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SHIFTS ON BANACH SPACES

1. Introduction

Throughout this paper X denotes an infinite-dimensional complex Banach space and $\mathcal{L}(X)$ the Banach algebra of all bounded linear operators on X. For $T \in \mathcal{L}(X)$ set $\alpha(T) = \dim N(T)$ and $\beta(T) = \operatorname{codim} T(X)$, where $N(T)$ is the kernel and $T(X)$ the range of T. Define the *generalized range* of T to be the subspace

$$\mathcal{R}(T) = \bigcap_{n \geq 1} T^n(X).$$

Write

$$\Phi_+(X) = \{T \in \mathcal{L}(X) : \alpha(T) < \infty,\ T(X) \text{ is closed}\}$$

and

$$\Phi_-(X) = \{T \in \mathcal{L}(X) : \beta(T) < \infty\}.$$

Observe that $T(X)$ is closed if $T \in \Phi_-(X)$ [5, Satz 55.4].

$$\Phi_\pm(X) = \Phi_+(X) \cup \Phi_-(X)$$

is the set of *semi-Fredholm operators* on X, while

$$\Phi(X) = \Phi_+(X) \cap \Phi_-(X)$$

is the set of *Fredholm operators* in $\mathcal{L}(X)$. If $T \in \Phi_\pm(X)$, $\operatorname{ind}(T) = \alpha(T) - \beta(T)$ is called the *index* of T.

Write

$$\Phi(T) = \{\lambda \in \mathbb{C} : T - \lambda I \in \Phi(X)\}$$

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for the *Fredholm region* of T. It is well known that $\Phi(T)$ is open.

$$\sigma_W(T) = \mathbb{C} \setminus \{\lambda \in \Phi(T) : \operatorname{ind}(T - \lambda I) = 0\}$$

is called the *Weyl spectrum* of T. We denote by $\sigma(T)$, $\varrho(T)$ and $r(T)$ the *spectrum*, the *resolvent set* and the *spectral radius* of T, respectively.

The following class of operators was introduced by Crownover [2]:

$T \in \mathcal{L}(X)$ is called a *shift* if $\alpha(T) = 0$, $\beta(T) = 1$ and $\mathcal{R}(T) = \{0\}$. If T is a shift and an isometry, T is called a *shift isometry*.

It is immediate that each shift T is a Fredholm operator with $\operatorname{ind}(T) = -1$.

In [10] we have proved the following

PROPOSITION 1. *Let* $T \in \Phi_{\pm}(X)$ *and* $\mathcal{R}(T) = \{0\}$. *Then*

(a) $T \in \Phi_{+}(X)$,

(b) $\operatorname{ind}(T - \lambda I) < 0$ *for all* $\lambda \in \sigma(T)$ *with* $T - \lambda I$ *is semi-Fredholm*,

(c) $\sigma(T) = \sigma_W(T)$ *is connected.*

COROLLARY. $T \in \mathcal{L}(X)$ *is a shift if and only if* $\beta(T) = 1$ *and* $\mathcal{R}(T) = \{0\}$. *In this case we have that* $\sigma(T) = \sigma_W(T)$ *is connected.*

Proof. If $\beta(T) = 1$ and $\mathcal{R}(T) = \{0\}$, we have that T is semi–Fredholm, hence, by Proposition 1(a), $\alpha(T) - \beta(T) = \alpha(T) - 1 = \operatorname{ind}(T) < 0$, thus $\alpha(T) < 1$, therefore $\alpha(T) = 0$. ∎

EXAMPLES:

(a) If $X = l^p$ $(1 \le p \le \infty)$ then the operator T given by

$$T(\xi_1, \xi_2, \ldots) = (0, \xi_1, \xi_2, \ldots)$$

is a shift isometry.

(b) If X is the disk algebra, the sup-norm algebra of functions f which are continuous on the disk $\{z \in \mathbb{C} : |z| \le 1\}$ and holomorphic for $|z| < 1$, then the operator T defined by

$$(Tf)(z) = zf(z)$$

is a shift isometry on X.

The following considerations are due to R. M. Crownover [2]. They justify the use of the term "shift". Let $T \in \mathcal{L}(X)$ be a shift. Then there exists some $x_0 \in X$ with $\|x_0\| = 1$ and

$$X = [x_0] \oplus T(X) \tag{1}$$

(where $[x_0] = \{\alpha x_0 : \alpha \in \mathbb{C}\}$). Let $x \in X$. Then we have $x = \alpha_0(x)x_0 + Tx_1$ for some $\alpha_0(x) \in \mathbb{C}$ and $x_1 \in X$. Similarly there exist $\alpha_1(x) \in \mathbb{C}$ and $x_2 \in X$

so that $x_1 = \alpha_1(x)x_0 + Tx_2$. Hence

$$x = \alpha_0(x)x_0 + \alpha_1(x)Tx_0 + T^2x_2.$$

Since $N(T) = \{0\}$, we get, by induction, unique sequences $(\alpha_n(x))_{n=0}^{\infty}$ and $(x_n)_{n=1}^{\infty}$ of scalars and vectors such that

$$x = \sum_{k=0}^{n} \alpha_k(x)T^kx_0 + T^{n+1}x_{n+1} \quad (n \geq 0). \tag{2}$$

The coefficients $\alpha_k(x)$ will be called the *Taylor coefficients* of x and equation (2) is called the *Taylor formula* for x *induced by the decomposition* (1).

Let X_s denote the space of sequences

$$X_s = \{(\alpha_n(x))_{n=0}^{\infty} : x \in X\}$$

with the norm given by

$$\|(\alpha_n(x))_{n=0}^{\infty}\| = \|x\|.$$

Therefore the mapping $x \mapsto (\alpha_n(x))_{n=0}^{\infty}$, which is linear and maps X onto X_s, is an isometric isomorphism. Equation (2) gives

$$Tx = \alpha_0(x)Tx_0 + \ldots + \alpha_n(x)T^{n+1}x_0 + T^{n+2}x_{n+1},$$

thus the sequence of Taylor coefficients of Tx is

$$(0, \alpha_0(x), \alpha_1(x), \ldots).$$

Hence T corresponds to the unilateral shift operator $T_s : X_s \to X_s$ given by

$$T_s(\alpha_0, \alpha_1, \alpha_2, \ldots) = (0, \alpha_0, \alpha_1, \alpha_2, \ldots).$$

2. Perturbation properties

For the rest of this paper T always denotes a shift in $\mathcal{L}(X)$ and Ω the connected component of $\Phi(T)$ which contains 0.

PROPOSITION 2.

(a) *$T - \lambda I$ is a shift for each $\lambda \in \Omega$.*

(b) *If $\lambda \in \Omega$, $A \in \mathcal{L}(X)$, $n \in \mathbb{N}$ and $A^n = T - \lambda I$, then $n = 1$ (hence $T - \lambda I$ has no nth root in $\mathcal{L}(X)$ for $n \geq 2$).*

Proof. (a) Let $\lambda \in \Omega$ and $\lambda \neq 0$. If $Tx = \lambda x$ then $T^n x = \lambda^n x$ for each $n \in \mathbb{N}$. Since $\lambda \neq 0$, $x = \frac{1}{\lambda^n}T^n x$ $(n \in \mathbb{N})$, thus $x \in \mathcal{R}(T) = \{0\}$. This shows that $\alpha(T - \lambda I) = 0$. Theorem 4.2 in [7] then gives $\mathcal{R}(T - \lambda I) = \{0\}$. Since $\operatorname{ind}(T - \lambda I)$ is a constant on Ω ([5, Satz 104.1]), we derive

$$\beta(T - \lambda I) = -\operatorname{ind}(T - \lambda I) = -\operatorname{ind}(T) = \beta(T) = 1.$$

(b) Use [5, Aufgabe 71.3] to derive $A \in \Phi(X)$. Thus, by the index-theorem ([5, Satz 71.3]), we get

$$-1 = \operatorname{ind}(T - \lambda I) = \operatorname{ind}(A^n) = n \operatorname{ind}(A),$$

thus $\operatorname{ind}(A) = -\frac{1}{n}$, therefore $n = 1$. ∎

Remark. Part (b) of Proposition 2 can be found in [2, p. 246] for shift isometries and for $|\lambda| < 1$. Crownover's proof uses Banach-algebraic techniques.

One more concept is useful at this point. For an operator $A \in \mathcal{L}(X)$, $A \neq 0$, the *minimum modulus* $\gamma(A)$ is given by

$$\gamma(A) = \inf\left\{\frac{\|Ax\|}{d(x, N(A))} : x \notin N(A)\right\},$$

where $d(x, N(A)) = \inf_{y \in N(A)} \|x - y\|$.

Our shift operator T is injective and $T(X)$ is closed so that $T^{-1} : T(X) \to X$ is bounded and

$$\gamma(T) = \|T^{-1}\|^{-1}.$$

Let $x_0 \in X$ as in the decomposition (1). In [2] Crownover introduced the Banach space $X \oplus \mathbb{C}$ with the norm

$$\|x \oplus \beta\| = \max\left\{\|x\|, \frac{|\beta|}{\gamma(T)}\right\}$$

and the operator $A_T : X \oplus \mathbb{C} \to X$ by

$$A_T(x \oplus \beta) = \beta x_0 + Tx.$$

It is easily seen that A_T is bijective. Crownover proved the following result ([2, Theorem 2]):

THEOREM 1. *Let* $\varrho = \|A_T^{-1}\|^{-1}$. *Then*

(a) *For* $|\lambda| < \varrho$ *the operator* $T - \lambda I$ *is a shift on* X *and*

(3) $$X = [x_0] \oplus (T - \lambda I)(X).$$

(b) *If* $x \in X$ *and* $(\alpha_n(x))_{n=0}^{\infty}$ *is the sequence of Taylor coefficients for* x *induced by* (1) *then*

$$\sum_{n=0}^{\infty} \alpha_n(x)\lambda^n \text{ converges for } |\lambda| < \varrho$$

and the representation of x induced by (3) has the form

$$x = \Big(\sum_{n=0}^{\infty} \alpha_n(x)\lambda^n\Big)x_0 + (T - \lambda I)y_\lambda$$

for some $y_\lambda \in X$ ($|\lambda| < \varrho$).

Crownover's proof of Theorem 1 is rather involved. In what follows we shall improve Theorem 1 with a simpler proof in the following sense:

We shall see that the assertions of Theorem 1 are valid for a radius which is in general larger than ϱ, and we shall derive a very simple representation for the Taylor coefficients for x. To this end we use the concept of relatively regular operators:

An operator $A \in \mathcal{L}(X)$ is said to be *relatively regular* if $ABA = A$ for some $B \in \mathcal{L}(X)$. It is well known that Fredholm operators are relatively regular (see [5]). Thus each shift on X is relatively regular.

PROPOSITION 3. (a) *Let $A, B \in \mathcal{L}(X) \setminus \{0\}$ and $ABA = A$, then*

$$\frac{1}{\|B\|} \le \gamma(A).$$

(b) *If T is a shift on X, then $ST = I$ for some $S \in \mathcal{L}(X)$. Furthermore we have*

$$I - TS \text{ and } TS \text{ are projections, } (I - TS)(X) = N(S),$$
$$(TS)(X) = T(X) \text{ and } X = N(S) \oplus T(X).$$

Proof. (a) [9, Prop. 4].

(b) Since T is relatively regular there is some $S \in \mathcal{L}(X)$ such that $TST = T$. It is easy to see that $(I - ST)(X) = N(T) = \{0\}$, thus $ST = I$. The rest is clear. ∎

Let T be a shift on X, hence T has a left inverse $S \in \mathcal{L}(X)$. Since $X = N(S) \oplus T(X)$ we have $\alpha(S) = 1$. Let $x_0 \in N(S)$ with $\|x_0\| = 1$. Then

$$X = [x_0] \oplus (TS)(X) = [x_0] \oplus T(X). \tag{4}$$

There is exactly one bounded linear functional f on X such that $f(x_0) = 1$ and $f(Tx) = 0$ for all $x \in X$. Finally, let the operator $A_T : X \oplus \mathbb{C} \to X$ be defined as before (with the above x_0).

Remarks.

(a) Since T is not invertible in $\mathcal{L}(X)$, the set of its left inverses is infinite. It is precisely the set $\{L + U(I - TL) : U \in \mathcal{L}(X)\}$, where L is any left inverse of T.

(b) The operator A_T, the value of $\|A_T^{-1}\|$ and f depend on the choice of x_0. Therefore they depend on the choice of the left inverse S.

PROPOSITION 4. *Let T, S, f and x_0 as above.*

(a) *For $x \in X$, the sequence of Taylor coefficients induced by* (4) *is*

$$(f(S^n x))_{n=0}^{\infty}$$

and the Taylor formula induced by (4) *is*

$$x = \sum_{k=0}^{n} f(S^k x)x_0 + T^{n+1}S^{n+1}x \quad (n \geq 0).$$

(b) $\|S\| \leq \|A_T^{-1}\|$.

Proof. (a) We have $x = \alpha_0(x)x_0 + Tx_1$, thus $Sx = \alpha_0(x)Sx_0 + STx_1 = x_1$ and $f(x) = \alpha_0(x)f(x_0) + f(Tx_1) = \alpha_0(x)$. Since $Sx = x_1 = \alpha_1(x)x_0 + Tx_2$, it follows that $S^2x = STx_2 = x_2$ and $f(Sx) = \alpha_1(x)$.

By induction we see that $x_n = S^n x$ for $n \geq 1$, $\alpha_n(x) = f(S^n x)$ for $n \geq 0$.

(b) By (1), we have $x = f(x)x_0 + TSx$, thus $x = A_T(Sx \oplus f(x))$ and $A_T^{-1}x = Sx \oplus f(x)$. This gives

$$\|Sx\| \leq \max\{\|Sx\|, \frac{|f(x)|}{\gamma(T)}\} = \|Sx \oplus f(x)\| = \|A_T^{-1}x\| \leq \|A_T^{-1}\|\|x\|.$$

This shows (b). ∎

We shall see that the assertions of Theorem 1 remain valid if we replace $\|A_T^{-1}\|^{-1}$ by $r(S)^{-1}$ (Theorem 3). But first we show by an example that the strict inequality $r(S) < \|A_T^{-1}\|$ may actually occur.

EXAMPLE. Let $X = l^2$ and $T \in \mathcal{L}(X)$ defined by

$$T(\xi_1, \xi_2, \ldots) = (0, \xi_1, \xi_2, \ldots).$$

If $L \in \mathcal{L}(X)$ is given by

$$L(\xi_1, \xi_2, \ldots) = (\xi_2, \xi_3, \ldots)$$

then we have $LT = I$. It is easy to see that the operator $S = L + I - TL$ is a left inverse for T. By induction we get ($x = (\xi_1, \xi_2, \ldots)$)

$$S^n x = \Big(\sum_{k=1}^{n+1} \xi_k, \xi_{n+2}, \xi_{n+3}, \ldots\Big)$$

for $n \in \mathbb{N}$. The Schwarz inequality gives

$$\Big|\sum_{k=1}^{n+1} \xi_k\Big| \le \sum_{k=1}^{n+1} |\xi_k| \le \Big(\sum_{k=1}^{n+1} |\xi_k|^2\Big)^{1/2} \sqrt{n+1},$$

hence

$$\|S^n x\| = \Big(\Big|\sum_{k=1}^{n+1} \xi_k\Big|^2 + \sum_{k=n+2}^{\infty} |\xi_k|^2\Big)^{1/2} \le \Big((n+1)\sum_{k=1}^{n+1} |\xi_k|^2 + \sum_{k=n+2}^{\infty} |\xi_k|^2\Big)^{1/2}$$
$$= \Big(n\sum_{k=1}^{n+1} |\xi_k|^2 + \sum_{k=1}^{\infty} |\xi_k|^2\Big)^{1/2} = \Big(\|x\|^2 + n\sum_{k=1}^{n+1} |\xi_k|^2\Big)^{1/2}.$$

Therefore we have $\|S^n x\| \le \sqrt{n+1}\|x\|$. Put $\tilde{x} = (\underbrace{1, 1, \ldots, 1}_{n+1}, 0, 0, 0, \ldots)$, then $\|S^n \tilde{x}\| = n+1$ and $\|\tilde{x}\| = \sqrt{n+1}$, therefore $\|S^n \tilde{x}\| = \sqrt{n+1}\|\tilde{x}\|$. We have shown that $\|S^n\| = \sqrt{n+1}$ for all $n \in \mathbb{N}$. This yields $r(S) = 1$, from which it follows that

$$r(S) < \sqrt{2} = \|S\| \le \|A_T^{-1}\|.$$

THEOREM 3. *Let T, S, f, x_0 and Ω as above. For $D = \{\lambda \in \mathbb{C} : |\lambda| < \frac{1}{r(S)}\}$ we have:*

(a) $D \subseteq \Omega$.

(b) $X = [x_0] \oplus (T - \lambda I)(X)$ *for all* $\lambda \in D$.

(c) *If $x \in X$ and $\big(\alpha_n(x)\big)_{n=0}^{\infty}$ is the sequence of Taylor coefficients for x induced by* (4) *then*

$$\sum_{n=0}^{\infty} \alpha_n(x)\lambda^n \text{ converges on } D$$

and

$$\sum_{n=0}^{\infty} \alpha_n(x)\lambda^n = f\big((I - \lambda S)^{-1}x\big) \text{ on } D.$$

(d) $x = f((I - \lambda S)^{-1}x)x_0 + (T - \lambda I)(I - \lambda S)^{-1}Sx$ *for all* $x \in X$ *and* $\lambda \in D$.

Proof. Put $F(\lambda) = (I - \lambda S)^{-1}S = S(I - \lambda S)^{-1} = \sum_{n=0}^{\infty} \lambda^n S^{n+1}$ $(\lambda \in D)$. Then it is easy to see that

$$F(\lambda) - F(\mu) = (\lambda - \mu)F(\lambda)F(\mu)$$

for λ, $\mu \in D$. This shows that $N(F(\lambda)) = N(F(\mu))$ for λ, $\mu \in D$. Thus

$$N(S) = N(F(0)) = N(F(\lambda)) \quad \text{for each } \lambda \in D.$$

An easy computation gives

$$F(\lambda)(T - \lambda I) = I \quad \text{on } D,$$

therefore

$$\begin{aligned} X &= N(F(\lambda)) \oplus (T - \lambda I)F(\lambda)(X) \\ &= N(S) \oplus (T - \lambda I)(X) = [x_0] \oplus (T - \lambda I)(X). \end{aligned}$$

Hence we have shown that (a) and (b) are valid.

(c) Since, for $\lambda \in D$, $(I - \lambda S)^{-1}x = \sum_{n=0}^{\infty} \lambda^n S^n x$ and $\alpha_n(x) = f(S^n x)$ (Proposition 4) we get

$$f((I - \lambda S)^{-1}x) = \sum_{n=0}^{\infty} \lambda^n f(S^n x) = \sum_{n=0}^{\infty} \alpha_n(x)\lambda^n.$$

(d) Let $x \in X$ and $\lambda \in D$. By (b) there exist $\beta \in \mathbb{C}$ and $y \in X$ such that $x = \beta x_0 + (T - \lambda I)y$. It follows that

$$F(\lambda)x = \beta F(\lambda)x_0 + F(\lambda)(T - \lambda I)y.$$

Recall that $x_0 \in N(S) = N(F(\lambda))$ and that $F(\lambda)(T - \lambda I) = I$. Hence $F(\lambda)x = y$, thus

$$\beta x_0 = x - (T - \lambda I)F(\lambda)x = x - TF(\lambda)x + \lambda F(\lambda)x.$$

Since $f(T(X)) = \{0\}$ we get

$$\begin{aligned} \beta &= f(\beta x_0) = f(x) + \lambda f((I - \lambda S)^{-1}Sx) \\ &= f(x) + f\Big(\sum_{n=0}^{\infty} \lambda^{n+1} S^{n+1} x\Big) = f(x) + f\Big(\sum_{n=0}^{\infty} \lambda^n S^n x - x\Big) \\ &= f\Big(\sum_{n=0}^{\infty} \lambda^n S^n x\Big) = f\Big((I - \lambda S)^{-1}x\Big). \end{aligned}$$

Remark. In section 3 of this paper we shall see that a poor choice of S can lead to an arbitrary small value of $r(S)^{-1}$.

We close this section with a further perturbation result for shifts.

THEOREM 4. *Let T and Ω as above. Then for every $\varepsilon > 0$ there exist $z_0 \in X \setminus \{0\}$ and a set $M \subseteq \Omega$ such that*

(a) *M is at most denumerable and has no accumulation points in Ω,*

(b) *for every* $\mu \in M$ *the distance from* μ *to the boundary of* Ω *is at most* ε,

(c) $X = [z_0] \oplus (T - \lambda I)(X)$ *for all* $\lambda \in \Omega \setminus M$.

Proof. Let $\varepsilon > 0$. By [8, Theorem 4.5.4] (see also [4, Sect. 3.1.3] and [11]) there exist a set M with the properties (a) and (b) and a holomorphic function $F : \Omega \setminus M \to \mathcal{L}(X)$ with

$$F(\lambda)(T - \lambda I) = I \quad \text{and} \quad F(\lambda) - F(\mu) = (\lambda - \mu)F(\lambda)F(\mu)$$

for all $\lambda, \mu \in \Omega \setminus M$. This gives $N(F(\lambda)) = N(F(\mu))$ and $X = N(F(\lambda)) \oplus (T - \lambda I)F(\lambda)(X)$ for $\lambda, \mu \in \Omega \setminus M$. Fix $\lambda_0 \in \Omega \setminus M$, put $z_0 \in N(F(\lambda_0))$, $z_0 \neq 0$, and observe that $(T - \lambda I)F(\lambda)(X) = (T - \lambda I)(X)$. This shows (c). ∎

3. Orthogonal decompositions

We begin with

PROPOSITION 5. *Let* $A \in \mathcal{L}(X) \setminus \{0\}$ *be relatively regular but not right invertible. Then*

$$\inf\{r(B)^{-1} : B \in \mathcal{L}(X),\ ABA = A\} = 0.$$

Proof. Fix some $C \in \mathcal{L}(X)$ such that $ACA = A$ and put $B = CAC$. Then $ABA = (ACA)CA = A$ and $BAB = C(ACA)CAC = C(ACA)C = CAC = B$. Set $P := I - AB$. Since A is not right invertible we have $P \neq 0$. Furthermore we have $BP = B - BAB = 0$. For $\alpha \in \mathbb{R}$ with $\alpha > r(B)$ set $B_\alpha = B + \alpha P$. We get $AB_\alpha A = ABA + \alpha APA = A + \alpha A(A - ABA) = A$.

We now show by induction that

$$B_\alpha^{n+1} = B^{n+1} + \alpha^{n+1} P \sum_{k=0}^{n} \left(\frac{B}{\alpha}\right)^k \quad \text{for } n \in \mathbb{N}. \tag{5}$$

(5) holds for $n = 1$, for $B_\alpha^2 = B^2 + \underbrace{\alpha BP}_{=0} + \alpha PB + \alpha^2 P = B^2 + \alpha^2 P\left(I + \frac{B}{\alpha}\right)$.

Let (5) be valid for some $n \in \mathbb{N}$. Then

$$\begin{aligned} B_\alpha^{n+2} &= \left(B^{n+1} + \alpha^{n+1} P \sum_{k=0}^{n} \left(\frac{B}{\alpha}\right)^k\right)(B + \alpha P) \\ &= B^{n+2} + \alpha^{n+1} P \sum_{k=0}^{n} \frac{B^{k+1}}{\alpha^k} + \underbrace{\alpha B^{n+1} P}_{=0} + \underbrace{\alpha^{n+2} P \sum_{k=0}^{n} \frac{B^k P}{\alpha^k}}_{=\alpha^{n+2} P} \end{aligned}$$

$$= B^{n+2} + \alpha^{n+2} P \sum_{k=0}^{n} \frac{B^{k+1}}{\alpha^{k+1}} + \alpha^{n+2} P = B^{n+2} + \alpha^{n+2} P \sum_{k=0}^{n+1} \left(\frac{B}{\alpha}\right)^k.$$

Thus (5) holds for $n+1$. From (5) we get

$$\frac{B_\alpha^{n+1}}{\alpha^{n+1}} = \frac{B^{n+1}}{\alpha^{n+1}} + P \sum_{k=0}^{n} \left(\frac{B}{\alpha}\right)^k.$$

Since $\alpha > r(B)$ we derive (see [5, Satz 95.3])

$$\lim_{n\to\infty} \frac{B^{n+1}}{\alpha^{n+1}} = 0 \quad \text{and} \quad \lim_{n\to\infty} P \sum_{k=0}^{n} \left(\frac{B}{\alpha}\right)^k = \alpha P(\alpha I - B)^{-1},$$

hence

$$\lim_{n\to\infty} \frac{\|B_\alpha^{n+1}\|}{\alpha^{n+1}} = \alpha \|P(\alpha I - B)^{-1}\|.$$

This gives

$$\lim_{n\to\infty} \frac{\|B_\alpha^{n+1}\|^{1/n+1}}{\alpha} = 1, \quad \text{thus} \quad r(B_\alpha) = \alpha.$$

It results that $\lim_{\alpha\to\infty} r(B_\alpha)^{-1} = 0$. ■

Remarks. (a) It is clear that the conclusion of Proposition 5 is also valid if A is relatively regular and not left invertible.

(b) Inspection of the proof just given shows that it uses no properties of $\mathcal{L}(X)$ which are not shared by every Banach algebra with identity.

Let $T \in \mathcal{L}(X)$ be a shift and Ω, S, f, x_0 and A_T as in section 2. The Propositions 3 and 4 give

$$\|A_T^{-1}\|^{-1} \le \|S\|^{-1} \le \gamma(T).$$

Since $\|A_T^{-1}\|^{-1} \le r(S)^{-1}$, Proposition 5 shows that a poor choice of S can lead to arbitrary small values of $\|A_T^{-1}\|^{-1}$ and $r(S)^{-1}$. On the other hand a perturbation result of J. Zemanek [11, Theorem 1] shows that

$$\Gamma(T) := \lim_{n\to\infty} \gamma(T^n)^{1/n} \text{ exists}, \quad \Gamma(T) = \sup_{n\ge 1} \gamma(T^n)^{1/n}$$

and

$$(6) \qquad \{\lambda \in \mathbb{C} : |\lambda| < \Gamma(T)\} \subseteq \Omega.$$

Since $S^nT^n = I$ for each $n \in \mathbb{N}$, Proposition 3(a) yields $\|S^n\|^{-1} \le \gamma(T^n)$ for all $n \in \mathbb{N}$, hence

$$(7) \qquad r(S)^{-1} \le \Gamma(T).$$

This together with (6) suggests that the assertions of Theorem 3 perhaps remain valid for $|\lambda| < \Gamma(T)$. We leave this as an open question.

We shall see below that if T is a shift isometry and if the decomposition $[x_0] \oplus T(X) = X$ is "orthogonal", we indeed have that $r(S)^{-1} = \Gamma(T)$.

Following R. C. James [6], we say that two vectors x and y are *orthogonal* if for each $\alpha \in \mathbb{C}$

$$\|x\| \leq \|x + \alpha y\| \quad \text{and} \quad \|y\| \leq \|y + \alpha x\|.$$

THEOREM 5. *Let T, S, f, x_0 as above. Suppose that x_0 is orthogonal to each Ty $(y \in X)$. Then*

(a) $\|A_T^{-1}\| = \|S\| = \frac{1}{\gamma(T)}$.

(b) $\|TS\| = \|f\| = 1$.

(c) *If T is a shift isometry then*

$$\|A_T^{-1}\| = \|S\| = r(S) = \gamma(T)^{-1} = \Gamma(T)^{-1}.$$

Proof. The equation $\|A_T^{-1}\| = \gamma(T)^{-1}$ is shown in [2, Theorem 3]. We give a different proof which also shows that (b) is valid.

(a) and (b): Let $x \in X$, then $x = \alpha x_0 + Tx_1$ $(\alpha \in \mathbb{C},\ x_1 \in X)$. Because of the orthogonality of αx_0 and Tx_1 we see that $|\alpha| = \|\alpha x_0\| \leq \|\alpha x_0 + Tx_1\| = \|x\|$. Since $f(x_0) = 1$ and $f(Tx_1) = 0$ we get

$$|f(x)| = |\alpha| \leq \|x\|,$$

hence $\|f\| \leq 1$. By $f(x_0) = 1 = \|x_0\|$ it follows that $\|f\| = 1$. Thus we obtain

$$\frac{|f(x)|}{\gamma(T)} \leq \frac{\|f\|\|x\|}{\gamma(T)} = \|x\|\gamma(T)^{-1} \leq \|x\|\|S\|,$$

this gives (observe that $Sx = STx_1 = x_1$ and $\alpha = f(x)$)

$$\|A_T^{-1}x\| = \|Sx \oplus f(x)\| = \max\left\{\|Sx\|, \frac{|f(x)|}{\gamma(T)}\right\} \leq \|S\|\|x\|,$$

therefore $\|A_T^{-1}\| \leq \|S\|$. Since we have already shown that $\|S\| \leq \|A_T^{-1}\|$, it follows that $\|S\| = \|A_T^{-1}\|$.

By the orthogonality of αx_0 and Tx_1 we also have that

$$\|TSx\| = \|Tx_1\| \leq \|\alpha x_0 + Tx_1\| = \|x\|,$$

therefore $\|TS\| \leq 1$. Because of $0 \neq TS = (TS)^2$ we also have $\|TS\| \geq 1$, hence $\|TS\| = 1$. It remains to show that $\|S\| \le \gamma(T)^{-1}$. We have

$$\|Sx\| = \|x_1\| = \|T^{-1}(Tx_1)\| \le \|T^{-1}\| \|Tx_1\| = \frac{1}{\gamma(T)}\|TSx\| \le \frac{\|TS\|}{\gamma(T)}\|x\| = \frac{1}{\gamma(T)}\|x\|.$$

This completes the proof of (a) and (b).

(c) If T is an isometry, T^n is an isometry for each $n \in \mathbb{N}$, thus $\gamma(T^n) = 1$ for all n, hence $\gamma(T) = \Gamma(T)$. Use (a) and (7) to get the result. ∎

4. Local spectra of shift isometries

Let us review some classical concepts of local spectral theory which are due to N. Dunford [3].

An operator $A \in \mathcal{L}(X)$ is said to have the *single valued extension property* (SVEP) in $\lambda_0 \in \mathbb{C}$ if for any analytic function $f : D \to X$, D an open neighbourhood of λ_0, with

$$(T - \lambda I)f(\lambda) = 0 \quad \text{on } D,$$

we have $f \equiv 0$. A is said to have the SVEP in $\mathbb{C}$ if A has the SVEP in each $\lambda_0 \in \mathbb{C}$.

Let $A \in \mathcal{L}(X)$ be arbitrary and fix $x \in X$. The *local resolvent set* $\delta_A(x)$ of A in x is defined by

$$\delta_A(x) = \{\lambda \in \mathbb{C} : \text{there is an open neighbourhood } U \text{ of } \lambda \text{ and an analytic function } f : U \to X \text{ with } (A - \mu I)f(\mu) = x \text{ for each } \mu \in U\}.$$

It is immediate that $\delta_A(x)$ is open. The complement $\gamma_A(x) = \mathbb{C} \setminus \delta_A(x)$ is called the *local spectrum* of A in x. It is clear that $\gamma_A(x)$ is closed and $\gamma_A(x) \subseteq \sigma(A)$. Observe that $\gamma_A(0) = \emptyset$. It follows from [3] that if A has the SVEP in $\mathbb{C}$, then

$$(8) \qquad \gamma_A(x) \ne \emptyset \text{ for each } x \in X \setminus \{0\} \quad \text{and} \quad \sigma(A) = \bigcup_{x \ne 0} \gamma_A(x).$$

PROPOSITION 6. *Let $T \in \mathcal{L}(X)$ be a shift and Ω as in the previous sections. Then*

(a) *T has the SVEP in $\mathbb{C}$.*
(b) *$\gamma_T(x)$ is connected for each $x \ne 0$.*
(c) *$\Omega \subseteq \bigcap_{x \ne 0} \gamma_T(x)$.*

Proof. [10, Theorems 1 and 2]. ∎

THEOREM 6. *Let T be a shift isometry and Ω as above. Then*

(a) $\sigma(T) = \{\lambda \in \mathbb{C} : |\lambda| \le 1\}$.

(b) $\Omega = \{\lambda \in \mathbb{C} : |\lambda| < 1\}$.
(c) $\gamma_T(x) = \sigma(T)$ *for each* $x \neq 0$.

Proof. (a) The spectrum of each non invertible isometry is the closed unit disk (see [2, p. 239]).

(b) Since $\Gamma(T) = 1$, (6) shows that

$$\{\lambda \in \mathbb{C} : |\lambda| < 1\} \subseteq \Omega \subseteq \sigma(T).$$

Since Ω is open, the result follows from (a).

(c) Let $x \neq 0$. Part (c) of Proposition 6 gives

$$\Omega \subseteq \gamma_T(x) \subseteq \sigma(T)$$

hence, since $\gamma_T(x)$ is closed, it follows that $\gamma_T(x) = \sigma(T)$. ∎

Let X^* denote the dual space of X and T^* the adjoint of $T \in \mathcal{L}(X)$.

THEOREM 7. *Let T be a shift on X and Ω as above.*

(a) T^* *is Fredholm and* $\alpha(T^*) = 1$, $\beta(T^*) = 0$.
(b) $\gamma_{T^*}(x^*) \subseteq \mathbb{C} \setminus \Omega$ *for all* $x^* \in X^*$.
(c) T^* *does not have the SVEP in* $\mathbb{C}$.
(d) *If T is a shift isometry, then*

$$\gamma_{T^*}(x^*) \subseteq \{\lambda \in \mathbb{C} : |\lambda| = 1\}.$$

Proof. (a) follows from [5, Satz 82.1].

(b) Since $T - \lambda I$ is a shift for each $\lambda \in \Omega$ (Proposition 2), $T - \lambda I$ is left invertible in $\mathcal{L}(X)$. By [1, Theorem 1], there is a holomorphic function $F : \Omega \to \mathcal{L}(X)$ such that

$$F(\lambda)(T - \lambda I) = I \quad \text{on } \Omega.$$

Thus $(T^* - \lambda I^*)F(\lambda)^* = I^*$ on Ω. This gives

$$(T^* - \lambda I^*)F(\lambda)^* x^* = x^* \quad \text{for all } \lambda \in \Omega \text{ and all } x^* \in X^*.$$

Therefore we have

$$\Omega \subseteq \delta_{T^*}(x^*) \quad \text{for each } x^* \in X^*. \tag{9}$$

(c) Fix $x_0^* \in N(T^*)$, $x_0^* \neq 0$. Put $\varphi(\lambda) = -\frac{x_0^*}{\lambda}$ for $\lambda \in \mathbb{C} \setminus \{0\}$. Then $(T^* - \lambda I^*)\varphi(\lambda) = x_0^*$ for $\lambda \neq 0$, hence $\mathbb{C} \setminus \{0\} \subseteq \delta_{T^*}(x_0^*)$. Use (9) to derive $\delta_{T^*}(x_0^*) = \mathbb{C}$. This gives $\gamma_{T^*}(x_0^*) = \emptyset$. By (8), T^* cannot have the SVEP.

(d) By (9) and Theorem 6 we get

$$\gamma_{T^*}(x^*) \subseteq \mathbb{C} \setminus \Omega = \{\lambda \in \mathbb{C} : |\lambda| \geq 1\}.$$

Since $\gamma_{T^*}(x^*) \subseteq \sigma(T^*) = \sigma(T) = \{\lambda \in \mathbb{C} : |\lambda| \leq 1\}$, the result follows. ∎

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