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NEIGHBOURHOODS OF CONVEX FUNCTIONS RELATED WITH PARABOLA

1. Introduction

Let us denote by A the class of functions f of the form $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ which are regular in the unit disk $K = \{z \in \mathbb{C} : |z| < 1\}$. For $\delta \geq 0$ we define the neighbourhood $N_\delta(f)$ of a function $f \in A$ as follows

$$(1)\qquad N_\delta(f) := \Big\{ g(z) = z + \sum_{k=2}^{\infty} b_k z^k \;\Big|\; \sum_{k=2}^{\infty} k|a_k - b_k| \leq \delta \Big\}.$$

The notion of neighbourhood was first introduced by Ruscheweyh [3]. Using convolution methods he obtained conditions such that for $f \in A$ all functions $g \in N_\delta(f)$ are in some class of univalent functions in K. Some applications and extensions of his results we can find in [2], see also [1], [5].

As usual by S we denote the set of functions $f \in A$ which are univalent in K. Let us consider the following subclasses of S

$$(2)\qquad SP(\alpha) = \left\{ f \in S : \left| \frac{zf'(z)}{f(z)} - \alpha \right| < \operatorname{Re} \frac{zf'(z)}{f(z)} + \alpha,\ z \in K \right\}, \qquad \alpha > 0,$$

$$(3)\qquad CP(\alpha) = \left\{ f \in S : \left| 1 + \frac{zf''(z)}{f'(z)} - \alpha \right| < \operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)} \right) + \alpha,\ z \in K \right\}, \qquad \alpha > 0.$$

The class $SP(\alpha)$ was introduced in [4]. For each $\alpha > 0$ holds $SP(\alpha) \subset S^*$ — the class of starlike functions. In fact, for $f \in SP(\alpha)$ the image of K under $p(z) = zf'(z)/f(z)$ lies in the parabolic region

$$(4)\qquad \Omega(\alpha) = \{w : |w - \alpha| < \operatorname{Re} w + \alpha\} = \{w = u + iv : v^2 < 4\alpha u\},$$

contained in the right half-plane. The class denoted by $CP(\alpha)$ was introduced in [6]. Obviously, an Alexander's type theorem relates classes $SP(\alpha)$ and $CP(\alpha)$. Thus, for each $\alpha > 0$, $CP(\alpha)$ is the subclass of S^c — the family of convex functions.

The aim of this paper is to give conditions such that for $f \in CP(\alpha)$ all functions $g \in N_\delta(f)$ are in the class $SP(\alpha)$.

2. Main results

The Hadamard product or convolution of two power series $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ and $g(z) = z + \sum_{k=2}^{\infty} b_k z^k$ is defined as $(f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k$. Using the properties of convolution we give the definition of the class $SP(\alpha)$ in a different way.

Note, that according to (4), for $f \in SP(\alpha)$ we have $zf'(z)/f(z) \neq t \pm 2\sqrt{\alpha t}\, i, z \in K$, $t \geq 0$. Now, for fix $\alpha > 0$ we define $SP'(\alpha)$ as a class of all functions H_t of the form

$$H_t(z) = \frac{f_2(z) - (t \pm 2\sqrt{\alpha t}\, i) f_1(z)}{1 - (t \pm 2\sqrt{\alpha t}\, i)}, \quad t \geq 0,\ z \in K, \tag{5}$$

where $f_1(z) = \frac{z}{1-z}$ and $f_2(z) = \frac{z}{(1-z)^2}$, $z \in K$.

THEOREM 1. *A function f is in the class $SP(\alpha)$, $\alpha > 0$, if and only if $\frac{1}{z}(f * H_t)(z) \neq 0$ in K for all $H_t \in SP'(\alpha)$.*

Proof. Assume that $\frac{1}{z}(f * H_t)(z) \neq 0$ in K for all $H_t \in SP'(\alpha)$. Thus

$$0 \neq \frac{1}{z}(f * H_t)(z) = \frac{f(z) * \frac{z}{(1-z)^2} - (t \pm 2\sqrt{\alpha t}\ i) f(z) * \frac{z}{1-z}}{z[1 - (t \pm 2\sqrt{\alpha t}\ i)]} =$$

$$= \frac{zf'(z) - (t \pm 2\sqrt{\alpha t}\ i) f(z)}{z[1 - (t \pm 2\sqrt{\alpha t}\ i)]}, \quad t \geq 0.$$

Hence $\frac{zf'(z)}{f(z)} \neq t \pm 2\sqrt{\alpha t}\, i$, $t \geq 0$. Since $\frac{zf'(z)}{f(z)} = 1$ at $z = 0$ and $\delta\Omega(\alpha) = \{t \pm 2\sqrt{\alpha t}\, i : t \geq 0\}$, we have $\frac{zf'(z)}{f(z)} \in \Omega(\alpha)$, $z \in K$. Therefore $f \in SP(\alpha)$.

Conversely, let $f \in SP(\alpha)$ for fixed $\alpha > 0$. Since $\frac{zf'(z)}{f(z)} \neq t \pm 2\sqrt{\alpha t}\, i$, $t \geq 0$, for $f \in SP(\alpha)$, then for all $H_t \in SP'(\alpha)$ holds

$$\frac{1}{z}(f * H_t)(z) = \frac{\frac{zf'(z)}{f(z)} - (t \pm 2\sqrt{\alpha t}\ i)}{1 - (t \pm 2\sqrt{\alpha t}\ i)} \frac{f(z)}{z} \neq 0.$$

This ends the proof.

We need the following

LEMMA 1. *If $H_t(z) = z + \sum_{k=2}^{\infty} h_k(t) z^k \in SP'(\alpha)$, $\alpha > 0$, then*

$$|h_k(t)| \leq \begin{cases} \frac{k}{2\sqrt{\alpha(1-\alpha)}} & \text{for } 0 < \alpha < 1/2, \\ k & \text{for } \alpha \geq 1/2, \end{cases}$$

for all $t \geq 0$.

Proof. Let for $\alpha > 0$ and $t \geq 0$

$$H_t(z) = \frac{1}{1-(t \pm 2\sqrt{\alpha t}\; i)}\left[\frac{z}{(1-z)^2} - (t \pm 2\sqrt{\alpha t}\; i)\frac{z}{1-z}\right] = z + \sum_{k=2}^{\infty} h_k(t) z^k.$$

Comparing the coefficients of both sides, we get

$$|h_k(t)| = \left|\frac{k-(t \pm 2\sqrt{\alpha t}\; i)}{1-(t \pm 2\sqrt{\alpha t}\; i)}\right|, \qquad t \geq 0.$$

Thus

$$|h_k(t)|^2 = \frac{(k-t)^2 + 4\alpha t}{(1-t)^2 + 4\alpha t} = 1 + \frac{(k-1)(k+1-2t)}{(1-t)^2+4\alpha t} \leq 1 + \frac{(k-1)(k+1)}{(1-t)^2 + 4\alpha t}$$

from $t \geq 0$. Now, if $t \geq 0$ then

$$(1-t)^2 + 4\alpha t \geq \begin{cases} 4\alpha(1-\alpha) & \text{for } 0 < \alpha < 1/2, \\ 1 & \text{for } \alpha \geq 1/2. \end{cases}$$

Hence

$$|h_k(t)|^2 \leq 1 + k^2 - 1 = k^2 \quad \text{for } \alpha \geq 1/2$$

and

$$|h_k(t)|^2 \leq 1 + \frac{k^2-1}{4\alpha(1-\alpha)} \leq \frac{k^2}{4\alpha(1-\alpha)} \qquad \text{for } 0 < \alpha < 1/2,$$

as desired.

For each complex number ϵ we define the function F_ϵ as follows

$$F_\epsilon(z) = \frac{f(z) + \epsilon z}{1+\epsilon}. \tag{6}$$

THEOREM 2. *Let $f \in A$ and $\delta > 0$. Assume that for ϵ such that $|\epsilon| < \delta$, holds $F_\epsilon \in SP(\alpha)$, where F_ϵ is defined by* (6). *Then for every $H_t \in SP'(\alpha)$, $\alpha > 0$, holds*

$$\left|\frac{1}{z}(f * H_t)(z)\right| \geq \delta, \qquad z \in K.$$

Proof. If $F_\epsilon \in SP(\alpha)$ for $|\epsilon| < \delta$, where $\delta > 0$ is fixed, then according to Theorem 1, holds $\frac{1}{z}(F_\epsilon * H_t)(z) \neq 0$ in K for all $H_t \in SP'(\alpha)$. Hence

$$\frac{(f * H_t)(z) + \epsilon z}{(1+\epsilon)z} \neq 0 \quad \text{or} \quad \frac{1}{z}(f * H_t)(z) \neq -\epsilon \quad \text{for } z \in K,$$

so $\left|\frac{1}{z}(f * H_t)(z)\right| \geq \delta$ follows from $|\epsilon| < \delta$.

THEOREM 3. *Let $f \in A$ and $\delta > 0$. Assume that for $\epsilon \in \mathbb{C}$, $|\epsilon| < \delta$, the function F_ϵ, defined by* (6), *is in $SP(\alpha)$, $\alpha > 0$. Then $N_{\delta'}(f) \subset SP(\alpha)$,* *where*

$$\delta' = \begin{cases} 2\sqrt{\alpha(1-\alpha)}\,\delta & \text{for } 0 < \alpha < 1/2, \\ \delta & \text{for } \alpha \ge 1/2. \end{cases}$$

Proof. Let $g(z) = z + \sum_{k=2}^{\infty} b_k z^k \in N_{\delta'}(f)$. Then for any $H_t \in SP'(\alpha)$ we have

$$\left|\frac{1}{z}(g * H_t)(z)\right| = \left|\frac{1}{z}(f * H_t)(z) + \frac{1}{z}((g-f) * H_t)(z)\right| \ge$$
$$\ge \left|\frac{1}{z}(f * H_t)(z)\right| - \left|\frac{1}{z}((g-f) * H_t)(z)\right|.$$

But by Theorem 2

$$\left|\frac{1}{z}(f * H_t)(z)\right| \ge \delta, \quad z \in K,$$

hence

$$\left|\frac{1}{z}(g * H_t)(z)\right| \ge \delta - \left|\sum_{k=2}^{\infty} \frac{(b_k - a_k)h_k(t)z^k}{z}\right| \ge$$
$$\ge \delta - |z| \sum_{k=2}^{\infty} |h_k(t)|\,|b_k - a_k| > \delta - \sum_{k=2}^{\infty} |h_k(t)||b_k - a_k|.$$

Next, in viev of Lemma 1 we have

$$\sum_{k=2}^{\infty} |h_k(t)||b_k - a_k| \le \begin{cases} \frac{1}{2\sqrt{\alpha(1-\alpha)}} \sum_{k=2}^{\infty} k|b_k - a_k| & \text{for } 0 < \alpha < 1/2, \\ \sum_{k=2}^{\infty} k|b_k - a_k| & \text{for } \alpha \ge 1/2. \end{cases}$$

From $g \in N_{\delta'}(f)$ it follows that

$$\left|\frac{1}{z}(g * H_t)(z)\right| \ge \begin{cases} \delta - \frac{\delta'}{2\sqrt{\alpha(1-\alpha)}} & \text{for } 0 < \alpha < 1/2, \\ \delta - \delta' & \text{for } \alpha \ge 1/2. \end{cases}$$

Therefore $\left|\frac{1}{z}(g * H_t)(z)\right| \ne 0$ in K for all $H_t \in SP'(\alpha)$ if

$$\delta' = \begin{cases} 2\sqrt{\alpha(1-\alpha)}\delta & \text{for } 0 < \alpha < 1/2, \\ \delta & \text{for } \alpha \ge 1/2. \end{cases}$$

By Theorem 1, this means that $g \in SP(\alpha)$, or equivalently $N_{\delta'}(f) \subset SP(\alpha)$, so the result follows.

For the proof of next theorem we need the following result obtained in [4].

LEMMA 2. *If $f \in S^c$ and $g \in SP(\alpha)$, $\alpha > 0$, then $f * g \in SP(\alpha)$.*

THEOREM 4. *If $f \in CP(\alpha)$, $\alpha > 0$, then the function F_ϵ, defined by* (6), *belongs to $SP(\alpha)$ for $|\epsilon| < 1/4$.*

Proof. Assume that $f(z) = z + \sum_{k=2}^{\infty} a_k z^k \in CP(\alpha)$, where $\alpha > 0$ is fixed. Then

$$F_\epsilon(z) = \frac{f(z) + \epsilon z}{1+\epsilon} = \frac{z(1+\epsilon) + \sum_{k=2}^{\infty} a_k z^k}{1+\epsilon} =$$
$$= \frac{f(z) * [z(1+\epsilon) + \sum_{k=2}^{\infty} z^k]}{1+\epsilon} = f(z) * \left(\frac{z - \frac{\epsilon}{1+\epsilon} z^2}{1-z} \right) = (f * h)(z),$$

where

$$h(z) = \frac{z(1-\rho z)}{1-z}, \quad z \in K, \quad \rho = \frac{\epsilon}{1+\epsilon}.$$

It is easy to see that if $|\epsilon| < 1/4$, then the function h is starlike. In fact, we have

$$\frac{zh'(z)}{h(z)} = \frac{1}{1-z} - \frac{\rho z}{1-\rho z},$$

hence

$$\operatorname{Re} \frac{zh'(z)}{h(z)} > 0 \quad \text{if} \quad |\rho| < \frac{1}{|z|^2 + 2|z|}.$$

The last inequality holds for $z \in K$, if $|\rho| < 1/3$, which is true for $|\epsilon| < 1/4$. Therefore for $|\epsilon| < 1/4$ the function

$$z \to \int_0^z \frac{h(t)}{t} \, dt = h(z) * \log \frac{1}{1-z}, \qquad z \in K,$$

is in S^c and we have

$$f \in CP(\alpha) \Rightarrow zf'(z) \in SP(\alpha).$$

But

$$(f * h)(z) = (h * f)(z) = h(z) * \left(zf'(z) * \log \frac{1}{1-z} \right) =$$
$$= zf'(z) * \left(h(z) * \log \frac{1}{1-z} \right).$$

Hence using Lemma 2 we get

$$F_\epsilon(z) = (f * h)(z) \in SP(\alpha) \quad \text{for } |\epsilon| < 1/4$$

and the proof is completed.

THEOREM 5. *If $f \in CP(\alpha)$, $\alpha > 0$, then $N_{\delta'}(f) \subset SP(\alpha)$, where*

$$\delta' = \begin{cases} \frac{\sqrt{\alpha(1-\alpha)}}{2} & \text{for } 0 < \alpha < 1/2, \\ 1/4 & \text{for } \alpha \geq 1/2. \end{cases}$$

Proof. Assume that $f \in CP(\alpha)$. Then from Theorem 4 it follows that the function F_ϵ, defined by (6), is in $SP(\alpha)$ for $|\epsilon| < 1/4$. Next, applying Theorem 3 with $\delta = 1/4$ we obtain desired result.

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