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A NOTE ON A SERIES OF PAPERS BY BURMEISTER AND WOJDYŁO

1. Introduction

In their series of papers [6], P. Burmeister and B. Wojdyło thoroughly study some basic category theoretical concepts in five categories "which are likely to be the most natural ones when dealing with partial algebras" (*loc. cit.*, Introduction). These categories have as objects all partial τ-algebras, for a given similarity type τ, and they differ in their morphisms. Two of them have as morphisms the two usual types of total homomorphisms: plain homomorphisms and closed homomorphisms. And the remaining three categories have as morphisms some special types of partial homomorphisms: quomorphisms (plain homomorphisms from a relative subalgebra), conformisms (closed homomorphisms from a weak subalgebra) and closed quomorphisms (closed homomorphisms from a relative subalgebra). Let these five categories be denoted, for a given similarity type τ, by $\mathfrak{Hom}(\tau)$, $\mathfrak{C}$-$\mathfrak{Hom}(\tau)$, $\mathfrak{Quom}(\tau)$, $\mathfrak{Conf}(\tau)$ and $\mathfrak{C}$-$\mathfrak{Quom}(\tau)$, respectively.

Recently, and through a work aimed at the generalization of graph grammars to transformation systems of partial algebras (see the survey [2]), it has appeared that there are two more types of partial homomorphisms that may have some interest, at least from the point of view of their use in single-pushout [10] and single-pullback [3] hypergraph algebraic transformation systems, and that seem not to have been considered previously in the literature. Namely, there are what we shall call *closed-domain quomorphisms* (*cd-quomorphisms* for short), plain homomorphisms from a closed subalgebra, and what we shall call *closed-domain closed quomorphisms* (*cdc-quomorphisms* for short), closed homomorphisms from a closed subalgebra (they are called "closed-domain quomorphic conformisms" in [2] and "partial closed homomorphisms" in [14]).

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We want to point out here that [2, Prop. 2.1] entails that cdc-quomorphisms are (as far as graph transformation goes) the right generalization to partial algebras of M. Löwe's partial homomorphisms for total algebras [10, 11], which are the current basis of the Berlin approach to single-pushout graph transformation; see [7] for a survey on this topic. These cdc-quomorphisms can also be understood as the partial morphisms associated to closed homomorphisms, in the sense of [9].

As it is done in [2] for the categories considered therein, we study here the existence of limits and colimits in the categories $\mathfrak{CD}$-$\mathfrak{Quom}(\tau)$ and $\mathfrak{CDC}$-$\mathfrak{Quom}(\tau)$ of partial τ-algebras with cd-quomorphisms and cdc-quomorphisms, respectively, as morphisms. This study is also motivated by the fact that, in the context of algebraic transformation systems, the main features one asks to a category of "partial homomorphisms" are related to completeness and cocompleteness ([10, 8, 13]). It turns out that $\mathfrak{CDC}$-$\mathfrak{Quom}(\tau)$ is complete and cocomplete iff [(1)] all operations in τ are unary (i.e., iff τ is a graph structure in the sense of [10, 11]), while $\mathfrak{CD}$-$\mathfrak{Quom}(\tau)$ is never (even finitely) complete or cocomplete, unless there is no operation in the similarity type.

So, this paper is to be understood as a sequel of [6] (and, in a single point, of [12], where the cartesian closedness of the categories considered in [6] is studied). Therefore, we shall make free use of the notations introduced in [6], and we shall even refer the reader, when possible or suitable, to proofs and examples from therein. There is one convention, not explicitly stated in [6], that shall be used systematically here, usually without any further mention: given a partial algebra denoted by a capital letter in boldface type (**A**, **B**, etc.), we shall always denote its universe by the same capital letter, but in slanted type (A, B, etc.).

The similarity types considered in this paper, as in [6], are one-sorted and with all its operations finitary, although there may be infinitely many such operations. Nevertheless, as it is also pointed out in *loc. cit.*, all results generalize in an easy way to more general settings.

We shall assume on the reader's side a working knowledge of the language of partial algebras, and we refer her/him to [4] or [5] for any concept not defined either here or in [6].

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(1) if and only if

led to a significant improvement of it. The counterexamples for the unary case in the proofs of Propositions 7.(b) and 13.(b) are essentially due to him.

2. Main results

Let $\tau = (n_\varphi)_{\varphi\in\Omega}$ be in the sequel a similarity type of partial algebras, with set of operations Ω. For every $n \in \mathbb{N}$, let $\Omega^{(n)}$ denote the set of operation symbols $\{\varphi \in \Omega \mid n_\varphi = n\}$.

Let $\mathbf{A} = (A, (\varphi^{\mathbf{A}})_{\varphi\in\Omega})$ and $\mathbf{B} = (B, (\varphi^{\mathbf{B}})_{\varphi\in\Omega})$ be two *partial τ-algebras* (partial algebras of similarity type τ), let $f : A \to B$ be a partial mapping with domain D, and let $\mathbf{D}$ be the relative subalgebra of $\mathbf{A}$ supported on D.

We shall say that f is a *closed-domain quomorphism*, *cd-quomorphism* for short, when $\mathbf{D}$ is a closed subalgebra of $\mathbf{A}$ and $f : \mathbf{D} \to \mathbf{B}$ is a (plain) homomorphism. In other words, when it satisfies the following condition:

for every $\varphi \in \Omega$ and $\underline{a} \in D^{n_\varphi}$, if $\underline{a} \in \operatorname{dom}\varphi^{\mathbf{A}}$ then $\varphi^{\mathbf{A}}(\underline{a}) \in D$, $f(\underline{a}) \in \operatorname{dom}\varphi^{\mathbf{B}}$ and $\varphi^{\mathbf{B}}(f(\underline{a})) = f(\varphi^{\mathbf{A}}(\underline{a}))$.

In particular, if $\varphi_0 \in \Omega^{(0)}$ and if $\varphi_0^{\mathbf{A}}$ is defined then $\varphi_0^{\mathbf{A}} \in D$, $\varphi_0^{\mathbf{B}}$ is defined and $f(\varphi_0^{\mathbf{A}}) = \varphi_0^{\mathbf{B}}$.

We shall say that f is a *closed-domain closed quomorphism*, *cdc-quomorphism* for short, when $\mathbf{D}$ is a closed subalgebra of $\mathbf{A}$ and $f : \mathbf{D} \to \mathbf{B}$ a closed homomorphism. In other words, when it satisfies the following condition:

for every $\varphi \in \Omega$ and $\underline{a} \in D^{n_\varphi}$, $\underline{a} \in \operatorname{dom}\varphi^{\mathbf{A}}$ iff $f(\underline{a}) \in \operatorname{dom}\varphi^{\mathbf{B}}$; and then $\varphi^{\mathbf{A}}(\underline{a}) \in D$ and $\varphi^{\mathbf{B}}(f(\underline{a})) = f(\varphi^{\mathbf{A}}(\underline{a}))$.

In particular, if $\varphi_0 \in \Omega^{(0)}$ then $\varphi_0^{\mathbf{A}}$ is defined iff $\varphi_0^{\mathbf{B}}$ is defined, and then $\varphi_0^{\mathbf{A}} \in D$ and $f(\varphi_0^{\mathbf{A}}) = \varphi_0^{\mathbf{B}}$.

It is straightforward to prove that cd-quomorphisms (resp. cdc-quomorphisms) are closed under composition and that the identity is a cd-quomorphism (resp. a cdc-quomorphism). Let $\mathfrak{CD}$-$\mathfrak{Quom}(\tau)$ and $\mathfrak{CDC}$-$\mathfrak{Quom}(\tau)$ be the categories with objects all partial τ-algebras and with morphisms the cd-quomorphisms and the cdc-quomorphisms, respectively.

The rest of this paper is devoted to establish the main properties of these categories.

PROPOSITION 1 (**Isomorphisms**). *The isomorphisms in $\mathfrak{CDC}$-$\mathfrak{Quom}(\tau)$ and in $\mathfrak{CD}$-$\mathfrak{Quom}(\tau)$ are the bijective closed homomorphisms.*

Proof. Any invertible partial homomorphism (of any kind) has to be totally defined, as well as its inverse, and therefore the isomorphisms in $\mathfrak{CDC}$-$\mathfrak{Quom}(\tau)$ and $\mathfrak{CD}$-$\mathfrak{Quom}(\tau)$ are exactly the isomorphisms in $\mathfrak{Hom}(\tau)$. ∎

PROPOSITION 2 (**Monomorphisms**). a) *The monomorphisms in $\mathfrak{CDC}$-$\mathfrak{Quom}(\tau)$ are the injective closed homomorphisms.*

b) *The monomorphisms in* $\mathfrak{CD}$-$\mathfrak{Quom}(\tau)$ *are the injective (plain) homomorphisms.*

Proof. a) Any injective closed homomorphism is a monomorphism in $\mathfrak{CDC}$-$\mathfrak{Quom}(\tau)$. Conversely, on one hand the proof given in [6, Prop. 2] for closed quomorphisms (with a slight change: with the notations therein, one must take now the domain of h_1, instead of the whole term algebra, as source algebra of h_1 and h_2) shows that any monomorphism in $\mathfrak{CDC}$-$\mathfrak{Quom}(\tau)$ has to be totally defined, i.e. a closed homomorphism. And, on the other hand, from *loc. cit.* we also know that a non-injective closed homomorphism cannot be a monomorphism in $\mathfrak{C}$-$\mathfrak{Hom}(\tau)$, and therefore neither in $\mathfrak{CDC}$-$\mathfrak{Quom}(\tau)$.

b) A proof similar to the previous one applies. ■

PROPOSITION 3 (**Epimorphisms**). a) *A cdc-quomorphism* $f : \mathbf{A} \to \mathbf{B}$ *is an epimorphism in* $\mathfrak{CDC}$-$\mathfrak{Quom}(\tau)$ *iff it is surjective.*

b) *A cd-quomorphism* $f : \mathbf{A} \to \mathbf{B}$ *is an epimorphism in* $\mathfrak{CD}$-$\mathfrak{Quom}(\tau)$ *iff for every subset* $X \subseteq f(A)$ *there is at most one closed subset* B_X *of* $\mathbf{B}$ *such that* $B_X \cap f(A) = X$.

Proof. a) The proof of the closed quomorphisms case in [6, Prop. 3] can be used here without any change, because the closed quomorphisms h_1 and h_2 used therein to prove that if f is not surjective then it cannot be an epimorphism, are cdc-quomorphisms if f is so.

b) As to the direct implication, assume that there exist two different closed subsets C_1 and C_2 of $\mathbf{B}$ such that $C_1 \cap f(A) = C_2 \cap f(A)$. Let $g_i : \mathbf{B} \to \mathbf{B}$, $i = 1, 2$, be the identity on C_i. Such g_1 and g_2 are two different cd-quomorphisms such that $g_1 \circ f = g_2 \circ f$. Therefore f is not an epimorphic cd-quomorphism.

Conversely, assume that for every subset $X \subseteq f(A)$ there is at most one closed subset of $\mathbf{B}$ intersecting $f(A)$ in X, and let $g_1, g_2 : \mathbf{B} \to \mathbf{C}$ be two cd-quomorphisms of partial τ-algebras such that $g_1 \circ f = g_2 \circ f$. Then $\operatorname{dom} g_1$ and $\operatorname{dom} g_2$ are closed subsets of $\mathbf{B}$ such that $(\operatorname{dom} g_1) \cap f(A) = (\operatorname{dom} g_2) \cap f(A)$ and therefore $\operatorname{dom} g_1 = \operatorname{dom} g_2$. Moreover, $\{b \in \operatorname{dom} g_1 = \operatorname{dom} g_2 \mid g_1(b) = g_2(b)\}$ is a closed subset of $\mathbf{B}$ (cf. [4, Prop. 3.5.3]) whose intersection with $f(A)$ is again $(\operatorname{dom} g_1) \cap f(A)$. Therefore $\{b \in \operatorname{dom} g_1 = \operatorname{dom} g_2 \mid g_1(b) = g_2(b)\} = \operatorname{dom} g_1$ and thus $g_1 = g_2$. ■

Remark. Every surjective cd-quomorphism is epimorphic, and every epimorphic cd-quomorphism is dense in the sense of [4, Prop. 3.6.1] and [6, Prop. 3], but it is easy to produce examples showing that these implications are strict.

PROPOSITION 4 (**Terminal object**). a) $\mathfrak{CDC}$-$\mathfrak{Quom}(\tau)$ *has a terminal object iff* $\Omega^{(0)} = \emptyset$; *and if it exists then it is the empty algebra.*

b) $\mathfrak{CD}$-$\mathfrak{Quom}(\tau)$ *has a terminal object iff* $\Omega^{(0)} = \emptyset$; *and if it exists then it is the empty algebra.*

Proof. a) Assume that $\varphi_0 \in \Omega^{(0)}$ and that $\mathbf{T}$ is a terminal algebra in $\mathfrak{CDC}$-$\mathfrak{Quom}(\tau)$. Then $\varphi_0^{\mathbf{T}}$ has to be defined (in order to accept a cdc-quomorphism from an algebra $\mathbf{A}$ with $\varphi_0^{\mathbf{A}}$ defined) as well as undefined (in order to accept a cdc-quomorphism from an algebra $\mathbf{B}$ with $\varphi_0^{\mathbf{B}}$ undefined). This shows that if $\mathfrak{CDC}$-$\mathfrak{Quom}(\tau)$ has terminal object then $\Omega^{(0)} = \emptyset$. And it is clear that if $\Omega^{(0)} = \emptyset$ then the empty algebra is a terminal object in $\mathfrak{CDC}$-$\mathfrak{Quom}(\tau)$.

b) Assume again that $\varphi_0 \in \Omega^{(0)}$ and that $\mathbf{T}$ is a terminal algebra in $\mathfrak{CD}$-$\mathfrak{Quom}(\tau)$. Let $\mathbf{A}$ be a partial τ-algebra with carrier $A = \{a, b\}$, with $\varphi_0^{\mathbf{A}} = a$ and with all other operations discrete. $\mathbf{T}$ being terminal, $\varphi_0^{\mathbf{T}}$ has to be defined, say $\varphi_0^{\mathbf{T}} = t \in T$. But then there are (at least) two cd-quomorphisms $h_1, h_2 : \mathbf{A} \to \mathbf{T}$; namely, h_1 with $\operatorname{dom} h_1 = \{a\}$ and $h_1(a) = t$, and h_2 with $\operatorname{dom} h_2 = \{a, b\}$ and $h_2(a) = h_2(b) = t$. This shows that if $\mathfrak{CD}$-$\mathfrak{Quom}(\tau)$ has terminal object then $\Omega^{(0)} = \emptyset$. As before, the converse implication is straightforward. ∎

PROPOSITION 5 (**Initial object**). a) $\mathfrak{CDC}$-$\mathfrak{Quom}(\tau)$ *has an initial object iff* $\Omega^{(0)} = \emptyset$; *and if it exists then it is the empty algebra.*

b) $\mathfrak{CD}$-$\mathfrak{Quom}(\tau)$ *has always an initial object, and it is the empty algebra.*

Proof. The proof of (a) is similar to that of point (a) in the last Proposition, and (b) is straightforward. ∎

COROLLARY 6 (**Zero object**). a) $\mathfrak{CDC}$-$\mathfrak{Quom}(\tau)$ *has a zero object iff* $\Omega^{(0)} = \emptyset$; *and if it exists then it is the empty algebra.*

b) $\mathfrak{CD}$-$\mathfrak{Quom}(\tau)$ *has a zero object iff* $\Omega^{(0)} = \emptyset$; *and if it exists then it is the empty algebra.* ∎

Since we have already dealt with terminal objects (Proposition 4), in the next proposition we only study the existence of products of families with non-empty index set.

PROPOSITION 7 (**Products (of non-empty families)**).

a) $\mathfrak{CDC}$-$\mathfrak{Quom}(\tau)$ *has all products of non-empty families of algebras iff* $\Omega = \Omega^{(1)}$. *And then, a product of a family of algebras is given by the construction described in* [6, Thm. 2] *of a product for them in* $\mathfrak{Conf}(\tau)$, *replacing everywhere in it "morphism" by cdc-quomorphism.*

b) $\mathfrak{CD}$-$\mathfrak{Quom}(\tau)$ *has all products of non-empty families of algebras iff* $\Omega = \Omega^{(0)}$. *And, in this case, a product of a family* $(\mathbf{A}_i)_{i \in I}$ *of partial* τ-*algebras* ($I \neq \emptyset$) *is obtained in the following way. Let* $(A^*, (pr_i : A^* \to A_i)_{i \in I})$ *be the product (described in [6, Thm. 2]) of the family of universes*

$(A_i)_{i\in I}$ in the category of sets with partial mappings. For every $\varphi \in \Omega^{(0)}$ set

$$\varphi^{\mathbf{A}^*} = \begin{cases} (\varphi^{\mathbf{A}_i})_{i\in I} & \text{if } \varphi^{\mathbf{A}_i} \text{ is defined for every } i \in I \\ \text{undefined} & \text{otherwise.} \end{cases}$$

Then the partial τ-algebra $\mathbf{A}^* = (A^*, (\varphi^{\mathbf{A}^*})_{\varphi\in\Omega})$ defined in this way, together with the projections pr_i, $i \in I$, is the product of $(\mathbf{A}_i)_{i\in I}$ in $\mathfrak{CD}$-$\mathfrak{Quom}(\tau)$.

Proof. a) When $\Omega^{(n)} \neq \emptyset$ for $n = 0$ or $n \geq 2$, the arguments in the proof of [6, Thm. 2] for closed homomorphisms can be used to show that there cannot exist the products in $\mathfrak{CDC}$-$\mathfrak{Quom}(\tau)$ of the partial τ-algebras $\mathbf{A}$ and $\mathbf{B}$ considered therein (in the case $n \geq 2$, one must replace in that proof $\mathbf{P}$ by the intersection of the domains of the projections).

So, in order that all (binary) products exist it must happen $\Omega = \Omega^{(1)}$. And in this case, a proof similar to the one for conformisms in *loc. cit.* shows that the same construction (taking as morphisms the cdc-quomorphisms) yields the product of a non-empty family of partial τ-algebras in $\mathfrak{CDC}$-$\mathfrak{Quom}(\tau)$.

b) Let us consider first the case $\Omega^{(n)} \neq \emptyset$, for some $n \geq 1$; as usual (in [6]), we shall assume without any loss of generality that $\Omega^{(1)} \neq \emptyset$. So, let $\varphi \in \Omega^{(1)}$, and consider a partial τ-algebra $\mathbf{A}$ with carrier $A = \{a, b\}$ and with all operations discrete except $\varphi^{\mathbf{A}}$, which is given by $\varphi^{\mathbf{A}}(a) = b$. Assume that the product $\mathbf{P}$ of $\mathbf{A}$ and $\mathbf{A}$ exists in $\mathfrak{CD}$-$\mathfrak{Quom}(\tau)$, together with projections $p_1, p_2 : \mathbf{P} \to \mathbf{A}$.

The family of cd-quomorphisms $f_1, f_2 : \mathbf{A} \to \mathbf{A}$ with $f_1 = \emptyset$ and $f_2 = \mathrm{Id}_{\mathbf{A}}$ entails the existence of an element $x \in P$ such that $p_1(x)$ is undefined and $p_2(x) = a$, and moreover such that $p_1(\varphi^{\mathbf{P}}(x))$ is again undefined and $p_2(\varphi^{\mathbf{P}}(x)) = b$. On the other hand, the family of cd-quomorphisms $g_1, g_2 : \mathbf{A} \to \mathbf{A}$, given by $g_1(b) = b$ and $g_2 = \mathrm{Id}_{\mathbf{A}}$, entails now the existence of an element $y \in P$ such that $p_1(y)$ is undefined and $p_2(y) = a$, and moreover such that $p_1(\varphi^{\mathbf{P}}(y)) = p_2(\varphi^{\mathbf{P}}(y)) = b$.

In particular, there must exist two different elements $x, y \in P$ such that $p_1(x)$ and $p_1(y)$ are undefined and $p_2(x) = p_2(y) = a$. But then there exist two different cd-quomorphisms $\mathbf{A} \to \mathbf{P}$ corresponding to the family of cd-quomorphisms $h_1, h_2 : \mathbf{A} \to \mathbf{A}$ given by $h_1 = \emptyset$ and $h_2(b) = a$. Namely, $h, h' : \mathbf{A} \to \mathbf{P}$ with $h(a)$ and $h'(a)$ undefined, and with $h(b) = x$ and $h'(b) = y$. This shows that the product of two copies of $\mathbf{A}$ in $\mathfrak{CD}$-$\mathfrak{Quom}(\tau)$ cannot exist in this case.

Now let us consider the case $\Omega = \Omega^{(0)}$. Given a non-empty family of partial τ-algebras $(\mathbf{A}_i)_{i\in I}$, let $\mathbf{A}^*$ be defined as in the statement. For every partial τ-algebra $\mathbf{C}$ and for every family of cd-quomorphisms $f_i : \mathbf{C} \to \mathbf{A}_i$, $i \in I$, there exists a unique partial mapping $f : C \to A^*$ such that $f_i = pr_i \circ f$

for every $i \in I$: namely, the one with $\operatorname{dom} f = \bigcup_{i \in I} \operatorname{dom} f_i$ and if $c \in \operatorname{dom} f$ and $I_c = \{i \in I \mid c \in \operatorname{dom} f_i\}$ then $f(c) = (f_i(c))_{i \in I_c} \in A^*$.

Since $\operatorname{dom} f_i$ is closed for every $i \in I$ and all operations in τ are nullary, $\operatorname{dom} f$ is a closed subset of $\mathbf{C}$. Moreover, if $\varphi \in \Omega^{(0)}$ is such that $\varphi^{\mathbf{C}}$ is defined then it belongs to $\operatorname{dom} f_i$ for every $i \in I$ and therefore $\varphi^{\mathbf{A}_i}$ is defined for every $i \in I$, which entails that $\varphi^{\mathbf{A}^*}$ is defined and $f(\varphi^{\mathbf{C}}) = \varphi^{\mathbf{A}^*}$. Therefore, f is a cd-quomorphism.

Since the (partial) projections $pr_i : A^* \to A_i$ are clearly cd-quomorphisms $\mathbf{A}^* \to \mathbf{A}_i$, $i \in I$, we finally conclude that $\mathbf{A}^*$, together with the projections pr_i, $i \in I$, is in this case the product of $(\mathbf{A}_i)_{i \in I}$ in $\mathfrak{CD}$-$\mathfrak{Quom}(\tau)$. ∎

Combining Propositions 4 and 7 we obtain the following.

COROLLARY 8 (**Products**). a) *$\mathfrak{CDC}$-$\mathfrak{Quom}(\tau)$ has all products iff $\Omega = \Omega^{(1)}$.*

b) *$\mathfrak{CD}$-$\mathfrak{Quom}(\tau)$ has never all products (unless $\Omega = \emptyset$).*

COROLLARY 9 (**Cartesian Closedness**). *$\mathfrak{CDC}$-$\mathfrak{Quom}(\tau)$ and $\mathfrak{CD}$-$\mathfrak{Quom}(\tau)$ are never cartesian closed.*

Proof. In order that $\mathfrak{CD}$-$\mathfrak{Quom}(\tau)$ is cartesian, it must happen that $\Omega = \emptyset$, and in order that $\mathfrak{CDC}$-$\mathfrak{Quom}(\tau)$ is cartesian it must happen that $\Omega = \Omega^{(1)}$. Therefore, when one of these categories is cartesian, it has a zero object as well as non-zero objects, and therefore it cannot be cartesian closed (see for instance [1, Ex. 27A]). ∎

PROPOSITION 10 (**Coproducts**). a) *$\mathfrak{CDC}$-$\mathfrak{Quom}(\tau)$ has all coproducts iff $\Omega = \Omega^{(1)}$. And, in this case, a coproduct of a family of algebras is given by its coproduct in $\mathfrak{Hom}(\tau)$ described in [4, §4.3].*

b) *$\mathfrak{CD}$-$\mathfrak{Quom}(\tau)$ has all coproducts iff either $\Omega^{(0)} = \emptyset$ or $\Omega^{(n)} = \emptyset$ for every $n \geq 2$. And, in this case, a coproduct of a family of algebras is given again by its coproduct in $\mathfrak{Hom}(\tau)$.*

Proof. a) In the case $\Omega^{(n)} \neq \emptyset$ for $n = 0$ or $n \geq 2$, an argument similar to the one used for closed homomorphisms in the proof of [6, Thm. 2d] shows that there do not exist all coproducts of pairs of partial algebras in $\mathfrak{CDC}$-$\mathfrak{Quom}(\tau)$.

Now, when $\Omega = \Omega^{(1)}$ it is straightforward to show that the usual "disjoint union" construction [4, Constr. 4.3.4.(a)] yields the coproduct of any nonempty family of partial algebras in $\mathfrak{CDC}$-$\mathfrak{Quom}(\tau)$. Since $\mathfrak{CDC}$-$\mathfrak{Quom}(\tau)$ has also an initial object in this case (Proposition 5), we conclude that it has all coproducts.

b) Let us consider first the case when $\Omega^{(0)} \neq \emptyset$ and $\Omega^{(n)} \neq \emptyset$ for some $n \geq 2$; as usual, we restrict ourselves to the case $n = 2$. So, let $\varphi_0 \in \Omega^{(0)}$ and $\varphi \in \Omega^{(2)}$.

Let $\mathbf{A}$ be a partial τ-algebra with carrier $A = \{a_0, a_1, a, a_0', a_1'\}$ and with all operations discrete except φ_0 and φ, which are defined as follows: $\varphi_0^{\mathbf{A}} = a_0$ and $\varphi^{\mathbf{A}}(a_0, a_0) = a_1$, $\varphi^{\mathbf{A}}(a_0, a) = a_0'$ and $\varphi^{\mathbf{A}}(a_1, a) = a_1'$. Let $\mathbf{B}$ be a partial τ-algebra with carrier $B = \{b\}$ and with all operations discrete except φ_0 and φ, which are given by $\varphi_0^{\mathbf{B}} = b$ and $\varphi^{\mathbf{B}}(b, b) = b$. Assume that the coproduct $\mathbf{C}$ of $\mathbf{A}$ and $\mathbf{B}$ exists in $\mathfrak{CD}$-$\mathfrak{Quom}(\tau)$, together with the natural injections (cd-quomorphisms) $\ell_1 : \mathbf{A} \to \mathbf{C}$ and $\ell_2 : \mathbf{B} \to \mathbf{C}$.

Since there is a (total) homomorphism from $\mathbf{A}$ to $\mathbf{B}$, both ℓ_1 and ℓ_2 would be totally defined, and then it should happen that $\ell_1(a_0) = \ell_1(a_1) = \ell_2(b)$ and $\ell_1(a_0') = \ell_1(a_1')$.

But now consider the partial τ-algebra $\mathbf{D}$ with carrier $D = \{d, d_0, d_1\}$ and all operations discrete except φ_0 and φ, which are given by $\varphi_0^{\mathbf{D}} = d$ and $\varphi^{\mathbf{D}}(d, d) = d$. Then the mappings $f : A \to D$, given by $f(a_0) = f(a_1) = d$, $f(a_0') = d_0$ and $f(a_1') = d_1$, and $g : B \to D$, given by $g(b) = d$, are cd-quomorphisms $f : \mathbf{A} \to \mathbf{D}$ and $g : \mathbf{B} \to \mathbf{D}$ and they would induce a cd-quomorphism $h : \mathbf{C} \to \mathbf{D}$ such that $h(\ell_1(a_0')) \neq h(\ell_1(a_1'))$. This yields a contradiction and therefore the coproduct $\mathbf{C}$ cannot exist.

Let us consider now the remaining cases. Since $\mathfrak{CD}$-$\mathfrak{Quom}(\tau)$ has always an initial object (Proposition 5), we shall only consider here coproducts of non-empty families. So, let $(\mathbf{A}_i)_{i\in I}$ be in the sequel a non-empty family of partial τ-algebras, and let $\bigoplus_{i\in I} \mathbf{A}_i$, together with homomorphisms $\ell_i : \mathbf{A}_i \to \bigoplus_{i\in I} \mathbf{A}_i$, $i \in I$, denote their coproduct in $\mathfrak{Hom}(\tau)$. Let $\mathbf{B}$ be any partial τ-algebra and let $f_i : \mathbf{A}_i \to \mathbf{B}$, $i \in I$, be a family of cd-quomorphisms. For every $i \in I$, let $\mathbf{D}_i$ be the closed subalgebra of $\mathbf{A}_i$ supported on $\operatorname{dom} f_i$. The total homomorphisms $f_i : \mathbf{D}_i \to \mathbf{B}$ induce a total homomorphism $f : \bigoplus_{i\in I} \mathbf{D}_i \to \mathbf{B}$.

In the case $\Omega^{(0)} = \emptyset$, we have that $\bigoplus_{i\in I} \mathbf{D}_i$ is a closed subalgebra of $\bigoplus_{i\in I} \mathbf{A}_i$. Therefore f yields a cd-quomorphism $f : \bigoplus_{i\in I} \mathbf{A}_i \to \mathbf{B}$ such that $f \circ \ell_i = f_i$ for every $i \in I$. And any cd-quomorphism $f' : \bigoplus_{i\in I} \mathbf{A}_i \to \mathbf{B}$ such that $f \circ \ell_i = f_i$ for every $i \in I$ must have $\bigoplus_{i\in I} \mathbf{D}_i$ as domain (because $\ell_i^{-1}(\operatorname{dom} f') = \operatorname{dom} f_i$ for every $i \in I$) and then such f' must be equal to f. Therefore, if $\Omega = \Omega^{(1)}$ then $\bigoplus_{i\in I} \mathbf{A}_i$ is indeed a coproduct of $(\mathbf{A}_i)_{i\in I}$ in $\mathfrak{CD}$-$\mathfrak{Quom}(\tau)$ too.

In the case $\Omega^{(0)} \neq \emptyset$, $\bigoplus_{i\in I} \mathbf{D}_i$ turns out again to be a closed subalgebra of $\bigoplus_{i\in I} \mathbf{A}_i$, but this fact is now not as straightforward as before, and we prove it in a Claim below. Knowing this claim to be true, one can continue the proof in this case as in the previous one, obtaining again that $\bigoplus_{i\in I} \mathbf{A}_i$ is a coproduct of $(\mathbf{A}_i)_{i\in I}$ in $\mathfrak{CD}$-$\mathfrak{Quom}(\tau)$.

CLAIM. *Let τ be a similarity type of algebras with $\Omega = \Omega^{(0)} \cup \Omega^{(1)}$, let $(\mathbf{A}_i)_{i\in I}$ be a non-empty family of partial τ-algebras, and for every $i \in I$*

let $\mathbf{D}_i$ *be a closed subalgebra of* $\mathbf{A}_i$. *Then* $\bigoplus_{i\in I}\mathbf{D}_i$ *is a closed subalgebra of* $\bigoplus_{i\in I}\mathbf{A}_i$.

Proof (of the Claim). If $\Omega^{(0)} = \emptyset$ then this is clear. Assume now $\Omega^{(0)} \neq \emptyset$ and let τ' be the similarity type obtained from τ by removing the operation symbols in $\Omega^{(0)}$. For every $i \in I$, let $\mathbf{A}'_i$ and $\mathbf{D}'_i$ denote respectively the τ'-reducts of $\mathbf{A}_i$ and $\mathbf{D}_i$. Then $\bigoplus_{i\in I}\mathbf{D}'_i$ is a closed τ'-subalgebra of $\bigoplus_{i\in I}\mathbf{A}'_i$.

Now let θ (resp. θ_0) denote the congruence on $\bigoplus_{i\in I}\mathbf{A}'_i$ (resp. on $\bigoplus_{i\in I}\mathbf{D}'_i$) generated by

$$X = \{((\varphi_0^{\mathbf{A}_i}, i), (\varphi_0^{\mathbf{A}_j}, j)) \mid \varphi_0 \in \Omega^{(0)} \text{ is defined both in } \mathbf{A}_i \text{ and } \mathbf{A}_j\}$$

(resp. by

$$X_0 = \{((\varphi_0^{\mathbf{D}_i}, i), (\varphi_0^{\mathbf{D}_j}, j)) \mid \varphi_0 \in \Omega^{(0)} \text{ is defined both in } \mathbf{D}_i \text{ and } \mathbf{D}_j\}).$$

Since every $\mathbf{D}_i$ is a closed subalgebra of the corresponding $\mathbf{A}_i$, we have that $X = X_0$, and since $\bigoplus_{i\in I}\mathbf{D}'_i$ is a closed τ'-subalgebra of $\bigoplus_{i\in I}\mathbf{A}'_i$, it finally turns out that, except for the diagonal pairs, $\theta = \theta_0$ (and, in particular, if $x \in \bigsqcup_{i\in I} D'_i$ then $[x]_\theta = [x]_{\theta_0}$ and if $x \notin \bigsqcup_{i\in I} D'_i$ then $[x]_\theta = \{x\}$) and $(\bigoplus_{i\in I}\mathbf{D}'_i)/_{\theta_0}$ is a closed τ'-subalgebra of $(\bigoplus_{i\in I}\mathbf{A}'_i)/_\theta$.

Recall now that $\bigoplus_{i\in I}\mathbf{A}_i$ is obtained from $(\bigoplus_{i\in I}\mathbf{A}'_i)/_\theta$ by adding the operations in $\Omega^{(0)}$ in the following way: a nullary operation φ_0 is defined in $\bigoplus_{i\in I}\mathbf{A}_i$ when it is defined in some $\mathbf{A}_i$, and in this case $\varphi_0^{\oplus_{i\in I}\mathbf{A}_i} = [(\varphi_0^{\mathbf{A}_i}, i)]_\theta$. And, of course, the same for $\bigoplus_{i\in I}\mathbf{D}_i$. But then $\varphi_0^{\oplus_{i\in I}\mathbf{A}_i}$ is defined iff $\varphi_0^{\oplus_{i\in I}\mathbf{D}_i}$, and they are the same. This finishes the proof that $\bigoplus_{i\in I}\mathbf{D}_i$ is a closed subalgebra of $\bigoplus_{i\in I}\mathbf{A}_i$, and the proof of the Proposition. ■

PROPOSITION 11 (**Equalizers**). a) $\mathfrak{CDC}$-$\mathfrak{Quom}(\tau)$ *has all equalizers iff* $\Omega = \Omega^{(0)} \cup \Omega^{(1)}$. *Moreover, if* $\Omega = \Omega^{(0)}$ *then an equalizer of a family* $(f_i : \mathbf{A} \to \mathbf{B})_{i\in I}$ *of cdc-quomorphisms is the same as its equalizer in* $\mathfrak{C}$-$\mathfrak{Quom}(\tau)$ *described in* [6, Thm. 3] (2), *while if* $\Omega^{(1)} \neq \emptyset$ *then an equalizer of* $(f_i : \mathbf{A} \to \mathbf{B})_{i\in I}$ *is given by the same construction as the one for its equalizer in* $\mathfrak{C}$-$\mathfrak{Quom}(\tau)$ *when all operations in* Ω *are unary, as described in* loc. cit.

b) $\mathfrak{CD}$-$\mathfrak{Quom}(\tau)$ *has always all equalizers, and they are given by the corresponding equalizers in* $\mathfrak{Quom}(\tau)$ *described in* [6, Thm. 3].

(2) The constructions in Theorems 3 and 3d and the corresponding Lemmas in [6], concerning equalizers and coequalizers, are given only for two morphisms, but, as it is pointed out therein, they are easily generalized to arbitrary families of morphisms. We shall refer in the sequel to these generalizations, rather than to the constructions actually displayed in *loc. cit.*

Proof. a) Let us consider first the case $\Omega = \Omega^{(0)}$. Let $\mathcal{F} = (f_i : \mathbf{A} \to \mathbf{B})_{i\in I}$ be a family of cdc-quomorphisms between two partial τ-algebras. Then all nullary operations defined in $\mathbf{A}$ belong to

$$C = \Big\{a \in \bigcap_{i\in I} \operatorname{dom} f_i \mid f_i(a) = f_j(a) \text{ for every } i,j \in I\Big\} \cup \Big(A - \bigcup_{i\in I} \operatorname{dom} f_i\Big),$$

(actually, they belong to the first set in this union) making the relative subalgebra $\mathbf{C}$ of $\mathbf{A}$ supported on C to be closed. And then a proof similar to the one for $\mathfrak{C}$-$\mathfrak{Quom}(\tau)$ in [6, Thm. 3] shows that the inclusion of $\mathbf{C}$ into $\mathbf{A}$ is the equalizer of $\mathcal{F}$ in $\mathfrak{CDC}$-$\mathfrak{Quom}(\tau)$.

In the case $\Omega = \Omega^{(0)} \cup \Omega^{(1)}$, with $\Omega^{(1)} \neq \emptyset$, let again $\mathcal{F} = (f_i : \mathbf{A} \to \mathbf{B})_{i\in I}$ be a family of cdc-quomorphisms between two partial τ-algebras and let

$$\begin{aligned} E = \{a \in A \mid & \text{ there are no } \mathbf{t} \in \mathbf{F}(\{x\}, \mathfrak{TAlg}(\tau)) \text{ and } i,j \in I \\ & \text{such that } a \in \operatorname{dom} \mathbf{t}^{\mathbf{A}} \text{ and} \\ \mathbf{t}^{\mathbf{A}}(a) \in (\operatorname{dom} f_i \cup \operatorname{dom} f_j) & - \{x \in (\operatorname{dom} f_i \cap \operatorname{dom} f_j) \mid f_i(x) = f_j(x)\}\}. \end{aligned}$$

As it is shown in the proof of [6, Th. 3], E is a closed subset of $\mathbf{A}$, and then an argument combining the arguments for closed quomorphisms in the pure nullary and pure unary cases in *loc. cit.* shows that the inclusion in $\mathbf{A}$ of its closed subalgebra $\mathbf{E}$ supported on E is the equalizer of $\mathcal{F}$ in $\mathfrak{CDC}$-$\mathfrak{Quom}(\tau)$.

Finally, in the case $\Omega^{(n)} \neq \emptyset$ for some $n \geq 2$, the example given in *loc. cit.* to show that in this case the equalizer of two closed quomorphisms need not exist can also be used to show that the equalizer of two cdc-quomorphisms need not exist either (the closed quomorphisms considered therein are actually cdc-quomorphisms).

b) The proof is essentially the same as the one for quomorphisms in *loc. cit.* ∎

Since a category is complete, i.e. it has all limits, iff it has all products and all equalizers [1, Th. 12.3], summarizing Corollary 8 and Proposition 11 we obtain:

COROLLARY 12 (**Completeness**). a) $\mathfrak{CDC}$-$\mathfrak{Quom}(\tau)$ *is complete iff* $\Omega = \Omega^{(1)}$.

b) $\mathfrak{CD}$-$\mathfrak{Quom}(\tau)$ *is never complete, unless* $\Omega = \emptyset$. ∎

PROPOSITION 13 (**Coequalizers**). a) $\mathfrak{CDC}$-$\mathfrak{Quom}(\tau)$ *has all coequalizers iff* $\Omega = \Omega^{(1)}$. *Moreover, in this case a coequalizer of a non-empty family* $\mathcal{F} = (f_i : \mathbf{A} \to \mathbf{B})_{i\in I}$ *of cdc-quomorphisms is obtained as follows. Let* $\theta(\mathcal{F})'$ *be the least equivalence relation on* $\mathbf{B}$ *containing all pairs of the form*

$(f_i(a), f_j(a))$ *with* $a \in \operatorname{dom} f_i \cap \operatorname{dom} f_j$, $i, j \in I$. *Let*

$$B(\mathcal{F}) = \Big\{ b \in B \mid \bigcup_{i \in I} f_i^{-1}([b]_{\theta(\mathcal{F})'}) \subseteq \bigcap_{i \in I} \operatorname{dom} f_i \Big\},$$

let $D(\mathcal{F})$ *be the greatest closed subset of* $\mathbf{B}$ *contained in* $B(\mathcal{F})$ *and let finally* $\theta(\mathcal{F})_0$ *be the restriction of* $\theta(\mathcal{F})'$ *to* $D(\mathcal{F})$. *Then* $\theta(\mathcal{F})_0$ *turns out to be a closed congruence on the relative subalgebra* $\mathbf{D}(\mathcal{F})$ *of* $\mathbf{B}$ *supported on* $D(\mathcal{F})$, *and the natural homomorphism* $nat_{\theta(\mathcal{F})_0} : \mathbf{B} \to \mathbf{D}(\mathcal{F})/_{\theta(\mathcal{F})_0}$ *is a coequalizer of* $\mathcal{F}$ *in* $\mathfrak{CDC}\text{-}\mathfrak{Quom}(\tau)$.

b) $\mathfrak{CD}\text{-}\mathfrak{Quom}(\tau)$ *has never all coequalizers, unless* $\Omega = \emptyset$.

Proof. a) Let us consider first the case when $\Omega^{(n)} \neq \emptyset$ for some $n \geq 2$, and as usual we restrict ourselves to the case $n = 2$. So, let $\varphi \in \Omega^{(2)}$, let $\mathbf{A}$ be a discrete τ-algebra with universe $A = \{a_1, a_2\}$ and let $\mathbf{B}$ be a partial τ-algebra with carrier $B = \{b_1, b_1', b_2\}$ and with all operations discrete except $\varphi^{\mathbf{B}}$, which is given by $\varphi^{\mathbf{B}}(b_1, b_1') = b_2$. Let $f : \mathbf{A} \to \mathbf{B}$ be the cdc-quomorphism given by $f(a_1) = b_1$ and $f(a_2) = b_2$, and let $g : \mathbf{A} \to \mathbf{B}$ be the cdc-quomorphism given by $g(a_1) = b_1$. Assume that the coequalizer of f and g in $\mathfrak{CDC}\text{-}\mathfrak{Quom}(\tau)$ exists, and that it is $h : \mathbf{B} \to \mathbf{C}$. Then it is clear that $b_1, b_1' \in \operatorname{dom} h$ and, since this domain has to be closed, $b_2 \in \operatorname{dom} h$ too. But then $h \circ f \neq h \circ g$. So, such a coequalizer does not exist.

Consider now the case when $\Omega^{(0)} \neq \emptyset$, and let $\varphi_0 \in \Omega^{(0)}$. Let $\mathbf{A}$ be a partial τ-algebra with universe $A = \{a, a'\}$ and with all operations in Ω discrete except φ_0, which is given by $\varphi_0^{\mathbf{A}} = a$, let $\mathbf{B}$ be a partial τ-algebra with universe $B = \{b\}$ and with all operations in Ω discrete except φ_0, which is given by $\varphi_0^{\mathbf{B}} = b$. Let $f : \mathbf{A} \to \mathbf{B}$ be the cdc-quomorphism given by $f(a) = f(a') = b$, and let $g : \mathbf{A} \to \mathbf{B}$ be the cdc-quomorphism given by $g(a) = b$. Then there is no cdc-quomorphism h coequalizing f and g.

So let us assume finally that $\Omega = \Omega^{(1)}$. From [6, Lemma 7.(i)] (for closed quomorphisms) we know that the restriction $\theta(\mathcal{F})$ of $\theta(\mathcal{F})'$ to $B(\mathcal{F})$ is a closed congruence on the relative subalgebra $\mathbf{B}(\mathcal{F})$ of $\mathbf{B}$ supported on $B(\mathcal{F})$. Then, $\theta(\mathcal{F})_0$ is a closed congruence on $\mathbf{D}(\mathcal{F})$ (the closed subalgebra of $\mathbf{B}$ supported on $D(\mathcal{F})$) and

$$\mathrm{nat}_{\theta(\mathcal{F})_0} : \mathbf{B} \to \mathbf{D}(\mathcal{F})/_{\theta(\mathcal{F})_0}$$

is a cdc-quomorphism. We must show now that it is the coequalizer of $\mathcal{F}$. We split this proof into several steps.

Let us first prove that $D(\mathcal{F})$ is a union of $\theta(\mathcal{F})$-classes. To do that, notice that a standard construction shows that

$$D(\mathcal{F}) = \{ b \in B(\mathcal{F}) \mid$$
$$\text{For every } \mathbf{t} \in \mathbf{F}(\{x\}, \mathfrak{TAlg}(\tau)), \text{ if } b \in \operatorname{dom} \mathbf{t}^{\mathbf{B}} \text{ then } \mathbf{t}^{\mathbf{B}}(b) \in B(\mathcal{F}) \}.$$

So, let $b \in D(\mathcal{F})$ and $b' \in B(\mathcal{F})$ such that $(b', b) \in \theta(\mathcal{F})$. Then there exist $f_1, \ldots, f_n, f'_1, \ldots, f'_n \in \mathcal{F}$ and $a_1, \ldots, a_n \in A$ such that $a_i \in \operatorname{dom} f_i \cap \operatorname{dom} f'_i$ for every $i = 1, \ldots, n$ and $f_1(a_1) = b'$, $f'_i(a_i) = f_{i+1}(a_{i+1})$ for every $i = 1, \ldots, n-1$, and $f'_n(a_n) = b$.

Let $\mathbf{t} \in \mathbf{F}(\{x\}, \mathfrak{TAlg}(\tau))$ such that $b' \in \operatorname{dom} \mathbf{t}^{\mathbf{B}}$. If $f_1(a_1) = b' \in \operatorname{dom} \mathbf{t}^{\mathbf{B}}$ then $a_1 \in \operatorname{dom} \mathbf{t}^{\mathbf{A}}$ and $f'_1(a_1) \in \operatorname{dom} \mathbf{t}^{\mathbf{B}}$. By induction we conclude that $b \in \operatorname{dom} \mathbf{t}^{\mathbf{B}}$. But, since $b \in D(\mathcal{F})$, it turns out that $\mathbf{t}^{\mathbf{B}}(b) \in B(\mathcal{F})$. And since $\theta(\mathcal{F})$ is a closed congruence on $\mathbf{B}(\mathcal{F})$, it implies that $\mathbf{t}^{\mathbf{B}}(b') \in B(\mathcal{F})$. We conclude that $b' \in D(\mathcal{F})$.

Therefore, $D(\mathcal{F})$ is a union of $\theta(\mathcal{F})$-classes, and it implies in particular that $\operatorname{nat}_{\theta(\mathcal{F})_0} \circ f_i = \operatorname{nat}_{\theta(\mathcal{F})_0} \circ f_j$, $i, j \in I$.

Now let $h : \mathbf{B} \to \mathbf{C}$ be another cdc-quomorphism coequalizing $\mathcal{F}$. Then $\operatorname{dom} h$ is a closed subset of $\mathbf{B}$, and by [6, Lemma 8] it is a union of $\theta(\mathcal{F})$-classes. It implies on one hand that $\operatorname{dom} h \subseteq B(\mathcal{F})$, and thus $\operatorname{dom} h \subseteq D(\mathcal{F})$, and on the other hand that $\theta(\mathcal{F})_h := \theta(\mathcal{F})_0 \cap (\operatorname{dom} h)^2$ is contained in $\ker h$ and that $(\operatorname{dom} h)/_{\theta(\mathcal{F})_h}$ is a closed subset of $\mathbf{D}(\mathcal{F})/_{\theta(\mathcal{F})_0}$.

Now, since $\theta(\mathcal{F})_h \subseteq \ker h$, there is a closed homomorphism $\tilde{h}$: $(\mathbf{dom}\, h)/_{\theta(\mathcal{F})_h} \to \mathbf{C}$ yielding a cdc-quomorphism $\tilde{h} : \mathbf{D}(\mathcal{F})_0/_{\theta(\mathcal{F})_0} \to \mathbf{C}$ such that $h = \tilde{h} \circ \operatorname{nat}_{\theta(\mathcal{F})_0}$. The unicity of such $\tilde{h}$ being clear, this achieves the proof that $\operatorname{nat}_{\theta(\mathcal{F})_0} : \mathbf{B} \to \mathbf{D}(\mathcal{F})/_{\theta(\mathcal{F})_0}$ is the coequalizer of $(f_i)_{i \in I}$ in $\mathfrak{CDC}$-$\mathfrak{Quom}(\tau)$.

b) Let us consider first the case when $\Omega^{(n)} \neq \emptyset$ for some $n \geq 1$. As usual, we restrict ourselves to the case $\Omega^{(1)} \neq \emptyset$, so let $\varphi \in \Omega^{(1)}$.

Let $\mathbf{A}$ be a discrete algebra with carrier $A = \{a\}$, let $\mathbf{B}$ be a partial τ-algebra with carrier $B = \{b_1, b_2, b'_1, b'_2\}$ and with all operations discrete except $\varphi^{\mathbf{B}}$, which is given by $\varphi^{\mathbf{B}}(b_1) = b'_1$ and $\varphi^{\mathbf{B}}(b_2) = b'_2$. Let $f, g : \mathbf{A} \to \mathbf{B}$ be cd-quomorphisms given by $f(a) = b_1$ and $g(a) = b_2$, and assume that there exists the coequalizer $h : \mathbf{B} \to \mathbf{C}$ of f and g. From the cd-quomorphism $p : \mathbf{B} \to \mathbf{B}$ given by $p(b_1) = p(b_2) = b_1$ and $p(b'_1) = p(b'_2) = b'_1$ we deduce that h is total, and since it must coequalize f and g it must also satisfy that $h(b_1) = h(b_2)$ and $h(b'_1) = h(b'_2)$. But now let $\cdot : \mathbf{B} \to \mathbf{B}$ be the cd-quomorphism given by the identity on the (closed) subalgebra of $\mathbf{B}$ supported on $\{b'_1, b'_2\}$. It also coequalizes f and g, and it entails that $h(b'_1) \neq h(b'_2)$, which yields a contradiction.

It remains to consider the case $\Omega = \Omega^{(0)} \neq \emptyset$. Let $\varphi_0 \in \Omega^{(0)}$, let $\mathbf{A}$ be a discrete algebra with carrier $A = \{a\}$, let $\mathbf{B}$ be a partial τ-algebra with carrier $B = \{b\}$ and with all operations discrete except φ_0, which is given by $\varphi_0^{\mathbf{B}} = b$, let $f : \mathbf{A} \to \mathbf{B}$ be the cd-quomorphism given by $f(a) = b$ and let $g : \mathbf{A} \to \mathbf{B}$ be the empty cd-quomorphism. Then there does not exist any cd-quomorphism with source algebra $\mathbf{B}$ coequalizing f and g (because it should be totally defined) and in particular there does not exist the coequalizer of f and g. ■

Remark: It is easy to produce examples showing that, in point (a) in the last Proposition, one has in general that $D(\mathcal{F}) \neq B(\mathcal{F})$. Therefore, the coequalizer of a family of cdc-quomorphisms in $\mathfrak{CDC}$-$\mathfrak{Quom}(\tau)$ is in general different from their pushout in $\mathfrak{C}$-$\mathfrak{Quom}(\tau)$ or $\mathfrak{Conf}(\tau)$.

Since a category is cocomplete, i.e. it has all colimits, iff it has all coproducts and all coequalizers, summarizing Propositions 9 and 13 we obtain:

COROLLARY 14 (**Cocompleteness**). a) $\mathfrak{CDC}$-$\mathfrak{Quom}(\tau)$ *is cocomplete iff* $\Omega = \Omega^{(1)}$.

b) $\mathfrak{CD}$-$\mathfrak{Quom}(\tau)$ *is never cocomplete, unless* $\Omega = \emptyset$. ■

Specific (and interesting) types of limits or colimits may exist in a category even when it is not complete or cocomplete. In addition to the general results on completeness and cocompleteness given so far (Corollaries 12 and 14), in the sequel we discuss the existence of pullbacks and pushouts of non-empty families of morphisms, as well as of inverse limits of non-empty inverse systems and direct limits of non-empty directed systems.

PROPOSITION 15 (**Pullbacks**). a) $\mathfrak{CDC}$-$\mathfrak{Quom}(\tau)$ *has all pullbacks of non-empty families of morphisms with common target algebra iff* $\Omega = \Omega^{(0)} \cup \Omega^{(1)}$. *Moreover, if* $\Omega = \Omega^{(1)}$ *then a pullback of such a family of cdc-quomorphisms is obtained by the usual combination of a product and an equalizer* (*cf.* [1, Prop. 11.11]), *while in the general case a pullback of a family* $(f_i : \mathbf{B}_i \to \mathbf{A})_{i\in I}$ *of cdc-quomorphisms of* τ*-algebras is obtained*, grosso modo, *by first computing the pullback* $(\mathbf{P}', (p_i : \mathbf{P}' \to \mathbf{B}'_i)_{i\in I})$ *of the family* $(f_i : \mathbf{B}'_i \to \mathbf{A}')_{i\in I}$ ($\mathbf{A}'$ *and* $\mathbf{B}'_i$ *the unary reducts of* $\mathbf{A}$ *and* $\mathbf{B}_i$, *resp.*), *and then defining in* $\mathbf{P}'$ *the nullary operations* (*obtaining a* τ*-algebra* $\mathbf{P}$) *in the only possible way so that the 'projections'* $p_i : \mathbf{P}' \to \mathbf{B}'_i$ *become cdc-quomorphisms of* τ*-algebras* $p_i : \mathbf{P} \to \mathbf{B}_i$ (see details in the proof).

b) $\mathfrak{CD}$-$\mathfrak{Quom}(\tau)$ *has all pullbacks of non-empty families of morphisms with common target algebra iff* $\Omega = \Omega^{(0)}$. *Moreover, a pullback of such a family of cdc-quomorphisms is obtained in this case by the usual combination of a product and an equalizer.*

Proof. a) Assume first $\Omega^{(n)} \neq \emptyset$ for some $n \geq 2$. Let τ' be the similarity type obtained from τ by removing the operation symbols in $\Omega^{(0)}$. Since there cannot exist any cdc-quomorphism from a τ-algebra having some nullary operation defined to a τ-algebra having no nullary operation defined, the τ'-reduct of the pullback in $\mathfrak{CDC}$-$\mathfrak{Quom}(\tau)$ of two empty cdc-quomorphisms $f : \mathbf{A} \to \emptyset$ and $g : \mathbf{B} \to \emptyset$ ($\mathbf{A}$ and $\mathbf{B}$ two τ-algebras with all their nullary operations undefined) would be the product of the τ'-reducts of $\mathbf{A}$ and $\mathbf{B}$

in $\mathfrak{CDC}$-$\mathfrak{Quom}(\tau')$. But $\mathfrak{CDC}$-$\mathfrak{Quom}(\tau')$ does not have all binary products (Proposition 8), and therefore $\mathfrak{CDC}$-$\mathfrak{Quom}(\tau)$ cannot have all pullbacks.

So, let us assume henceforth that $\Omega^{(n)} = \emptyset$ for every $n \geq 2$, and let as before τ' be the similarity type obtained from τ by removing its nullary operations. Let $(f_i : \mathbf{B}_i \to \mathbf{A})_{i\in I}$, $I \neq \emptyset$, be a family of cdc-quomorphisms of τ-algebras. Let

$$\Omega_{\mathbf{A}}^{(0)} = \{\varphi_0 \in \Omega^{(0)} \mid \varphi_0^{\mathbf{A}} \text{ is defined}\}.$$

We have that, for every $\mathbf{B}_i$, the nullary operations defined in $\mathbf{B}_i$ are exactly those in $\Omega_{\mathbf{A}}^{(0)}$.

For every $i \in I$, let $\mathbf{D}_i$ be the closed subalgebra of $\mathbf{B}_i$ supported on $\operatorname{dom} f_i$, and let $f_i : \mathbf{D}_i \to \mathbf{A}$ still denote the corresponding (totally defined) closed homomorphism. Since $\mathfrak{C}$-$\mathfrak{Hom}(\tau)$ is closed under arbitrary pullbacks [6, Thm. 5], there exists the pullback of this family of closed homomorphisms. Let it be $\mathbf{D}$, together with closed homomorphisms $d_i : \mathbf{D} \to \mathbf{D}_i$, $i \in I$, which we shall also understand as closed homomorphisms $d_i : \mathbf{D} \to \mathbf{B}_i$. In particular, the nullary operations defined in $\mathbf{D}$ are exactly those in $\Omega_{\mathbf{A}}^{(0)}$.

Now let $(f_i : \mathbf{B}'_i \to \mathbf{A}')_{i\in I}$ be the family of cdc-quomorphisms under consideration, but understood as between the τ'-reducts of the corresponding algebras. Since $\mathfrak{CDC}$-$\mathfrak{Quom}(\tau')$ is complete (Corollary 12), there exists the pullback of this family in $\mathfrak{CDC}$-$\mathfrak{Quom}(\tau')$. Let it be $\mathbf{P}'$, together with cdc-quomorphisms (of τ'-algebras) $p_i : \mathbf{P}' \to \mathbf{B}'_i$, $i \in I$.

By the universal property of pullbacks, there exists one and only one cdc-quomorphism (of τ'-algebras) $d : \mathbf{D}' \to \mathbf{P}'$, where $\mathbf{D}'$ is the τ'-reduct of $\mathbf{D}$, such that $p_i \circ d = d_i$. Such d is totally defined, because the d_i are so.

Finally, let $\mathbf{P}$ be the τ-algebra obtained from $\mathbf{P}'$ by adding the operations in $\Omega^{(0)}$ in the following way: a nullary operation φ_0 is defined in $\mathbf{P}$ iff $\varphi_0 \in \Omega_{\mathbf{A}}^{(0)}$, and in this case $\varphi_0^{\mathbf{P}} = d(\varphi_0^{\mathbf{D}})$.

With this definition, it is clear that $p_i : \mathbf{P} \to \mathbf{B}_i$ is a cdc-quomorphism (of τ-algebras) for every $i \in I$.

And it turns out that the τ-algebra $\mathbf{P}$, together with these cdc-quomorphisms, is the pullback in $\mathfrak{CDC}$-$\mathfrak{Quom}(\tau)$ of $(f_i : \mathbf{B}_i \to \mathbf{A})_{i\in I}$. Indeed, let $(g_i : \mathbf{C} \to \mathbf{B}_i)_{i\in I}$ be an I-indexed family of cdc-quomorphisms of τ-algebras such that $f_i \circ g_i = f_j \circ g_j$ for every $i, j \in I$. Let $\mathbf{C}'$ be the τ'-reduct of $\mathbf{C}$, and let $(g_i : \mathbf{C}' \to \mathbf{B}'_i)_{i\in I}$ denote the same family of cdc-quomorphisms, but now taken between the corresponding τ'-reducts.

By the universal property of pullbacks in $\mathfrak{CDC}$-$\mathfrak{Quom}(\tau')$, there exists one and only one cdc-quomorphism $g : \mathbf{C}' \to \mathbf{P}'$ such that $g_i = p_i \circ g$ for every $i \in I$. It is enough now to check that g is a cdc-quomorphism of τ-algebras $g : \mathbf{C} \to \mathbf{P}$: i.e., to show that, for every $\varphi_0 \in \Omega_{\mathbf{A}}^{(0)}$, $\varphi_0^{\mathbf{C}} \in \operatorname{dom} g$ and $g(\varphi_0^{\mathbf{C}}) =$

$\varphi_0^{\mathbf{P}}$. To do that, set $E = \operatorname{dom} f_i \circ g_i$ for any $i \in I$, and let $\mathbf{E}$ be the closed subalgebra of $\mathbf{C}$ supported on E. By the universal property of pullbacks in $\mathfrak{C}$-$\mathfrak{Hom}(\tau)$, there exists one, and only one, closed homomorphism $h : \mathbf{E} \to \mathbf{D}$ such that $d_i \circ h = g_i|_E$ for every $i \in I$. Reasoning with the τ'-reducts, we have that $g|_E = d \circ h$ and then it is clear that $\varphi_0^{\mathbf{C}} \in \operatorname{dom} g|_E \subseteq \operatorname{dom} g$ and $g(\varphi_0^{\mathbf{C}}) = d(h(\varphi_0^{\mathbf{C}})) = \varphi_0^{\mathbf{P}}$.

Finally, $\mathbf{P}$ is the only τ-algebra with τ'-reduct $\mathbf{P}'$ and such that $p_i : \mathbf{P} \to \mathbf{B}_i$, $i \in I$, are cdc-quomorphisms. Indeed, if there exists another τ-algebra $\widehat{\mathbf{P}}$ with τ'-reduct $\mathbf{P}'$ and so that the mappings $p_i : P \to B_i$ are cdc-quomorphisms $\widehat{\mathbf{P}} \to \mathbf{B}_i$ then the only cdc-quomorphism $p : \widehat{\mathbf{P}} \to \mathbf{P}$ corresponding to this family of cdc-quomorphisms should be the identity on the τ'-reducts, and therefore the identity.

b) Arguing as in the case $\Omega^{(n)} \neq \emptyset$, $n \geq 2$, for cdc-quomorphisms, we easily prove that if $\Omega^{(n)} \neq \emptyset$, for some $n \geq 1$, then $\mathfrak{CD}$-$\mathfrak{Quom}(\tau)$ does not have all pullbacks of non-empty families. And if $\Omega = \Omega^{(0)}$ then $\mathfrak{CD}$-$\mathfrak{Quom}(\tau)$ has all non-empty products (Proposition 8) and all equalizers (Proposition 11) and therefore all pullbacks of non-empty families. ∎

PROPOSITION 16 (**Pushouts**). a) $\mathfrak{CDC}$-$\mathfrak{Quom}(\tau)$ *has all of pushouts of non-empty families of morphisms with common source algebra iff* $\Omega = \Omega^{(1)}$. *Moreover, a pushout of a family of cdc-quomorphisms is obtained in this case by the usual combination of a coproduct and a coequalizer (cf. dual of* [1, Prop. 11.11]).

b) $\mathfrak{CD}$-$\mathfrak{Quom}(\tau)$ *has never all pushouts, unless* $\Omega = \emptyset$.

Proof. a) In the case $\Omega^{(n)} \neq \emptyset$ for some $n \geq 2$, an argument similar (but dual) to the one used in the same case for pullbacks of cdc-quomorphisms in the previous Proposition, shows that $\mathfrak{CDC}$-$\mathfrak{Quom}(\tau)$ does not have all pushouts, while in the case $\Omega^{(0)} \neq \emptyset$, the example for the nullary case in Proposition 13.(a) can also be used to show that $\mathfrak{CDC}$-$\mathfrak{Quom}(\tau)$ does not have all pushouts either. And if $\Omega = \Omega^{(1)}$ then $\mathfrak{CDC}$-$\mathfrak{Quom}(\tau)$ is cocomplete by Corollary 14.

b) Since $\mathfrak{CD}$-$\mathfrak{Quom}(\tau)$ has always an initial object (Proposition 5) but never all coequalizers (unless $\Omega = \emptyset$; Proposition 13), it never has all pullbacks (unless $\Omega = \emptyset$). ∎

PROPOSITION 17 (**Inverse limits**). $\mathfrak{CDC}$-$\mathfrak{Quom}(\tau)$ *and* $\mathfrak{CD}$-$\mathfrak{Quom}(\tau)$ *have always all inverse limits of non-empty inverse systems. Moreover, an inverse limit of a non-empty inverse system of cdc-quomorphisms or cd-quomorphisms is given by its inverse limit in* $\mathfrak{Quom}(\tau)$ *described in* [6, Th. 6].

Proof. Let $\mathbf{I} = (I, \leq)$ be any non-empty upward directed set and let

$$\mathcal{A} = ((\mathbf{A}_i)_{i\in I}, (f_{i,j} : \mathbf{A}_i \to \mathbf{A}_j)_{\substack{i,j\in I \\ i\geq j}})$$

be any inverse system of partial τ-algebras in $\mathfrak{Quom}(\tau)$. Let $\mathbf{A}$, together with quomorphisms $f_i : \mathbf{A} \to \mathbf{A}_i$, $i \in I$, be the inverse limit of $\mathcal{A}$ in $\mathfrak{Quom}(\tau)$ described in *loc. cit.* In particular, for every system of quomorphisms $(g_i : \mathbf{B} \to \mathbf{A}_i)_{i\in I}$ compatible with $\mathcal{A}$ there exists one (and only one) quomorphism $g : \mathbf{B} \to \mathbf{A}$ such that $g_i = f_i \circ g$ for every $i \in I$.

In *loc. cit.* it is shown that if all quomorphisms $f_{i,j}$ and g_i are closed then all quomorphisms f_i, as well as g, are also closed. This implies that $(\mathbf{A}, (f_i : \mathbf{A} \to \mathbf{A}_i)_{i\in I})$ is the inverse limit of $\mathcal{A}$ in $\mathfrak{C}$-$\mathfrak{Quom}(\tau)$ when all quomorphisms $f_{i,j}$ are closed.

Now a similar (but easier) argument shows that if all quomorphisms $f_{i,j}$ and g_i have their domain closed then all quomorphisms f_i, as well as g, have also their domain closed. And this implies that $(\mathbf{A}, (f_i : \mathbf{A} \to \mathbf{A}_i)_{i\in I})$ is the inverse limit of $\mathcal{A}$ in $\mathfrak{CD}$-$\mathfrak{Quom}(\tau)$ (resp. $\mathfrak{CDC}$-$\mathfrak{Quom}(\tau)$) when all quomorphisms $f_{i,j}$ are cd-quomorphisms (resp. cdc-quomorphisms). ∎

A similar proof also applies for direct limits, yielding the following result.

PROPOSITION 18 (**Direct limits**). *$\mathfrak{CDC}$-$\mathfrak{Quom}(\tau)$ and $\mathfrak{CD}$-$\mathfrak{Quom}(\tau)$ have always all direct limits of non-empty directed systems. Moreover, a direct limit of a non-empty directed system of cdc-quomorphisms or cd-quomorphisms is given by its direct limit in $\mathfrak{C}$-$\mathfrak{Quom}(\tau)$ described in* [6, Th. 6d]. ∎

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