\mathcal{U}^{-1}$. Moreover, $\mathcal{T}(\mathcal{U})$ and $\mathcal{T}(\mathcal{U}^{-1})$ will denote topologies on Y induced by $\mathcal{U}$ and $\mathcal{U}^{-1}$, respectively. Recall that $(Y,\mathcal{U})$ is said to be *locally symmetric* [7] if for each $U \in \mathcal{U}$ and each point $y \in Y$ there is a symmetric $V \in \mathcal{U}$ such that $V^2(y) \subset U(y)$. It is well-known that a topological space Y can admit a compatible locally symmetric quasiuniformity if and only if it is regular. Finally, $(Y,\mathcal{U})$ is called *small-set symmetric* [8] provided that for each $U \in \mathcal{U}$ and each open set $A \subset Y$ we have $\overline{A} \subset U(A)$. It was shown in [12] that $(Y,\mathcal{U})$ is small-set symmetric if and only if $\mathcal{T}(\mathcal{U}^{-1}) \subset \mathcal{T}(\mathcal{U})$.

DEFINITION 4.1. [5] Let X be a topological space and $(Y,\mathcal{U})$ be a quasiuniform space. A net $\{F_i : i \in D\}$ of multifunctions from X into $(Y,\mathcal{U})$ is said to be *convergent almost quasiuniformly* to F, denoted by $F_i \xrightarrow{aqc} F$, if for each $U \in \mathcal{U}$ and each point $x \in X$ there exists $i_0 \in D$ such that for each $i \geq i_0$, there is a neighbourhood G_i of x such that $F_i(z) \subset U(F(z))$ and $F(z) \subset U^{-1}(F_i(z))$ for all $z \in G_i$.

THEOREM 4.2. *Let $\{F_i : i \in D\}$ be a net of multifunctions from a topological space X into a small-set symmetric quasiuniform space $(Y,\mathcal{U})$. If $F \in C_m(X,Y)$ and $F_i \xrightarrow{cc} F$, then $F_i \xrightarrow{aqc} F$.*

Proof. For any $x \in X$ and any $U \in \mathcal{U}$, choose $V \in \mathcal{U}$ with $V^2 \subset U$. Since $(Y,\mathcal{U})$ is small-set symmetric and $\{Int(V(y) \cap V^{-1}(y)) : y \in F(x)\}$ is an open cover of $F(x)$ then there is a finite set $\{y_1, y_2, \ldots, y_n\} \subset F(x)$ such that $F(x) \subset \bigcup_{k=1}^{n}(V(y_k) \cap V^{-1}(y_k))$. Since $F_i \xrightarrow{cc} F$ then there exist $i_0 \in D$ and a neighbourhood G_1 of x such that for all $i \geq i_0$, $F_i(G_1) \subset \bigcup_{k=1}^{n} V(y_k)$ and $F_i(z) \cap V(y_k) \neq \emptyset$ for all $z \in G_1$ and all k, $1 \leq k \leq n$. Furthermore, by continuity of F, there exists a neighbourhood G_2 of x such that $F(G_2) \subset \bigcup_{k=1}^{n} V^{-1}(y_k)$ and $F(z) \cap V^{-1}(y_k) \neq \emptyset$ for all $z \in G_2$ and all k, $1 \leq k \leq n$. For any point $z \in G = G_1 \cap G_2$ we have

$$F_i(z) \subset \bigcup_{k=1}^{n} V(y_k) \subset V^2(F(z)) \subset U(F(z))$$

and

$$F(z) \subset \bigcup_{k=1}^{n} V^{-1}(y_k) \subset V^{-1} \circ V^{-1}(F_i(z)) \subset U^{-1}(F_i(z)),$$

whenever $i \geq i_0$. This implies that $F_i \xrightarrow{aqc} F$.

EXAMPLE 4.3. The small-set symmetry of $(Y,\mathcal{U})$ in Theorem 4.2. cannot be dropped. Let $X = Y = [0, +\infty)$. Topologize X by defining the basic neighbourhood system for $x = 0$ as: $B_n(0) = \{0\} \cup \{x \in X : x \geq n\}$ for all $n \in N$; and the basic neighbourhood system for $x \neq 0$ as: $B_n(x) = \{x\}$ for all $n \in N$. Let $U_n = \Delta(Y) \cup \{(0,y) : y \in Y, y \geq n\}$ for all $n \in N$ and $\mathcal{U}$ be the quasiuniformity on Y generated by $\{U_n : n \in N\}$. Since $\mathcal{T}(\mathcal{U}^{-1})$ is the discrete topology, $(Y,\mathcal{U})$ is not small-set symmetric. Define a sequence of multifunctions $F_n : X \to Y$ $(n \in N)$ as

$$F_n(x) = \begin{cases} \{x\}, & \text{if } x \leq n; \\ [n, +\infty), & \text{otherwise.} \end{cases}$$

and $F(x) = \{x\}$ for all $x \in X$. Then $F_n \xrightarrow{cc} F$, but $F_n \overset{aqc}{\not\longrightarrow} F$.

Recall that in [4] a net $\{F_i : i \in D\}$ of multifunctions from a topological space X into a quasiuniform space $(Y,\mathcal{U})$ is said to be *convergent quasiuniformly* to F, denoted by $F_i \xrightarrow{qc} F$, if for each $U \in \mathcal{U}$ there exists $i_0 \in D$ such that $F_i(x) \subset U(F(x))$ and $F(x) \subset U^{-1}(F_i(x))$ for all $x \in X$ whenever $i \geq i_0$.

THEOREM 4.4. *Let $\{F_i : i \in D\}$ be a net of multifunctions from a topological space X into a quasiuniform space $(Y,\mathcal{U})$. If $F \in C_m(X,Y)$ and $F_i \xrightarrow{qc} F$, then $F_i \xrightarrow{cc} F$.*

Proof. (1) $F_i \xrightarrow{ucc} F$. Take an arbitrary $x \in X$ and let W be an open neighbourhood of $F(x)$ in Y. Since $F(x)$ is compact then there exists $U \in \mathcal{U}$ such that $F(x) \subset U^2(F(x)) \subset W$. By the upper continuity of F, there exists an open neighbourhood G of x such that $F(G) \subset U(F(x))$. Since $F_i \xrightarrow{qc} F$, there exists $i_0 \in D$ such that $F_i(z) \subset U(F(z))$ for all $z \in X$ whenever $i \geq i_0$. The latter implies that

$$F_i(G) \subset U(F(G)) \subset U^2(F(x)) \subset W$$

for all $i \geq i_0$. Hence $F_i \xrightarrow{ucc} F$.

(2) $F_i \xrightarrow{lcc} F$. Fix $x \in X$ and let W be an open subset of Y with $F(x) \cap W \neq \emptyset$. Choose $y \in W \cap F(x)$ and $U \in \mathcal{U}$ such that $U^2(y) \subset W$. Since F is lsc then there exists a neighbourhood G of x such that $F(z) \cap U(y) \neq \emptyset$ for all $z \in G$. On the other hand, since $F_i \xrightarrow{qc} F$, there exists $i_0 \in D$ such that $F(z) \subset U^{-1}(F_i(z))$ for all $z \in X$, whenever $i \geq i_0$. For each $z \in G$, take $p \in F(z) \cap U(y)$. Then $p \in U^{-1}(F_i(z))$ but this shows that

$$\emptyset \neq F_i(z) \cap U(p) \subset F_i(z) \cap U^2(y) \subset F_i(z) \cap W,$$

whenever $i \geq i_0$. Therefore $F_i \xrightarrow{lcc} F$.

COROLLARY 4.5. *Let $\{F_i : i \in D\}$ be a net of multifunctions from a topological space X into a small-set symmetric quasiuniform space $(Y,\mathcal{U})$. If $F \in C_m(X,Y)$, then $F_i \xrightarrow{qc} F \Rightarrow F_i \xrightarrow{cc} F \Rightarrow F_i \xrightarrow{aqc} F$.*

EXAMPLE 4.6. The hypothesis of continuity of F in Theorem 4.4 cannot be dropped. Let $X = [0,+\infty)$ with the usual Euclidean topology. Let $Y = [0,+\infty)$ and $U_n = \Delta(Y) \cup \{(0,y) : y \in Y, y \geq n\} \cup \{(y,0) : y \in Y\}$ for all $n \in N$. Let $\mathcal{U}$ be the quasiuniformity on Y generated by $\{U_n : n \in N\}$. Define

$$F_n(x) = \begin{cases} \{0\}, & \text{if x=0;} \\ (0,x], & \text{if } x \leq n; \\ \{0\}, & \text{if } x > n. \end{cases}$$

and

$$F(x) = \begin{cases} \{0\}, & \text{if x=0;} \\ (0,x], & \text{if } x \neq 0. \end{cases}$$

Then $F_n \xrightarrow{qc} F$. But F is neither usc nor lsc and $F_n \overset{cc}{\not\longrightarrow} F$.

DEFINITION 4.7. A family $\mathcal{F}$ of multifunctions from a topological space X into a quasiuniform space $(Y,\mathcal{U})$ is called:

(1) *upper quasi-equicontinuous* at $x \in X$ if for each $U \in \mathcal{U}$, there exists a neighbourhood G of x such that $F(G) \subset U(F(x))$ for all $F \in \mathcal{F}$.

(2) *lower quasi-equicontinuous* at $x \in X$ if for each $U \in \mathcal{U}$, there exists a neighbourhood G of x such that $F(z) \cap U(y) \neq \emptyset$ for all $F \in \mathcal{F}$, $z \in G$ and $y \in F(x)$.

We say that $\mathcal{F}$ is *quasi-equicontinuous* at $x \in X$ if it is both upper and lower quasi-equicontinuous at $x \in X$. Moreover, $\mathcal{F}$ is called *quasi-equicontinuous* if it is quasi-equicontinuous at every point $x \in X$. The following result is a consequence of Corollary 4.5 and the remark in section 4 of [5].

COROLLARY 4.8. *Let $\mathcal{F}$ be a family of quasi-equicontinuous and point-compact multifunctions from a compact space X into a small-set symmetric quasiuniform space $(Y,\mathcal{U})$. If $\{F, F_i : i \in D\} \subset \mathcal{F}$, then the following statements are equivalent*:

(1) $F_i \xrightarrow{aqc} F$,

(2) $F_i \xrightarrow{cc} F$,

(3) $F_i \xrightarrow{qc} F$.

Our next result is a generalization of Theorem 2 of Beer [3] and Theorem 3 of Kowalczyk [11].

THEOREM 4.9. *Let $\{F_i : i \in D\}$ be a net of multifunctions from a topological space X into a regular topological space Y and assume that $\{F_i : i \in D\}$ is $\mathcal{U}$-upper quasi-equicontinuous with respect to some compatible locally symmetric quasiuniformity $\mathcal{U}$ on Y. Then*

(1) $F_i \xrightarrow{ptc} F$ *if and only if* $F_i \xrightarrow{gtc} F$.

(2) *If* $F_i \xrightarrow{pc} F$ *and F is point-compact, then* $F_i \xrightarrow{gtc} F$.

Proof. (1) "⇒". Suppose that $F_i \xrightarrow{ptc} F$. Then $Gr(F) = \bigcup_{x\in X}\{x\} \times F(x) = \bigcup_{x\in X}\{x\} \times LiF_i(x) \subset LiGr(F_i)$, so it is sufficient to prove that $LsGr(F_i) \subset Gr(F)$. To do this, fix $(x,y) \in LsGr(F_i)$ and let W be any neighbourhood of y in Y. Let $\mathcal{U}$ be a compatible locally symmetric quasiuniformity on Y such that $\{F_i : i \in D\}$ is $\mathcal{U}$-upper quasi-equicontinuous.

Choose a symmetric $V \in \mathcal{U}$ such that $V^2(y) \subset W$. Then there exists a neighbourhood G of x such that for all $i \in D$ we have $F_i(G) \subset V(F_i(x))$. Since $(x,y) \in LsGr(F_i)$ then there is a cofinal subset E of D such that $(G \times V(y)) \cap Gr(F_j) \neq \emptyset$ for all $j \in E$. Pick what $x_j \in G$ and $y_j \in V(y) \cap F_j(x_j)$. Then $y_j \in V(y) \cap V(F_j(x))$ which implies that $\emptyset \neq F_j(x) \cap V^2(y) \subset F_j(x) \cap W$ for all $j \in E$. Thus $y \in LsF_i(x) = F(x)$ and $(x,y) \subset Gr(F)$.

"$\Leftarrow$". Suppose that $F_i \xrightarrow{gtc} F$. Fix $x \in X$. Then for any point $y \in LsF_i(x)$, we have $(x,y) \in LsGr(F_i) = Gr(F)$ which implies that $LsF_i(x) \subset F(x)$. Now we have to show that $F(x) \subset LiF_i(x)$. Let $\mathcal{U}$ be chosen as in the above. Fix $y \in F(x)$ and let W be an arbitrary neighbourhood of y in Y. Then there exists a symmetric $V \in \mathcal{U}$ such that $V^2(y) \subset W$. By upper quasi-equicontinuity of $\{F_i : i \in D\}$, there exists a neighbourhood G of x such that for all $i \in D$ we have $F_i(G) \subset V(F_i(x))$. Since $(x,y) \in Gr(F) = LiGr(F_i)$ then there exists $i_0 \in D$ such that $(G \times V(y)) \cap Gr(F_i) \neq \emptyset$ for all $i \geq i_0$. Thus we can choose $(x_i, y_i) \in (G \times V(y)) \cap Gr(F_i)$ for each $i \geq i_0$ which implies that $y_i \in F_i(x_i) \subset V(F_i(x))$. Therefore $\emptyset \neq F_i(x) \cap V(y_i) \subset F_i(x) \cap V^2(y) \subset F_i(x) \cap W$ whenever $i \geq i_0$, i.e., $y \in LiF_i(x)$.

(2) It is similar to the first part of the proof of (1), so we omit it.

COROLLARY 4.10. [3] *Let $\{f_n\}$ be a pointwise equicontinuous sequence from a metric space X into a metric space Y. If $\{f_n\}$ is pointwise convergent to a continuous function f, then $f_n \xrightarrow{gtc} f$.*

COROLLARY 4.11. [11] *Let Y be a completely regular space and $\mathcal{U}$ be any compatible uniformity on Y. Let $\{F_i : i \in D\}$ be a net of multifunctions from a space X into $(Y, \mathcal{U})$.*

(1) If $\{F_i : i \in D\}$ is $\mathcal{U}$-upper equicontinuous and $F_i \xrightarrow{ptc} F$, then $F_i \xrightarrow{gtc} F$.

(2) If $\{F_i : i \in D\}$ is $\mathcal{U}$-upper equicontinuous at $x \in X$ and $F_i \xrightarrow{gtc} F$, then $LtF_i(x) = F(x)$.

COROLLARY 4.12. *Let $\{F_i : i \in D\}$ be a net of multifunctions from a locally connected topological space X into a rimcompact Hausdorff topological space Y. If $\{F_i : i \in D\}$ is $\mathcal{U}$-upper quasi-equicontinuous with respect to some compatible locally symmetric quasiuniformity $\mathcal{U}$ on Y, then the following statements are equivalent:*

(1) $F_i \xrightarrow{pc} F$;

(2) $F_i \xrightarrow{gtc} F$;

(3) $F_i \xrightarrow{cc} F$;

(4) $F_i \xrightarrow{ptc} F$.

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DEPARTMENT OF MATHEMATICS
THE UNIVERSITY OF AUCKLAND
Private Bag 92019
AUCKLAND 1, NEW ZEALAND

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