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Adam Lecko

A GENERALIZATION OF ANALYTIC CONDITION FOR CONVEXITY IN ONE DIRECTION

For $\delta \in [-\pi/2, \pi/2]$, ξ_1, $\xi_2 \in \mathbf{C}$, $|\xi_1| \le 1$, $|\xi_2| \le 1$, we introduce the classes $C(\delta, \xi_1, \xi_2)$ defined as follows: a function f regular in $U = \{z \in \mathbf{C} : |z| < 1\}$ of the form $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, $z \in U$, belongs to $C(\delta, \xi_1, \xi_2)$ if

$$\mathrm{Re}\{e^{i\delta}(1 - \xi_1 z)(1 - \xi_2 z)f'(z)\} \ge 0 \text{ for } z \in U.$$

If $|\xi_1| = |\xi_2| = 1$, then the functions $f \in C(\delta, \xi_1, \xi_2)$ for δ suitable choosen according to ξ_1 and ξ_2 are convex in the direction of the imaginary axis. For $\xi_1 = \xi_2 = 0$ the functions $f \in C(\delta, 0, 0)$ are of bounded boundary rotation. Some geometric properties of functions in $C(\delta, \xi_1, \xi_2)$ are examined. There are given coefficient formulae and estimates in the class $C(\delta, \xi_1, \xi_2)$.

0. Introduction

In this paper there are considered subclasses $C(\delta, \xi_1, \xi_2)$, $\delta \in [-\pi/2, \pi/2]$, $\xi_1, \xi_2 \in \overline{U}$, of close-to-convex functions. For each $\xi_1, \xi_2 \in \overline{U}$, $\xi_1 \ne \xi_2$ and $\delta \in (-\pi/2, \pi/2)$ suitable choosen according to ξ_1 and ξ_2 functions $f \in C(\delta, \xi_1, \xi_2)$ have certain geometric property concerning to the way of mapping corresponding hyperbolic family of arcs with vertexes at $1/\xi_1$ and $1/\xi_2$. Analogously, if $\xi_1 = \xi_2 = \xi_0$, then for suitable choosen $\delta \in (-\pi/2, \pi/2)$ functions $f \in C(\delta, \xi_0, \xi_0)$ have similar geometric property concerning to the way of mapping corresponding parabolic family of arcs with vertex at $1/\xi_0$. In the case when $|\xi_1| = |\xi_2| = 1$ or $|\xi_0| = 1$, this geometric property proved in Section 2 and formulated in Corollary 2.1 is equivalent to the convexity in the direction of the imaginary axis of $f(U)$. Therefore the definition condition (1.2) of the class $C(\delta, \xi_1, \xi_2)$ generalizes the well known Roberston

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condition for convexity in one direction [10] proved finally by Royster and Ziegler [11].

In Section 4 there are found coefficient formulae and estimates in the class $\mathcal{C}(\delta, \xi_1, \xi_2)$.

1. Preliminaries

Let $U = \{z \in \mathbf{C} : |z| < 1\}$ denote the unit disk in the complex plane $\mathbf{C}$, $T = \partial U$ the unit circle and $\overline{U} = U \cup T$. By P we denote the class of functions p of the form $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n$, $z \in U$, which are regular in U and have positive real part.

A function f regular in U is called subordinate to a function F regular in U if F is univalent in U, $f(0) = F(0)$ and $f(U) \subset F(U)$. We write then $f \prec F$ or $f(z) \prec F(z)$, $z \in U$.

A function f regular in U with $f(0) = f'(0) - 1 = 0$, is said to be starlike if $\operatorname{Re}\{zf'(z)/f(z)\} > 0$ for $z \in U$, and is said to be convex if $\operatorname{Re}\{1 + zf''(z)/f'(z)\} > 0$ for $z \in U$. It is well known that every starlike and every convex function is univalent in U.

DEFINITION 1.1. A function f of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in U, \tag{1.1}$$

regular in U belongs to the class $\mathcal{C}(\delta, \xi_1, \xi_2)$, $\delta \in [-\pi/2, \pi/2]$, ξ_1, $\xi_2 \in \overline{U}$, if

$$\operatorname{Re}\{e^{i\delta}(1 - \xi_1 z)(1 - \xi_2 z)f'(z)\} \geq 0, \quad z \in U. \tag{1.2}$$

From (1.2) it follows that the assumption $|\xi_1| > 1$ or $|\xi_2| > 1$ implies that $\mathcal{C}(\delta, \xi_1, \xi_2) = \emptyset$ for every $\delta \in [-\pi/2, \pi/2]$.

If (1.2) holds in U and the left hand side of (1.2) is equal to zero at some point in U, then by the minimum principle for harmonic functions it vanishes identically in U. For this reason every function f in $\mathcal{C}(\delta, \xi_1, \xi_2)$ satisfy then the identity

$$e^{i\delta}(1 - \xi_1 z)(1 - \xi_2 z)f'(z) \equiv ai, \quad z \in U, \quad a \in \mathbf{R} \setminus \{0\}.$$

Thus by the normalization of f we see that $\delta = -\pi/2$ and $a = -1$ or $\delta = \pi/2$ and $a = 1$. Therefore from the above we have

Remark 1.2. For every fixed $\xi_0 \in \overline{U}$ the classes $\mathcal{C}(-\pi/2, \xi_0, \xi_0)$ and $\mathcal{C}(\pi/2, \xi_0, \xi_0)$ contain only the function

$$f_{-\pi/2,\xi_0,\xi_0}(z) = f_{\pi/2,\xi_0,\xi_0}(z) = \frac{z}{1 - \xi_0 z}, \quad z \in U. \tag{1.3}$$

For every fixed ξ_1, $\xi_2 \in \overline{U}$ such that $\xi_1 \neq \xi_2$ the classes $C(-\pi/2, \xi_1, \xi_2)$ and $C(\pi/2, \xi_1, \xi_2)$ contain only the function

$$(1.4)\quad f_{-\pi/2,\xi_1,\xi_2}(z) = f_{\pi/2,\xi_1,\xi_2}(z) = \frac{1}{\xi_2 - \xi_1} \log \frac{1 - \xi_1 z}{1 - \xi_2 z},\ \log 1 = 0,\ z \in U.$$

Setting

$$(1.5)\qquad \xi_1 = \alpha e^{-i(\mu+\nu)},\quad \xi_2 = \beta e^{-i(\mu-\nu)},\quad \alpha, \beta \in [0,1],\quad \mu, \nu \in [0, \pi],$$

we can rewrite (1.2) as

$$(1.6)\quad \operatorname{Re}\{e^{i\delta}(1 - (\alpha e^{-i\nu} + \beta e^{i\nu})e^{-i\mu} z + \alpha\beta e^{-2i\mu} z^2) f'(z)\} \geq 0, \quad z \in U.$$

Hence we can formulate the following:

DEFINITION 1.3. A function f of the form (1.1) regular in U belongs to the class $C(\delta, \alpha, \beta, \mu, \nu)$, $\delta \in [-\pi/2, \pi/2]$, $\alpha, \beta \in [0,1]$, $\mu, \nu \in [0, \pi]$, if (1.6) is satisfied.

Of course, $C(\delta, \xi_1, \xi_2) = C(\delta, \alpha, \beta, \mu, \nu)$ for parameters described by (1.5).

From (1.2) it follows that if $f \in C(\delta, \xi_1, \xi_2)$ for $\delta \in (-\pi/2, \pi/2)$, then

$$e^{i\delta}(1 - \xi_1 z)(1 - \xi_2 z) f'(z) = q(z), \quad z \in U,$$

where q is a function regular in U, $\operatorname{Re} q(z) > 0$ for $z \in U$ and $q(0) = e^{i\delta}$. Thus there exists a function $p \in P$ such that $q(z) = p(z)\cos\delta + i \sin\delta$ and consequently

$$(1.7)\qquad e^{i\delta}(1 - \xi_1 z)(1 - \xi_2 z) f'(z) = p(z)\cos\delta + i\sin\delta, \quad z \in U.$$

By (1.7) and by the fact that $p(z) \prec (1+z)/(1-z)$, $z \in U$, we see that $f \in C(\delta, \xi_1, \xi_2)$ for $\delta \in (-\pi/2, \pi/2)$ if and only if

$$(1.8)\qquad (1 - \xi_1 z)(1 - \xi_2 z) f'(z) \prec \frac{1 + e^{-2i\delta} z}{1 - z}, \quad z \in U.$$

For each ξ_1, $\xi_2 \in \overline{U}$ let us define the function

$$(1.9)\qquad h(\xi_1, \xi_2; z) = \frac{z}{(1 - \xi_1 z)(1 - \xi_2 z)},\ z \in U.$$

Hence and by (1.7) we have

$$(1.10)\qquad z f'(z) = e^{-i\delta}(p(z)\cos\delta + i\sin\delta) h(\xi_1, \xi_2; z), \quad z \in U.$$

The inequality (1.6) with additional conditions on parameters δ, α, β, μ, ν, reduces to well known conditions for subclasses of univalent functions.

For $\alpha = \beta = 1$ and $\delta = \mu - \pi/2$ the inequality (1.6) is due to Robertson [10] and characterizes the class $C(\mu - \pi/2, 1, 1, \mu, \nu)$ as the subclass of the class $CV(i)$ of functions convex in the direction of the imaginary axis. The equivalence of analytic condition (1.6) for such choosen parameters and geometric definition of the class $CV(i)$ have done Royster and Ziegler [11] (see also Burniak, Lewandowski, Pituch [1] and Koepf [5]). Directly, they proved that

$$CV(i) = \bigcup_{\mu,\nu\in[0,\pi]} C(\mu - \pi/2, 1, 1, \mu, \nu).$$

In [3] Hengartner and Schober examined the subclasses $C(0,1,1,\pi/2,\pi/2)$, $C(-\pi/2,1,1,0,0)$ and $C(-\pi/2,1,1,0,\pi)$ distinguished in $CV(i)$.

If $\alpha = \beta = 0$, then in view of (1.5) we have $\xi_1 = \xi_2 = 0$. For fixed $\delta \in (-\pi/2, \pi/2)$ the functions $f \in \mathcal{C}(\delta, 0, 0)$ are called of bounded rotation with argument δ and are univalent in U. This criterium of univalence is due to Noshiro [9] and Warshawski [12].

The classes $C(\mu - \pi/2, \alpha, \alpha, \mu, \nu)$ and their subclasses where investigated in [6–7].

2. Geometric properties

In this section we deal with some geometric properties of the classes $\mathcal{C}(\delta, \xi_1, \xi_2)$ for $\delta \in (-\pi/2, \pi/2)$.

Let us fix ξ_1 and ξ_2 in $\overline{U}$. We will consider the following cases:

1. $\xi_k \neq 0$ for every $k \in \{1, 2\}$ and $\operatorname{Re}\xi_1 \neq \operatorname{Re}\xi_2$.
2. $0 = \xi_2 \neq \xi_1$ and $\operatorname{Re}\xi_1 \neq 0$ or $0 = \xi_1 \neq \xi_2$ and $\operatorname{Re}\xi_2 \neq 0$.
3. $\xi_1 = \xi_2 \neq 0$.
4. $\xi_1 = \xi_2 = 0$.

1. Assume first that $\xi_k \neq 0$ for every $k \in \{1, 2\}$ and $\operatorname{Re}\xi_2 \neq \operatorname{Re}\xi_1$.

Let us consider the hyperbolic family $\widetilde{\Gamma}_h(\xi_1, \xi_2)$ of all circles $\widetilde{\gamma}$ going through the fixed points $1/\xi_1$ and $1/\xi_2$. By $\Gamma_h(\xi_1, \xi_2)$ we denote the family of entire circular arcs γ obtained by the restriction of the circles $\widetilde{\gamma} \in \widetilde{\Gamma}_h(\xi_1, \xi_2)$ to the disk U.

1^0 Suppose now that

$$\operatorname{Re}\xi_2 > \operatorname{Re}\xi_1 \text{ and } 0 < \arg\{\xi_2/\xi_1\} \leq \pi. \tag{2.1}$$

Observe that there exists an arc $\widetilde{\gamma}^\circ \in \widetilde{\Gamma}_h(\xi_1, \xi_2)$ such that $\widetilde{\gamma}^\circ \cap U = \emptyset$. Let us parametrize each circle $\widetilde{\gamma}$ in $\widetilde{\Gamma}_h(\xi_1, \xi_2)$ as follows:

$$\widetilde{\gamma} = \widetilde{\gamma}_\tau : z = z_\tau(t) = \frac{1/\xi_1 - 1/\xi_2\ te^{i\tau}}{1 - te^{i\tau}}, \qquad \tau \in (-\tau_0, \pi - \tau_0],\ t \in (-\infty, \infty], \tag{2.2}$$

where $\tau_0 \in [0, \pi)$ is choosen in a such way in order to $\widetilde{\gamma}^\circ = \widetilde{\gamma}_{\tau_0}$. Every circle $\widetilde{\gamma}_\tau \in \widetilde{\Gamma}_h(\xi_1, \xi_2)$, $\tau \in (-\tau_0, \pi - \tau_0]$, achieve the points $1/\xi_1$ and $1/\xi_2$ for $t = 0$ and $t = \infty$ respectively.

We parametrize every arc $\gamma \in \Gamma_h(\xi_1, \xi_2)$ also by (2.2), where $\tau \in \mathcal{I} \subseteq (-\tau_0, \pi - \tau_0]$ and $t \in \mathcal{J}(\tau) \subseteq (-\infty, \infty]$ for every fixed $\tau \in \mathcal{I}$. Since parameter

t has constant sign for all points $z_\tau(t)$ lying either in the disk or outside of the disk with the boundary of $\tilde{\gamma}_{\tau_0}$, we see that $\mathcal{J}(\tau) \subseteq (0, \infty]$ or $\mathcal{J}(\tau) \subseteq (-\infty, 0)$ for each $\tau \in \mathcal{I}$. But for each ξ_1, $\xi_2 \in \overline{U}$ there exists $\gamma_{\tau_1} \in \Gamma_h(\xi_1, \xi_2)$ such that $z_{\tau_1}(t_0) = 0$ for some $t_1 \in \mathcal{J}(\tau_1)$. By (2.2) and (2.1) we have $\tau_1 = \arg\{\xi_2/\xi_1\} \in (0, \pi]$ and $t_0 = |\xi_2/\xi_1|$. For this reason $\mathcal{J}(\tau) \subseteq (0, \infty]$ for every fixed $\tau \in \mathcal{I}$.

By (2.1) we have $\operatorname{Re}\{1/(\xi_2 - \xi_1)\} > 0$ and therefore we set

$$\delta = \arg \frac{1}{\xi_2 - \xi_1} \in (-\pi/2, \pi/2). \tag{2.3}$$

From (2.2) it follows that

$$(1 - \xi_1 z_\tau(t))(1 - \xi_2 z_\tau(t)) = \frac{(\xi_2 - \xi_1)^2}{\xi_1 \xi_2} \frac{t e^{i\tau}}{(1 - t e^{i\tau})^2}. \tag{2.4}$$

Consequently, since $t \in \mathcal{J}(\tau)$ is positive, we see by (1.2), (2.2) and (2.4) that for every function $f \in \mathcal{C}(\delta, \xi_1, \xi_2)$, where δ is given by (2.3), and for every arc $\gamma_\tau \in \Gamma_h(\xi_1, \xi_2)$ holds

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{dt}} \operatorname{Re} f(z_\tau(t)) \\ &= \operatorname{Re}\left\{\frac{\mathrm{d}}{\mathrm{dt}}\left(\frac{1/\xi_1 - (1/\xi_2) t e^{i\tau}}{1 - t e^{i\tau}}\right) f'(z_\tau(t))\right\} \\ &= \operatorname{Re}\left\{\frac{\xi_2 - \xi_1}{\xi_1 \xi_2} \frac{e^{i\tau}}{(1 - t e^{i\tau})^2} f'(z_\tau(t))\right\} \\ &= \frac{1}{t} \operatorname{Re}\left\{\frac{1}{\xi_2 - \xi_1} \frac{(\xi_2 - \xi_1)^2}{\xi_1 \xi_2} \frac{t e^{i\tau}}{(1 - t e^{i\tau})^2} f'(z_\tau(t))\right\} \\ &= \frac{1}{t|\xi_2 - \xi_1|} \operatorname{Re}\{e^{i\delta}(1 - \xi_1 z_\tau(t))(1 - \xi_2 z_\tau(t)) f'(z_\tau(t))\} > 0, \\ &\qquad\qquad \tau \in \mathcal{I},\ t \in \mathcal{J}(\tau). \end{aligned} \tag{2.5}$$

If $\pi < \arg\{\xi_2/\xi_1\} \leq 2\pi$, then we parametrize circles $\tilde{\gamma} \in \tilde{\Gamma}_h(\xi_1, \xi_2)$ also by (2.2) but now we set $\tau \in (\pi - \tau_0, 2\pi - \tau_0]$ where τ_0 is choosen in the interval $[\pi, 2\pi)$. In the same manner as in the above we deduce that (2.5) is satisfied for this case.

2^0 If $\operatorname{Re}\xi_2 < \operatorname{Re}\xi_1$, then repeating exactly considerations from Part 1^0, with ξ_1 in place of ξ_2 and vice versa, we have that (2.5) holds also.

In consequence, from (2.5) it follows that every arc $\gamma_\tau \in \Gamma_h(\xi_1, \xi_2)$ is mapped by every function $f \in \mathcal{C}(\delta, \xi_1, \xi_2)$, for δ given by (2.3), onto an analytic arc $f(\gamma_\tau)$ which has with every vertical line at most one common point.

In the case when $|\xi_1| = |\xi_2| = 1$ the above geometric property implies convexity in the direction of the imaginary axis of the domain $f(U)$ (see [3]).

2. Let $0 = \xi_2 \neq \xi_1$ and $\operatorname{Re}\xi_1 \neq 0$. Let us set

$$\delta = \begin{cases} \arg\{1/\xi_1\}, & \text{if } \arg\{1/\xi_1\} \in (-\pi/2, \pi/2) \\ \arg\{1/\xi_1\} - \pi, & \text{if } \arg\{1/\xi_1\} \in (\pi/2, 3\pi/2). \end{cases} \tag{2.6}$$

Let us consider the degenerate hyperbolic family $\widetilde{\Gamma}_h(\xi_1, 0)$ of all lines $\widetilde{\gamma}$ going through the fixed point $1/\xi_1$. By $\Gamma_h(\xi_1, 0)$ we denote the family of entire segments γ obtained by the restriction of the lines $\widetilde{\gamma} \in \widetilde{\Gamma}_h(\xi_1, 0)$ to the disk U.

If $\arg\{1/\xi_1\} \in (-\pi/2, \pi/2)$, then we parametrize the family $\widetilde{\Gamma}_h(\xi_1, 0)$ as follows:

$$\widetilde{\gamma} = \widetilde{\gamma}_\tau : \; z = z_\tau(t) = \frac{1}{\xi_1}(1 - tie^{i\tau}), \quad \tau \in (0, \pi], \;\; t \in (-\infty, \infty]. \tag{2.7}$$

The family $\Gamma_h(\xi_1, 0)$ will be also parametrized by (2.7), where by an easy computation of the equation $|z_\tau(t)| = 1$, we obtain $\tau \in (\tau_0, \pi - \tau_0)$, $\tau_0 = \arccos(\alpha^2)$, and $t \in (t_0(\tau), t_1(\tau))$, where $t_0(\tau) = -\sin\tau - \sqrt{\alpha^2 - \cos^2(\tau)}$, $t_1(\tau) = -\sin\tau + \sqrt{\alpha^2 - \cos^2(\tau)}$.

Since $t \in (t_0(\tau), t_1(\tau))$ is negative for each $\tau \in (\tau_0, \pi - \tau_0)$, by (2.7) and (1.2) we deduce that for every function $f \in \mathcal{C}(\delta, \xi_1, 0)$, where δ is given by (2.6), and for every arc $\gamma_\tau \in \Gamma_h(\xi_1, 0$ holds

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{dt}} \operatorname{Re} f(z_\tau(t)) &= \operatorname{Re}\left\{ \frac{\mathrm{d}}{\mathrm{dt}} \left(\frac{1}{\xi_1}(1 - tie^{i\tau}) \right) f'(z_\tau(t)) \right\} \\ &= \operatorname{Re}\left\{ -\frac{1}{\xi_1} ie^{i\tau} f'(z_\tau(t)) \right\} \\ &= -\frac{1}{t|\xi_1|} \operatorname{Re}\left\{ e^{i\delta}(1 - \xi_1 z_\tau(t)) f'(z_\tau(t)) \right\} > 0 \end{aligned} \tag{2.8}$$

for $t \in (t_0(\tau), t_1(\tau))$.

If $\arg\{1/\xi_1\} \in (\pi/2, 3\pi/2)$, then we parametrize the family $\widetilde{\Gamma}_h(\xi_1, 0)$ also by (2.7) but now we set $\tau \in (\pi, 2\pi]$. Hence, every arc $\gamma \in \Gamma_h(\xi_1, 0)$ is also parametrized by (2.7), where $\tau \in (\pi + \tau_0, 2\pi - \tau_0)$ and $t \in (t_0(\tau), t_1(\tau))$. Since now $t \in (t_0(\tau), t_1(\tau))$ is positive for each $\tau \in (\pi + \tau_0, 2\pi - \tau_0)$, it follows that (2.8) is satisfied for δ given by (2.6).

Repeating the above considerations with ξ_1 in place of ξ_2 and vice versa, we see that (2.8) is also true for the case $0 = \xi_1 \neq \xi_2$ and $\operatorname{Re}\xi_2 \neq 0$.

3. Let us assume that $\xi_1 = \xi_2 = \xi_0 \neq 0$. By (1.5) we first set $\xi_0 = \alpha e^{-i\mu}$, $\alpha \in (0, 1]$, $\mu \in [0, \pi]$, i.e $\nu = 0$. Fix $\theta \in [\mu + \arcsin\alpha, \mu + \pi - \arcsin\alpha] \setminus \{2\mu \pm \pi/2\}$. From this, $2\mu - \theta \in [\mu + \arcsin\alpha - \pi, \mu - \arcsin\alpha] \setminus \{\pm\pi/2\} \subset$

$(-\pi,\pi)\setminus\{\pm\pi/2\}$. Let us set

$$\delta = \begin{cases} 2\mu-\theta+\pi, & \text{if } 2\mu-\theta \in (-\pi,-\pi/2) \\ 2\mu-\theta, & \text{if } 2\mu-\theta \in (-\pi/2,\pi/2) \\ 2\mu-\theta-\pi, & \text{if } 2\mu-\theta \in (\pi/2,\pi). \end{cases} \tag{2.9}$$

We consider now the parabolic family $\widetilde{\Gamma}_p(\theta,\xi_0)$ of all circles $\widetilde{\gamma}$ containing fixed point $1/\xi_0$ and tangent at $1/\xi_0$ to the line having the direction θ. Let us denote by $\Gamma_p(\theta,\xi_0)$ the family of entire circular arcs γ which are the restrictions of the circles $\widetilde{\gamma} \in \widetilde{\Gamma}_p(\theta,\xi_0)$ to the disk U.

We parametrize the family $\widetilde{\Gamma}_p(\theta,\xi_0)$ as follows:

$$\widetilde{\gamma} = \widetilde{\gamma}_\tau : z = z_\tau(t) = c\tau(1+e^{it})ie^{i\theta} + 1/\xi_0, \qquad \tau \in (-\infty,\infty),\ \ t \in [0,2\pi), \tag{2.10}$$

where $c = 1$ or $c = -1$. Let every arc $\gamma_\tau \in \Gamma_p(\theta,\xi_0)$ be also parametrized by (2.10) where $\tau \in \mathcal{I} \subseteq (-\infty,\infty)$ and $t \in \mathcal{J}(\tau) \subseteq [0,2\pi)$ for every fixed $\tau \in \mathcal{I}$. Since the line of the direction θ has no common points with the disk U we see that $\mathcal{I} \subseteq (-\infty,0)$ or $\mathcal{I} \subseteq (0,\infty)$.

Assume now that $2\mu-\theta \in (-\pi,-\pi/2) \cup (\pi/2,\pi)$. We set then $c = -1$ in (2.10). By (2.10) we see that the open halfline $\{z_\tau(0) : \tau \in (-\infty,0)\}$ and the disk U lie in the same halfplane which has the line of the direction θ as its boundary. In consequence, $\mathcal{I} \subseteq (-\infty,0)$.

From (2.10) it follows that

$$(1-\xi_0 z_\tau(t))^2 = -4\xi_0^2\tau^2\cos^2(t/2)e^{it}e^{2i\theta}. \tag{2.11}$$

Therefore, for every arc $\gamma_\tau \in \Gamma_p(\theta,\xi_0)$ and for every function $f \in \mathcal{C}(\delta,\xi_0,\xi_0)$, where δ is given by (2.9), we conclude from (2.10), (2.11), (1.2) and from the fact that $\tau \in \mathcal{I}$ is negative that

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{dt}}\operatorname{Re} f(z_\tau(t)) \quad & \\ &= \operatorname{Re}\left\{\frac{\mathrm{d}}{\mathrm{dt}}(-\tau(1+e^{it})ie^{i\theta}+1/\xi_0)f(z_\tau(t))\right\} \\ &= \operatorname{Re}\left\{\tau e^{i\theta}e^{it}f'(z_\tau(t))\right\} \\ &= \operatorname{Re}\left\{\left(\frac{-e^{-i(\theta+\delta)}}{4\tau\xi_0^2\cos^2(t/2)}\right)e^{i\delta}(-4\xi_0^2\tau^2\cos^2(t/2)e^{it}e^{2i\theta})f'(z_\tau(t))\right\} \\ &= \frac{-1}{4|\xi_0|^2\tau\cos^2(t/2)}\operatorname{Re}\{e^{i\delta}(1-\xi_0 z_\tau(t))^2 f'(z_\tau(t))\} > 0,\ t \in \mathcal{J}(\tau). \end{aligned} \tag{2.12}$$

In the case when $2\mu-\theta \in (-\pi/2,\pi/2)$, we set $c = 1$ in (2.10). For this reason the open halfline $\{z_\tau(0) : \tau \in (0,\infty)\}$ and the disk U lie in the same halfplane which has the line of the direction θ as the boundary. Consequently, $\mathcal{I} \subseteq (0, \infty)$. On account of this, (2.12) is also satisfied for this case.

The same inequality (2.12) can be drawn for $\xi_0 = -\alpha e^{-i\mu}$, $\alpha \in (0,1]$, $\mu \in [0, \pi]$, i.e. for $\nu = \pi$, by similar considerations to that above.

4. Let $\xi_1 = \xi_2 = 0$ and $\delta \in (-\pi/2, \pi/2)$ be arbitrary. Let us take into account the family $\Gamma_p(\delta, 0)$ of the entire segments γ parametrized as follows:

$$\begin{gathered} \gamma = \gamma_\tau : z = z_\tau(t) = e^{i\delta}(t + i\tau), \\ \tau \in (-1,1),\ t \in (-t_0, t_0),\ t_0 = \sqrt{1-\tau^2} \end{gathered} \tag{2.13}$$

($\Gamma_p(\delta, 0)$ is the subfamily of the degenerate parabolic family $\widetilde{\Gamma}_p(\delta, 0)$ which contains all lines of the direction δ).

For every arc $\gamma_\tau \in \Gamma_p(\delta, 0)$ and for every function $f \in C(\delta, 0, 0)$ by (2.13) and (1.2) we have

$$\frac{\mathrm{d}}{\mathrm{d}t} \operatorname{Re} f(z_\tau(t)) = \operatorname{Re}\left\{e^{i\delta} f'(z_\tau(t))\right\} > 0,\ t \in (-t_0, t_0). \tag{2.14}$$

Finally, the inequalities (2.5), (2.8), (2.12) and (2.14) may be summarized geometrically by saying that

COROLLARY 2.1. 1. *Every function $f \in C(\delta, \xi_1, \xi_2)$, for fixed ξ_1, $\xi_2 \in \overline{U}$ such that $\operatorname{Re}\xi_1 \neq \operatorname{Re}\xi_2$ and for $\delta \in (-\pi/2, \pi/2)$ suitable choosen according to ξ_1 and ξ_2 maps a certain hyperbolic family of circular arcs lying in the disk U and dependent on ξ_1 and ξ_2 onto the family of analytic arcs each of them have with every vertical line at most one common point.*

2. *Every function $f \in C(\delta, \xi_0, \xi_0)$ for fixed $\xi_0 \in \overline{U}$ and $\delta \in (-\pi/2, \pi/2)$ suitable choosen according to ξ_0 and fixed direction $\theta \in [0, 2\pi)$ maps a certain parabolic family of circular arcs lying in the disk U and dependent on θ and ξ_0 onto the family of analytic arcs each of them have with every vertical line at most one common point.*

THEOREM 2.2. *If $f \in C(\delta, \xi_1, \xi_2)$, $\delta \in [-\pi/2, \pi/2]$, $\xi_1, \xi_2 \in \overline{U}$, then f is univalent in U.*

Proof. By the fact that $|\xi_1| \le 1$ and $|\xi_2| \le 1$ it follows from (1.9) that

$$\begin{aligned} \operatorname{Re}\left\{\frac{zh'(\xi_1,\xi_2;z)}{h(\xi_1,\xi_2;z)}\right\} &= \operatorname{Re}\frac{1-\xi_1\xi_2 z^2}{(1-\xi_1 z)(1-\xi_2 z)} \\ &= \frac{1}{2}\operatorname{Re}\left\{\frac{1+\xi_1 z}{1-\xi_1 z} + \frac{1+\xi_2 z}{1-\xi_2 z}\right\} > 0, \quad z \in U. \end{aligned}$$

Hence the function $h(\xi_1, \xi_2; z)$, $z \in U$, is starlike and univalent in U for every fixed ξ_1, $\xi_2 \in \overline{U}$.

For each $\xi_0 \in \overline{U}$ the function $f_{-\pi/2,\xi_0,\xi_0}$ defined by (1.3) is evidently convex and univalent in U. Therefore, by Remark 1.2 the classes $C(-\pi/2,\xi_0,\xi_0)$ and $C(\pi/2,\xi_0,\xi_0)$ contain only univalent functions.

For every $\xi_1,\ \xi_2 \in \overline{U}$ such that $\xi_1 \neq \xi_2$ by (1.4) holds

$$f_{-\pi/2,\xi_1,\xi_2}(z) = \int_0^z \frac{h(\xi_1,\xi_2;u)}{u}du,\ z \in U.$$

In consequence, by Alexander's Theorem ([2], vol. I, p. 115), the function $f_{-\pi/2,\xi_1,\xi_2}$ is convex and univalent in U. Hence, and on account of Remark 1.2 the classes $C(-\pi/2,\xi_1,\xi_2)$ and $C(\pi/2,\xi_1,\xi_2)$ contain only univalent functions.

Let us assume now that $\delta \in (-\pi/2,\pi/2)$. By this, for every function $f \in C(\delta,\xi_1,\xi_2)$ holds

$$\mathrm{Re}\left\{e^{i\delta}\frac{zf'(z)}{h(\xi_1,\xi_2;z)}\right\} = \mathrm{Re}\{e^{i\delta}(1-\xi_1 z)(1-\xi_2 z)f'(z)\} > 0,\ z \in U.$$

Hence it follows that f is close-to-convex and univalent in U (see [4], [2, vol. II, p. 51]).

Putting $p_\varepsilon(z) = (1+\varepsilon z)/(1-\varepsilon z)$, $\varepsilon \in T$, $z \in U$, to (1.7) we see that for each $\delta \in (-\pi/2,\pi/2)$, $\xi_1,\ \xi_2 \in \overline{U}$, the function $f_{\delta,\xi_1,\xi_2}(\varepsilon;z)$, where $\varepsilon \in T$ is fixed and $z \in U$, being the solution of the equation

$$f'_{\delta,\xi_1,\xi_2}(\varepsilon;z) = \frac{1+e^{-2i\delta}\varepsilon z}{(1-\xi_1 z)(1-\xi_2 z)(1-\varepsilon z)}, \tag{2.15}$$

belongs to the class $C(\delta,\xi_1,\xi_2)$.

For each $\delta \in [-\pi/2,\pi/2]$ we obtain from (2.15) by a simple integration the following

COROLLARY 2.3. 1. *If* $\xi_1,\ \xi_2 \in \overline{U},\ \xi_1 \neq \xi_2,\ \varepsilon \in T$ *and* $\varepsilon \neq \xi_i,\ i = 1,\ 2,$ *then*

$$f_{\delta,\xi_1,\xi_2}(\varepsilon;z) = e^{-i\delta}\left\{\frac{1}{\xi_1-\xi_2}\left(\frac{\xi_2 e^{i\delta}+\varepsilon e^{-i\delta}}{\xi_2-\varepsilon}\log(1-\xi_2 z)\right.\right. \tag{2.16}$$
$$\left.\left.-\frac{\xi_1 e^{i\delta}+\varepsilon e^{-i\delta}}{\xi_1-\varepsilon}\log(1-\xi_1 z)\right)\right) - \frac{2\varepsilon\cos\delta}{(\varepsilon-\xi_1)(\varepsilon-\xi_2)}\log(1-\varepsilon z)\Big\},\ \log 1 = 0,\ z \in U.$$

2. *If* $\xi_1 \in T,\ \xi_2 \in \overline{U},\ \xi_1 \neq \xi_2,\ \varepsilon \in T$ *and* $\varepsilon = \xi_1$, *then*

$$f_{\delta,\xi_1,\xi_2}(\xi_1;z) = \frac{e^{-i\delta}}{\xi_1-\xi_2}\left\{\frac{2\xi_1 z\cos\delta}{1-\xi_1 z} + \frac{\xi_2 e^{i\delta}+\xi_1 e^{-i\delta}}{\xi_1-\xi_2}\log\frac{1-\xi_1 z}{1-\xi_2 z}\right\}, \tag{2.17}$$

$\log 1 = 0,\ z \in U.$

If $\xi_2 \in T,\ \xi_1 \in \overline{U},\ \xi_1 \neq \xi_2,\ \varepsilon \in T$ *and* $\varepsilon = \xi_2$, *then* $f_{\delta,\xi_1,\xi_2}(\xi_2;z)$ *is of the form* (2.17) *with* ξ_1 *in place of* ξ_2 *and vice versa.*

3. *If* $\xi_0 \in \overline{U}$, $\varepsilon \in T$ *and* $\varepsilon \neq \xi_0$, *then*

$$(2.18)\qquad f_{\delta,\xi_0,\xi_0}(\varepsilon;z)=\frac{e^{-i\delta}}{\xi_0-\varepsilon}\left\{(\xi_0e^{i\delta}+\varepsilon e^{-i\delta})\frac{z}{1-\xi_0z}+\frac{2\varepsilon\cos\delta}{\xi_0-\varepsilon}\log\frac{1-\xi_0z}{1-\varepsilon z}\right\},$$

$\log 1 = 0$, $z \in U$.

4. *If* ξ_0, $\varepsilon \in T$ *and* $\varepsilon = \xi_0$, *then*

$$(2.19)\qquad f_{\delta,\xi_0,\xi_0}(\xi_0;z)=\frac{z-i\xi_0e^{-i\delta}z^2\sin\delta}{(1-\xi_0z)^2},\ z\in U.$$

3. Coefficient formulae and estimates

In this section we deal with coefficient formulae and estimates for functions in the class $C(\delta,\xi_1,\xi_2)$.

For each $\delta \in [-\pi/2,\pi/2]$, α, $\beta \in [0,1]$ and $\nu \in [0,\pi]$ let us introduce

$$C_{\delta,\alpha,\beta}(\nu)=\bigcup_{\mu\in[0,\pi]}C(\delta,\alpha,\beta,\mu,\nu),\ C_{\delta,\alpha,\beta}=\bigcup_{\mu,\nu\in[0,\pi]}C(\delta,\alpha,\beta,\mu,\nu).$$

It is easy to check that for each ξ_1, $\xi_2 \in \overline{U}$ such that $\xi_1 \neq \xi_2$ the function (1.9) is of the form

$$(3.1)\qquad h(\xi_1,\xi_2;z)=z+\sum_{n=2}^{\infty}A_nz^n,\ z\in U,$$

where

$$(3.2)\qquad A_n=\frac{\xi_1^n-\xi_2^n}{\xi_1-\xi_2}=\frac{\beta^ne^{in\nu}-\alpha^ne^{-in\nu}}{\beta e^{i\nu}-\alpha e^{-i\nu}}e^{-i(n-1)\mu},\quad n=2,3,\ldots$$

For each $\xi_0 \in \overline{U}$ we have

$$(3.3)\qquad h(\xi_0,\xi_0;z)=\frac{z}{(1-\xi_0z)^2}=z+\sum_{n=2}^{\infty}n\xi_0^{n-1}z^n,\ z\in U.$$

Let $M[0,2\pi]$ denote the set of real-valued nondecreasing functions $m = m(t)$, $t \in [0,2\pi]$, such that $\int_0^{2\pi} dm(t) = 2\pi$.

THEOREM 3.1. *If* $f \in C(\delta,\xi_1,\xi_2)$ *for* $\delta \in [-\pi/2,\pi/2]$, ξ_1, $\xi_2 \in \overline{U}$ *such that* $\xi_1 \neq \xi_2$, *and* f *is of the form* (1.1), *then for* $n = 2,3,\ldots$, *holds*

$$(3.4)\qquad a_n=\frac{A_n}{n}+\frac{e^{-i\delta}\cos\delta}{\pi n}\int_0^{2\pi}\left(e^{-i(n-1)t}+\sum_{k=2}^{n-1}A_ke^{-i(n-k)t}\right)dm(t),$$

where $m \in M[0,2\pi]$ *and* A_k *are given by* (3.2).

If $f \in C(\delta, \xi_0, \xi_0)$, $\delta \in [-\pi/2, \pi/2]$, $\xi_0 \in \overline{U}$, *and* f *is of the form* (1.1), *then*

$$(3.5)\qquad a_n = \xi_0^{n-1} + \frac{e^{-i\delta}\cos\delta}{\pi n}\int_0^{2\pi}\Big(e^{-i(n-1)t} + \sum_{k=2}^{n-1} k\xi_0^{k-1}e^{-i(n-k)t}\Big)dm(t),$$

$n = 2, 3, \ldots, m \in M[0, 2\pi]$.

Proof. 1. From (1.4) we have that for each ξ_1, $\xi_2 \in \overline{U}$ such that $\xi_1 \neq \xi_2$ holds $f'_{-\pi/2,\xi_1,\xi_2}(z) = h(\xi_1, \xi_2; z)/z$, $z \in U$. Therefore (3.1) and (3.2) yields that the coefficients of the functions $f_{-\pi/2,\xi_1,\xi_2}$ and $f_{\pi/2,\xi_1,\xi_2}$ are of the form (3.4) for $\delta = -\pi/2$ and $\delta = \pi/2$, respectively.

In the same manner, from (1.3) it follows that for each $\xi_0 \in \overline{U}$ holds $f'_{-\pi/2,\xi_0,\xi_0}(z) = h(\xi_0, \xi_0; z)/z$, $z \in U$, which gives in view of (3.3) that the coefficients of the functions $f_{-\pi/2,\xi_0,\xi_0}$ and $f_{\pi/2,\xi_0,\xi_0}$ are of the form (3.5) for $\delta = -\pi/2$ and $\delta = \pi/2$, respectively.

2. Let now f be of the form (1.1) and $f \in C(\delta, \xi_1, \xi_2)$ for fixed $\delta \in (-\pi/2, \pi/2)$, ξ_1, $\xi_2 \in \overline{U}$ such that $\xi_1 \neq \xi_2$. Then there exists $p \in P$ of the form $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n$, $z \in U$, such that (1.10) is satisfied. Hence using (3.1) we obtain

$$(3.6)\qquad z + \sum_{n=2}^{\infty} n a_n z^n = \Big(z + \sum_{n=2}^{\infty} A_n z^n\Big)\Big(1 + e^{-i\delta}\cos\delta\sum_{n=1}^{\infty} p_n z^n\Big).$$

Comparing coefficients in (3.6) we get

$$2a_2 = A_2 + p_1 e^{-i\delta}\cos\delta, \ldots,$$

$$(3.7)\qquad n a_n = A_n + e^{-i\delta}\cos\delta\Big(p_{n-1} + \sum_{k=2}^{n-1} A_k p_{n-k}\Big), \quad n = 2, 3, \ldots$$

Using well known formulae

$$p_n = \frac{1}{\pi}\int_0^{2\pi} e^{-int}dm(t), \quad n \in \mathrm{N}, \quad m \in M[0, 2\pi],$$

for the coefficients p_n of the functions $p \in P$ (see [2], vol. I, p. 96), we have by (3.7),

$$n a_n = A_n + \frac{e^{-i\delta}\cos\delta}{\pi}\int_0^{2\pi}\Big(e^{-i(n-1)t} + \sum_{k=2}^{n-1} A_k e^{-i(n-k)t}\Big)dm(t)$$

$n = 2,\ 3, \ldots,\ m \in M[0, 2\pi]$. This gives (3.4).

3. For any function $f \in C(\delta, \xi_0, \xi_0)$, $\delta \in (-\pi/2, \pi/2)$, $\xi_0 \in \overline{U}$, we argue similar to Part 2 using now (1.10) and (3.3). Consequently we get

$$(3.8)\qquad na_n = n\xi_0^{n-1} + e^{-i\delta}\cos\delta\Big(p_{n-1} + \sum_{k=2}^{n-1} k\xi_0^{k-1}p_{n-k}\Big), \quad n = 2,3,\ldots$$

Hence we obtain (3.5).

This completes the proof of the theorem.

Let now $|\xi_1| = |\xi_2| = \alpha \in [0,1]$. If $\xi_1 \neq \xi_2$, then taking into account (1.5) we see that $\alpha \in (0,1]$ and $\nu \in (0,\pi)$. For this reason and by (3.2) we have

$$(3.9)\qquad A_n = \alpha^{n-1}e^{-i(n-1)\mu}\frac{\sin n\nu}{\sin\nu}.$$

If $\xi_1 = \xi_2 \neq 0$, then $\nu = 0$ or $\nu = \pi$ in (1.5). Thus by (3.9) and Theorem 3.1 we get the following

Corollary 3.2. *If $f \in C(\delta,\alpha,\alpha,\mu,\nu)$, $\delta \in [-\pi/2,\pi/2]$, $\alpha \in [0,1]$, $\mu \in [0,\pi]$, $\nu \in (0,\pi)$, and f is of the form* (1.1), *then*

$$(3.10)\qquad a_n = \frac{1}{n}\Bigg(\frac{\sin n\nu}{\sin\nu}\alpha^{n-1}e^{-i(n-1)\mu} + \frac{e^{-i\delta}\cos\delta}{\pi} \times \int_0^{2\pi}\Big(e^{-i(n-1)t} + \sum_{k=2}^{n-1}\frac{\sin k\nu}{\sin\nu}\alpha^{k-1}e^{-i(k-1)\mu}e^{-i(n-k)t}\Big)dm(t)\Bigg),$$

$n = 2,3,\ldots, m \in M[0,2\pi]$.

If $f \in C(\delta,\alpha,\alpha,\mu,0)$, $\delta \in [-\pi/2,\pi/2]$, $\alpha \in [0,1]$, $\mu \in [0,\pi]$, and f is of the form (1.1), *then*

$$(3.11)\qquad a_n = \alpha^{n-1}e^{-i(n-1)\mu} + \frac{e^{-i\delta}\cos\delta}{\pi n}\int_0^{2\pi}\Big(e^{-i(n-1)t} + \sum_{k=2}^{n-1}k\alpha^{k-1}e^{-i(k-1)\mu}e^{-i(n-k)t}\Big)dm(t),$$

$n = 2,3,\ldots, m \in M[0,2\pi]$.

If $f \in C(\delta,\alpha,\alpha,\mu,\pi)$, $\delta \in [-\pi/2,\pi/2]$, $\alpha \in [0,1]$, $\mu \in [0,\pi]$, and f is of the form (1.1), *then*

$$(3.12)\qquad a_n = (-1)^{n+1}\alpha^{n-1}e^{-i(n-1)\mu} + \frac{e^{-i\delta}\cos\delta}{\pi n}\int_0^{2\pi}\Big(e^{-i(n-1)t} + \sum_{k=2}^{n-1}(-1)^{k+1}k\alpha^{k-1}e^{-i(k-1)\mu}e^{-i(n-k)t}\Big)dm(t),$$

$n = 2,3,\ldots, m \in M[0,2\pi]$.

Setting $\delta = \mu - \pi/2$ and $\alpha = 1$ into formulae (3.10) - (3.12) we obtain fomulae for the coefficients in the class $C(\mu - \pi/2,1,1,\mu,\nu)$. Setting $\alpha = 0$ into (3.10)–(3.12) we get formulae for the coefficients in the class $C(\delta,0,0)$.

Especially, putting $\nu = \pi/2$ into the formula (3.10) it follows the following

COROLLARY 3.3. *If $f \in C(\delta,\alpha,\alpha,\mu,\pi/2)$, $\delta \in [-\pi/2,\pi/2]$, $\alpha \in [0,1]$, $\mu \in [0,\pi]$, and f is of the form* (1.1), *then*

$$a_{2k} = \frac{e^{-i\delta}\cos\delta}{2k\pi}\int_0^{2\pi}\Big(e^{-i(2k-1)t} + \sum_{j=2}^{k}((-1)^{j+1}\alpha^{2(j-1)}e^{-2(j-1)i\mu}e^{-i(2k-(2j-1))t})\Big)dm(t),$$

$$a_{2k+1} = \frac{1}{2k+1}\Big((-1)^k\alpha^{2k}e^{-2ki\mu} + \frac{e^{-i\delta}\cos\delta}{\pi} \times\int_0^{2\pi}\Big(e^{-2kit} + \sum_{j=2}^{k}(-1)^{j+1}\alpha^{2(j-1)}e^{-2(j-1)i\mu}e^{-2(k-j+1)it}\Big)dm(t)\Big),$$

$k \in \mathrm{N}$, $m \in M[0,2\pi]$.

As a consequence of Theorem 3.1 we can find the set of variability of the system (a_2,a_3) of the coefficients of the functions f in $C(\delta,\xi_1,\xi_2)$.

COROLLARY 3.4. *The region V_3 of values of the system (a_2,a_3) of the coefficients of the functions $f \in C(\delta,\xi_1,\xi_2)$, $\delta \in (-\pi/2,\pi/2)$, $\xi_1, \xi_2 \in \overline{U}$, of the form* (1.1) *is the closed convex hull of the curve whose equation is following*

$$w_2 = e^{-it}e^{-i\delta}\cos\delta + \frac{1}{2}(\xi_1+\xi_2),$$

$$w_3 = \frac{1}{3}(2e^{-2it}e^{-i\delta}\cos\delta + 2e^{-it}e^{-i\delta}(\xi_1+\xi_2)\cos\delta + \xi_1^2 + \xi_1\xi_2 + \xi_2^2),\ t \in [0,2\pi].$$

Using results obtained above we will find now coefficient estimates in $C(\delta,\xi_1,\xi_2)$. From Theorem 3.1 it follows immediately

THEOREM 3.5. *If $f \in C(\delta,\xi_1,\xi_2)$ for $\delta \in [-\pi/2,\pi/2]$, $\xi_1, \xi_2 \in \overline{U}$ such that $\xi_1 \neq \xi_2$, and f is of the form* (1.1), *then*

$$(3.13)\quad |a_n| \le \frac{1}{|\xi_1-\xi_2|n}\Big(|\xi_1^n-\xi_2^n| + 2\cos\delta\sum_{k=1}^{n-1}|\xi_1^k-\xi_2^k|\Big),\quad n = 2,3,\dots$$

If $f \in C(\delta,\xi_0,\xi_0)$, $\delta \in [-\pi/2,\pi/2]$, $\xi_0 \in \overline{U}$, and f is of the form (1.1), *then*

$$(3.14)\quad |a_n| \le \begin{cases} |\xi_0|^{n-1} + 2\cos\delta\dfrac{1-n|\xi_0|^{n-1}+(n-1)|\xi_0|^n}{(1-|\xi_0|)^2 n}, & \text{for } |\xi_0| \neq 1 \\ 1+(n-1)\cos\delta, & \text{for } |\xi_0| = 1, \end{cases}$$

$n = 2,3,\dots$

Estimates (3.13)–(3.14) are not sharp in general. They are sharp only for some systems of parameters δ, ξ_1 and ξ_2 or for some coefficients. Trivially, on account of Remark 1.2 and (3.1) - (3.3) the bounds (3.13)–(3.14) are sharp for $\delta = \pm\pi/2$.

Estimates (3.14) are sharp for $\delta = 0$ and for each $\xi_0 \in \overline{U}$. In this case, setting $\xi_0 = |\xi_0|e^{i\varphi}$, $\varphi \in [0, 2\pi)$, we get in view of (3.8) that the equality in (3.14) is attained when $p_n = 2e^{-in\varphi}$ for all $n = 2, 3, \ldots$ For this reason the extremal function is of the form (2.18) or (2.19) for $\varepsilon = e^{-i\varphi}$.

By the same argument as the above the bound (3.13) is sharp for $\delta = 0$, $\xi_2 = 0$ and for each $\xi_1 \in \overline{U}$, $\xi_1 \neq 0$ (or for $\xi_1 = 0$ and for each $\xi_2 \in \overline{U}$, $\xi_2 \neq 0$). By (3.2) we get $A_n = \xi_1^{n-1}$. If now $\xi_1 = |\xi_1|e^{i\varphi}$, $\varphi \in [0, 2\pi)$, then (3.7) yields that the extremal function is of the form (2.16) or (2.17) for $\varepsilon = e^{-i\varphi}$.

For $\xi_1 = \xi_2 = 0$ we obtain from (3.14) the following

COROLLARY 3.6. *If* $f \in \mathcal{C}(\delta, 0, 0)$, $\delta \in [-\pi/2, \pi/2]$, *and* f *is of the form* (1.1), *then*

$$|a_n| \le \frac{2}{n}\cos\delta, \quad n = 2, 3, \ldots \tag{3.15}$$

The case $\delta = 0$ in (3.15) was proved by MacGregor [8]. From (3.8) it follows immediately that bounds (3.15) are sharp for all $\delta \in [-\pi/2, \pi/2]$. Equality is realized by the function (2.18) for $\xi_0 = 0$ and $\varepsilon = 1$.

From Theorem 3.5 we have the following estimate of the second coefficient in the class $\mathcal{C}(\delta, \xi_1, \xi_2)$ which is sharp for all δ, ξ_1 and ξ_2.

COROLLARY 3.7. *If* $f \in \mathcal{C}(\delta, \xi_1, \xi_2)$, $\delta \in [-\pi/2, \pi/2]$, $\xi_1, \xi_2 \in \overline{U}$, *and* f *is of the form* (1.1), *then*

$$|a_2| \le \frac{1}{2}|\xi_1 + \xi_2| + \cos\delta. \tag{3.16}$$

By (3.2) we have $A_2 = \xi_1 + \xi_2 = |\xi_1 + \xi_2|e^{i\varphi}$, $\varphi \in [0, 2\pi)$. Then by (3.7) equality in (3.16) is realized when $p_1 = 2e^{i(\varphi+\delta)}$. For this reason the extremal function for the estimate (3.16) is according to ξ_1 and ξ_2 one of the form (2.16)–(2.19) for $\varepsilon = e^{i(\varphi+\delta)}$.

From Corollary 3.2 we get estimates of the coefficients in the class $\mathcal{C}(\delta, \xi_1, \xi_2)$, where $|\xi_1| = |\xi_2| = \alpha \in [0, 1]$, especially for $\xi_1 = -\xi_2$. We formulate these results for the classes $C_{\delta,\alpha,\alpha}(\nu)$.

COROLLARY 3.8. *If* $f \in C_{\delta,\alpha,\alpha}(\nu)$, $\delta \in [-\pi/2, \pi/2]$, $\alpha \in [0, 1]$, $\nu \in (0, \pi)$, *and* f *is of the form* (1.1), *then*

$$(3.17) \qquad |a_n| \leq \frac{1}{n}\left(\left|\frac{\sin n\nu}{\sin\nu}\right|\alpha^{n-1} + 2\cos\delta\left(1 + \sum_{k=2}^{n-1}\left|\frac{\sin k\nu}{\sin\nu}\right|\alpha^{k-1}\right)\right),$$

$n = 2, 3, \ldots$

If $\nu = 0$ or $\nu = \pi$, then the estimates in the classes $C_{\delta,\alpha,\alpha}(0)$ and $C_{\delta,\alpha,\alpha}(\pi)$ reduce to (3.14), where $|\xi_0| = \alpha \in [0,1]$.

From (3.14) and (3.17) it follows that (3.14) give estimates in the classes $C_{\delta,\alpha,\alpha}$. Putting $\delta = \mu - \pi/2$ into (3.14) and (3.17) we get estimates in the classes $C(\mu - \pi/2, \alpha, \alpha, \mu, \nu)$. Especially, for $\alpha = 1$ we have bounds of the coefficients in the classes $C(\mu - \pi/2, 1, 1, \mu, \nu)$, thus in the class $CV(i)$ (Robertson [10], Royster and Ziegler [11]). Using a Lemma due to Gronwall and (3.17) Robertson examined also an asymptotic bound for $|a_n|$. He proved that

$$\overline{\lim_{n\to\infty}}|a_n| \leq \frac{4}{\pi}\frac{\sin\mu}{\sin\nu}, \quad \nu \in (0,\pi),$$

where a_n are the coefficients of functions in the class $CV(i)$.

From (3.13) and (3.14) we deduce

Corollary 3.9. 1. *If $f \in C(\delta, \xi_1, \xi_2)$, $\delta \in [-\pi/2, \pi/2]$, $\xi_1 \in T$, $\xi_2 \in U$ (or $\xi_2 \in T$ and $\xi_1 \in U$), and f is of the form* (1.1), *then*

$$\overline{\lim_{n\to\infty}}|a_n| \leq \frac{2\cos\delta}{|\xi_1 - \xi_2|}.$$

2. *If $f \in C(\delta, \xi_1, \xi_2)$, $\delta \in [-\pi/2, \pi/2]$, ξ_1, $\xi_2 \in U$, and f is of the form* (1.1), *then*

$$\lim_{n\to\infty}|a_n| = 0.$$

3. *If $f \in C(\delta, \xi_0, \xi_0)$, $\delta \in [-\pi/2, \pi/2]$, $\xi_0 \in U$, and f is of the form* (1.1), *then*

$$\lim_{n\to\infty}|a_n| = 0.$$

Setting $\nu = \pi/2$ we obtain from Corollary 3.8 the following

Corollary 3.10. *If $f \in C_{\delta,\alpha,\alpha}(\pi/2)$, $\delta \in [-\pi/2, \pi/2]$, $\alpha \in [0,1]$, and f is of the form* (1.1), *then, for all $k \in \mathbf{N}$,*

$$(3.18) \qquad |a_{2k}| \leq \begin{cases} \dfrac{1-\alpha^{2k}}{(1-\alpha^2)k}\cos\delta, & \text{for } \alpha \in [0,1) \\ \cos\delta, & \text{for } \alpha = 1, \end{cases}$$

$$(3.19) \quad |a_{2k+1}| \leq \begin{cases} \dfrac{2\cos\delta + (1-2\cos\delta)\alpha^{2k} - \alpha^{2(k+1)}}{(1-\alpha^2)(2k+1)}, & \text{for } \alpha \in [0,1) \\ \dfrac{2k\cos\delta + 1}{2k+1}, & \text{for } \alpha = 1. \end{cases}$$

The case $\alpha = 1$ in (3.18) - (3.19) is due to Hengartner and Schober [3].

Estimates (3.18) are sharp. For $\alpha \in (0,1)$ the function (2.16) for $\xi_1 = \alpha$, $\xi_2 = -\alpha$ ($\mu = \pi/2$ in (1.5)) and $\varepsilon = 1$, is extremal. For $\alpha = 0$ the function (2.18) for $\xi_0 = 0$ and $\varepsilon = 1$ is extremal. For $\alpha = 1$ the function (2.17) for $\xi_1 = 1$, $\xi_2 = -1$ and $\varepsilon = 1$ is extremal.

From (3.19) we get the sharp bound for the third coefficient a_3 for all $\alpha \in [0,1]$,

$$|a_3| \leq \frac{1}{3}(2\cos\delta + \alpha^2).$$

The extremal function is one of the form (2.16)–(2.18) for $\varepsilon = e^{i\delta/2}$.

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DEPARTMENT OF MATHEMATICS
TECHNICAL UNIVERSITY OF RZESZÓW
ul. W. Pola 2
35-959 RZESZÓW, POLAND
E-mail: alecko@prz.rzeszow.pl

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