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FIXED POINT THEOREMS FOR WEAKLY CONDENSING AND WEAKLY COMPACT MAPS

1. Introduction

Let f be a self-map of a metric space (X,d). Set $C_f = \{g : X \to X : fg = gf\}$. G. Jungck [6] proved some fixed point theorems for C_f. Janos, Ko and Tan obtained an interesting result on fixed point (see [5, Corollary 2]). Next Shih and Yeh [9] attempted to extend the result of Janos, Ko and Tan to more general case. Recently, Z.-Q. Liu [8] proved some fixed point theorems for condensing and compact maps, which extends results of Janos [4], Leader [7], Shih and Yeh [9].

In this paper we shall discuss the existence of some fixed point theorems for weakly condensing and weakly compact maps. Our considerations extend some results of [4,7,8,9].

Throughout this paper we will denote by E a Banach space and by E_ω the space E with the weak topology $\sigma(E,E^*)$. If S is a subset of E, then $d(S)$ and $\overline{S}^\omega$ denote the diameter and the weak closure of S respectively. A function $f : E \to E$ will be called weakly continuous if it is continuous from E_ω into E_ω, and weakly sequentially continuous if for each weakly convergent sequence (x_n), the sequence $(f(x_n))$ is weakly convergent. There exist many important examples of mappings, which are weakly sequentially continuous but not weakly continuous. The relationship between strong, weak and weak sequential continuity for mappings is studied in detailin [1].

Let $B = \{x \in E : \|x\| \le 1\}$ and consider a nonempty bounded subset A of E. The measure of weak noncompactness $\beta(A)$ is defined by

$$\beta(A) = \inf\{\varepsilon > 0 : \text{there exists a weakly compact set } P \text{ such that } A \subset P + \varepsilon B\}.$$

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The properties of measure of weak noncompactness β are analogous to the properties of measure of noncompactness (see [2]).

We say that $f : E \to E$ is weakly condensing if f is weakly continuous and for any nonempty bounded subset A of E we have $\beta(f(A)) < \beta(A)$. Also f is said to be weakly compact if there exists a weakly compact subset Y of E such that $f(E) \subset Y$.

Let N denote the set of positive integers and $N_0 = N \cup \{0\}$.

For $f : E \to E$, define

$$H_f = \Big\{g : E \to E : g\Big(\bigcap_{n=0}^{\infty} f^n(E)\Big) \subset \bigcap_{n=0}^{\infty} f^n(E)\Big\},$$

and

$$CIS_f = \{g : E \to E : \text{for every nonempty weakly compact } f\text{-invariant subset } A \text{ of } E,\ g(A) \subset A\}.$$

Now we state and prove our main results.

2. Main results

THEOREM 1. *Let f be a weakly continuous self-map of a bounded weakly complete metrizable topological vector space (X, τ) and let d be a metric on X. Suppose that there exists $m \in N$ such that f^m is weakly condensing and*

$$(1) \qquad d(f^m x, f^m y) < d(\{gz : z \in \{x, y\} \text{ and } g \in CIS_f\})$$

for all $x, y \in X$ with $x \neq y$. Then

(i) *f has a unique fixed point $v \in X$ such that $f^n x \to v$ for every $x \in X$;*

(ii) *for every nonempty weakly compact f-invariant subset Y of X we have $\bigcap_{n=0}^{\infty} f^n(Y) = \{v\}$;*

(iii) *v is a unique common fixed point of the family CIS_f.*

Proof. Fix $x \in X$ and put $O(x) = \{f^n x : n \in N_0\}$. Then

$$(*) \qquad \beta(O(x)) = \max\{\beta(\{x, fx, \dots, f^{m-1}x\}), \beta(O(f^m x))\} = \\ = \beta(O(f^m x)) = \beta(f^m(O(x))).$$

Since f^m is weakly condensing, $O(x)$ is bounded, and from the weakly completness of (X, τ), we have $\overline{O(x)}^{\omega}$ is weakly compact. By the weakly continuity of f, we conclude that $f(\overline{O(x)})^{\omega} \subset \overline{f(O(x))}^{\omega} \subset \overline{O(x)}^{\omega}$; i.e., $\overline{(O(x))}^{\omega}$ is f-invariant. Hence $f^{n+1}\overline{(O(x))}^{\omega} \subset f^n\overline{(O(x))}^{\omega}$ for $n \in N$. From the weakly compactness of $\overline{O(x)}^{\omega}$ and the weakly continuity of f,it follows that $\{f^n\overline{(O(x))}^{\omega} : n \in N\}$ has the finite intersection property. Thus $\bigcap_{n=1}^{\infty} f^n\overline{(O(x))}^{\omega}$ is a nonempty weakly compact subset of $\overline{(O(x))}^{\omega}$.

Let $D = \bigcap_{n=1}^{\infty} f^n\overline{(O(x))}^{\omega}$. We prove that $f(D) = D$. For every $v \in D$, there exists $x_n \in f^{n-1}\overline{(O(x))}^{\omega}$ such that $fx_n = v$, $n \in N$. From the weakly compactness of $\overline{(O(x))}^{\omega}$ we may (by selecting a subsequence, if necessary) assume that $x_n \to u \in \overline{(O(x))}^{\omega}$.

Since $\{x_{n+1}, x_{n+2}, \ldots\} \subset f^n\overline{(O(x))}^{\omega}$ and $f^n\overline{(O(x))}^{\omega}$ is weakly compact then $u \in f^n\overline{(O(x))}^{\omega}$ for $n \in N$. Therefore $u \in D$ and by $fu = v$, we have $D \subset f(D)$. Clearly $f(D) \subset D$. Hence $f(D) = D$. Thus $f^m(D) = D$.

We shall show that D is a singleton. Otherwise $d(D) > 0$, so by the weakly compactness, there exists $p, q \in D$ with $p \neq q$ such that $d(p,q) = d(D)$. Since $f^m(D) = D$ there exist $x, y \in D$ with $x \neq y$ such that $p = f^m x$, $q = f^m y$; i.e., $d(f^m x, f^m y) = d(D)$. From (1) we have

$$0 < d(D) = d(f^m x, f^m y) < d(\{gz : z \in \{x,y\} \text{ and } g \in CIS_f\}).$$

By the definition of CIS_f

$$\{gz : z \in \{x,y\} \text{ and } g \in CIS_f\} \subset D.$$

Then

$$0 < d(D) = d(f^m x, f^m y) < d(\{gz : z \in \{x,y\} \text{ and } g \in CIS_f\}) \leq d(D).$$

A contradiction. Thus D is a singleton, say $D = \{v\}$. Hence v is a fixed point of f.

Let $b(\neq v)$ be another fixed point of f. Put $A = \{v, b\}$.
It is easy to see that $\{gz : z \in \{v,b\} \text{ and } g \in CIS_f\} \subset A$. By (1), we have

$$d(v,b) = d(f^m v, f^m b) < d(\{gz : z \in \{v,b\} \text{ and } g \in CIS_f\}) \leq d(v,b),$$

a contradiction. Hence v is the unique fixed point of f.

Since $f^n x \in f^n\overline{(O(x))}^{\omega}$ and $\bigcap_{n=1}^{\infty} f^n\overline{(O(x))}^{\omega} = \{v\}$. Therefore $d(f^n x, v) \leq d(f^n(\overline{O(x)})^{\omega}) \to 0$ as $n \to \infty$; i.e., $f^n x \to v$ as $n \to \infty$. This proves (i).

Similarly we can show that (ii) holds.

Now we prove that (iii) holds. Since for any $g \in CIS_f$, it follows from (i) and the definition of CIS_f that $g(\{v\}) \subset \{v\}$; i.e., $g(v) = v$. Hence v is a fixed point of CIS_f. Note that $f \in CIS_f$. By (i), v is the only fixed point of CIS_f. This completes the proof.

Similar as in [8] (th. 4.1) we can prove the following theorem:

THEOREM 2. *Let f be a weakly sequentially continuous self-map of a metrizable locally convex space (X, τ) and let d be a metric on X. Suppose that there exists $m \in N$ such that f^m is weakly compact and*

$$d(f^m x, f^m y) < d(\{gz : z \in \{x,y\} \text{ and } g \in H_f\}) \tag{2}$$

for all $x, y \in X$ with $x \neq y$. Then

(i) *f has a unique fixed point $v \in X$ such that $f^n x \to v$ for every $x \in X$;*
(ii) *H_f has a unique fixed point v.*

As there are weakly sequentially continuous mappings, which are not continuous and conversely (see [1]) so we may prove the following theorem:

THEOREM 3. *Let f be a weakly sequentially continuous self-map of a bounded weakly complete metrizable topological vector space (X,τ) and let d be a metric on X. Suppose that there exists $m \in N$ such that f^m is weakly condensing and satisfies* (1). *Then* (i), (ii) *and* (iii) *of Theorem* 1 *hold.*

Proof. Similarly as in Theorem 1, we can prove that $\overline{O(x)}^{\omega}$ is weakly compact. As f is weakly sequentially continuous we prove that f is weakly continuous on $\overline{O(x)}^{\omega}$. Let A be a weakly closed subset of $\overline{O(x)}^{\omega}$. As f is sequentially continuous $f^{-1}(A)$ is sequentially closed in $\overline{O(x)}^{\omega}$ so, by Eberlein–Šmulian theorem ([3], th. 8.12.4) $f^{-1}(A)$ is weakly closed. Hence f is weakly continuous on $\overline{O(x)}^{\omega}$.

The theorem now follows easily by the use of similar argument to that of the corresponding part of Theorem 1.

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