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FIXED COEFFICIENTS FOR NEW CLASSES OF UNIVALENT FUNCTIONS WITH NEGATIVE COEFFICIENTS

In this paper we consider the class $R^*_{n,c}$ consisting of analytic and univalent functions with negative coefficients and fixed second coefficient. The object of the present paper is to show coefficient estimates, convex linear paper is to show coefficient estimates, convex linear combination, some distortion theorems and radii of starlikeness and convexity for $f(z)$ in the class $R^*_{n,c}$. The results are generalized to families with finitely many fixed coefficients.

1. Introduction

Let A stands for the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1.1}$$

which are analytic in the unit disc $U = \{z : |z|(1\}$. We denote by S the subclass of univalent functions $f(z)$ in A. The Hadamard product of two functions $f(z) \in A$ and $g(z) \in A$ will be denoted by $f * g(z)$, that is, if $f(z)$ is given by (1.1) and $g(z)$ is given by

$$g(z) = z + \sum_{k=2}^{\infty} b_k z^k, \tag{1.2}$$

then

$$f * g(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k. \tag{1.3}$$

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Let

$$D^n f(z) = \frac{z[z^{n-1}f(z)]^{(n)}}{n!}, \tag{1.4}$$

for $n \in \mathbb{N}_0 = \mathbb{N} \cup (0)$ and $z \in U$, where $\mathbb{N} = \{1, 2, \ldots\}$. This symbol $D^n f(z)$ has been named by Al-Amiri [1] the n-th order Ruscheweyh derivative of $f(z)$. We note that $D^0 f(z) = f(z)$ and $D^1 f(z) = zf'(z)$.

Introduce, using the Hadamard product, operator

$$D^\beta f(z) = \frac{z}{(1-z)^{\beta+1}} * f(z), \quad \beta \geq -1. \tag{1.5}$$

Ruscheweyh in [3] observed that (1.4) and (1.5) are equivalent when $\beta = n \in \mathbb{N}_0$.

It is easy to see that

$$D^n f(z) = z + \sum_{k=2}^{\infty} \delta(n,k) a_k z^k, \tag{1.6}$$

where

$$\delta(n,k) = \binom{n+k-1}{n} \tag{1.7}$$

Denote by T the subclass of S consisting of functions $f(z)$ having the form

$$f(z) = z - \sum_{k=2}^{\infty} a_k z^k, \quad a_k \geq 0. \tag{1.8}$$

In [2] Owa studied the classes R_n^* defined by

$$R_n^* = \left\{ f \in T : \operatorname{Re}\left\{ \frac{D^{n+1}f(z)}{D^n f(z)} \right\} > \frac{n}{n+1}, \text{ for } z \in U \right\}. \tag{1.9}$$

For the classes R_n^*, Owa [2] showed the following lemma

LEMMA 1. *Let the function $f(z)$ satisfies* (1.8). *Then $f \in R_n^*$, $n \in \mathbb{N}_0$, if and only if*

$$\sum_{k=2}^{\infty} k\delta(n,k) a_k \leq 1. \tag{1.10}$$

The above results is sharp. In particuler it means that for the functions $f \in R_n^*$ we have

$$a_2 \leq \frac{1}{2(n+1)}. \tag{1.11}$$

Denote by $R^*_{n,c}$ the class of functions $f(z)$ of the form

$$f(z) = z - \frac{c}{2(n+1)}z^2 - \sum_{k=3}^{\infty} a_k z^k; \qquad a_k \geq 0, \tag{1.12}$$

where $0 \leq c \leq 1$ is fixed and observe that by (1.1)

$$R^*_{n,c} \subset R^*_n.$$

2. Coefficient estimates

THEOREM 1. *For the class $R^*_{n,c}$ defined by (1.12) we have the following characterization: Function $f(z) \in R^*_{n,c}$ if and only if*

$$\sum_{k=3}^{\infty} k\delta(n,k)a_k \leq 1 - c. \tag{2.1}$$

Proof. Putting

$$a_2 = \frac{c}{2(n+1)}, \qquad 0 \leq c \leq 1, \tag{2.2}$$

in (1.10) and simplifying we get the result.

The result is sharp. In particular, it means that functions

$$f(z) = z - \frac{c}{2(n+1)}z^2 - \frac{(1-c)}{k\delta(n,k)}z^k, \qquad k \geq 3, \tag{2.3}$$

are elements of $R^*_{n,c}$. Moreover, we have

COROLLARY 1. *Let the function $f(z)$ be defined by* (1.12). *Then*

$$a_k \leq \frac{(1-c)}{k\delta(n,k)}, \qquad k \geq 3. \tag{2.4}$$

3. Convexity theorems

One can easily observe that the class $R^*_{n,c}$ is convex.

This situation can be, in a sense, reversed. Namely, we have the following

THEOREM 2. *Let*

$$f_2(z) = z - \frac{c}{2(n+1)}z^2 \tag{3.1}$$

and

$$f_k(z) = z - \frac{c}{2(n+1)}z^3 - \frac{(1-c)}{k\delta(n,k)}z^k \tag{3.2}$$

for $k = 3, 4, \ldots$ *Then* $f(z)$ *is in the class* $R^*_{n,c}$ *if and only if it can be expressed in the form*

$$f(z) = \sum_{k=2}^{\infty} \lambda_k f_k(z), \tag{3.3}$$

where $\lambda_k \geq 0$ *and* $\sum_{k=2}^{\infty} \lambda_k = 1$.

Proof. Suppose that $f(z)$ can be expressed in the form (3.3). Then we have

$$f(z) = z - \frac{c}{2(n+1)} z^2 - \sum_{k=3}^{\infty} \frac{(1-c)\lambda_k}{k\delta(n,k)} z^k. \tag{3.4}$$

Since

$$\sum_{k=3}^{\infty} \frac{(1-c)\lambda_k}{k\delta(n,k)} \cdot k\delta(n,k) = (1-c)(1-\lambda_2) \leq 1-c, \tag{3.5}$$

then it follows from (2.1) that $f(z)$ is in the class $R^*_{n,c}$.

Conversely, let $f(z)$ defined by (1.12) is in the class $R^*_{n,c}$. Then, by using (2.4) we get

$$a_k \leq \frac{(1-c)}{k\delta(n,k)}, \quad k \geq 3. \tag{3.6}$$

Setting

$$\lambda_k = \frac{k\delta(n,k)}{(1-c)} a_k, \quad k \geq 3 \tag{3.7}$$

and

$$\lambda_2 = 1 - \sum_{k=3}^{\infty} \lambda_k, \tag{3.8}$$

we have (3.3). This completes the proof of the theorem.

COROLLARY 3. *The extreme points of the class* $R^*_{n,c}$ *are the functions* $f_k(z) (k \geq 2)$ *given by Theorem* 2.

4. Distortion theorems

We begin with the following

LEMMA 2. *Let the function* $f_3(z)$ *be defined by* (3.2) *and define* c_0, r_0 *putting*

$$c_0 = \frac{1}{4}\{-(3n^2+25n+20) + \sqrt{(3n^2+25n+20)^2 + 28(n+1)}\} \tag{4.1}$$

and

(4.2) $r_0 =$

$$= \frac{-16(1-c)(n+1)+\sqrt{256(1-c)^2(n+1)^2+24c^2(1-c)(n+1)(n+2)}}{4c(n-c)}.$$

Then, for $0 \le r < 1$ and $0 \le c \le 1$, we have

$$|f_3(re^{i\theta})| \ge r - \frac{c}{2(n+1)}r^2 - \frac{2(1-c)}{3(n+1)(n+2)}r^3 \tag{4.3}$$

with eauality for $\theta = 0$.

For either $0 \le c < c_0$ and $0 \le r \le r_0$ or $c_0 \le c \le 1$.

$$|f_3(re^{i\theta})| \le r + \frac{c}{2(n+1)}r^2 - \frac{2(1-c)}{3(n+1)(n+2)}r^3 \tag{4.4}$$

with eauality for $\theta = \pi$.

Further, for $0 \le c < c_0$ and $r_0 \le r < 1$,

$$|f_3(re^{i\theta})| = \tag{4.5}$$

$$\le r\left\{\left[1+\frac{3c^2(n+2)}{32(1-c)(n+1)}\right]+\left[\frac{4(1-c)}{3(n+1)(n+2)}+\frac{c^2}{8(n+1)^2}\right]r^2 + \left[\frac{4(1-c)^2}{9(n+1)^2(n+2)^2}+\frac{c^2(1-c)}{24(n+1)^3(n+2)}\right]r^4\right\}^{\frac{1}{2}}$$

with eauality for

$$\theta = \pm\cos^{-1}\left(\frac{2c(1-c)r^2-3c(n+1)(n+2)}{16(1-c)(n+1)r}\right). \tag{4.6}$$

Proof. We employ the same technique as it was used by Silverman and Silvia [4]. Since

$$\frac{\partial|f_3(re^{i\theta})|^2}{\partial\theta} = \frac{r^3\sin\theta}{(n+1)}\left[c+\frac{16(1-c)}{3(n+2)}r\cos\theta - \frac{2c(1-c)}{3(n+1)(n+2)}r^2\right] \tag{4.7}$$

we can see that

$$\frac{\partial|f_3(re^{i\theta})|^2}{\partial\theta} = 0 \tag{4.8}$$

for $\theta_1 = 0, \theta_2 = \pi$, and

$$\theta_3 = \pm\cos^{-1}\left(\frac{2c(1-c)r^2-3c(n+1)(n+2)}{16(1-c)(n+1)r}\right). \tag{4.9}$$

Since θ_3 is a valid root only when $-1 \le cos\theta_3 \le 1$. Hence we have a third root if and only if $r_0 \le r < 1$ and $0 \le c < c_0$. Thus the results of the theorem follows from comparing the extremal values $|f_3(re^{i\theta})|$ $(l = 1,2,3)$, on the appropriate intervals.

LEMMA 3. *Let the functions* $f_k(z)$ *be defined by* (3.2) *and* $k \geq 4$. *Then*

$$|f_k(re^{i\theta})| \leq |f_4(-r)|. \tag{4.10}$$

and

$$|f_k'(re^{i\theta})| \leq |f_4'(-r)|. \tag{4.11}$$

Proof. Since $f_k(z) = z - \frac{c}{2(n+1)}z^2 - \frac{(1-c)}{k\delta(n,k)}z^k$ and $\frac{(1-c)r^k}{k\delta(n,k)}$ is a decreasing function of k, we have

$$\begin{aligned}|f_k(re^{i\theta})| &\leq r + \frac{c}{2(n+1)}r^2 + \frac{(1-c)}{k\delta(n,k)}r^k \\ &\leq r + \frac{c}{2(n+1)}r^2 + \frac{6(1-c)}{4(n+1)(n+2)(n+3)}r^4 = -f_4(-r)\end{aligned}$$

which shows (4.10).

In the same way we obtain (4.11).

THEOREM 3. *Let the function* $f(z)$ *belongs to the class* $R^*_{n,c}$. *Then for* $0 \leq r < 1$, *we have*

$$|f(re^{i\theta})| \geq r - \frac{c}{2(n+1)}r^2 - \frac{2(1-c)}{3(n+1)(n+2)}r^3 \tag{4.12}$$

with equality for $f_3(z)$ *at* $z = r$, *and*

$$|f(re^{i\theta})| \leq \text{Max}\{\underset{\theta}{\text{Max}}|f_3(re^{i\theta})|, -f_4(-r)\}, \tag{4.13}$$

where $\text{Max}_\theta\, f_3(re^{i\theta})|$ *is given by Lemma* 2.

The proof of Theorem 3 is obtained by comparing the bounds in Lemma 2 and Lemma 3.

Remark. Putting $c = 1$ in Theorem 3 we obtain the following result due to Owa [2]:

COROLLARY 4. *Let the function* $f(z)$ *defined by* (1.8) *be in the class* R^*_n. *Then for* $|z| = r < 1$, *we have*

$$r - \frac{1}{2(n+1)}r^2 \leq |f(z) \leq r + \frac{1}{2(n+1)}r^2. \tag{4.14}$$

LEMMA 4. *Let the function* $f_3(z)$ *be defined by* (3.2) *and let*

$$c_1 = \frac{1}{4}\{-(n^2 + 11n + 8) + \sqrt{(n^2 + 11n + 8)^2 + 64(n+1)}\} \tag{4.15}$$

and

(4.16) $r_1 =$

$$= \frac{-8(1-c)(n+1) + \sqrt{[8(1-c)(n+1)]^2 + 8c^2(1-c)(n+1)(n+2)}}{4c(1-c)}.$$

Then, for $0 \le r < 1$ *and* $0 \le c \le 1$, *we have*

$$|f_3'(re^{i\theta})| \ge 1 - \frac{c}{(n+1)}r - \frac{2(1-c)}{(n+1)(n+2)}r^2 \tag{4.17}$$

with eouality for $\theta = 0$.

For either $0 \le c < c_1$ *and* $0 \le r \le r_1$ *or* $c_1 \le c \le 1$,

$$|f_3'(re^{i\theta})| \le 1 + \frac{c}{(n+1)}r - \frac{2(1-c)}{(n+1)(n+2)}r^2 \tag{4.18}$$

with eouality for $\theta = \pi$.

Further, for $0 \le c < c_1$ *and* $r_1 \le r < 1$,

$$|f'(re^{i\theta})| \le \left\{ \left[1 + \frac{c^2(n+2)}{8(1-c)(n+1)}\right] + \left[\frac{c^2}{2(n+1)^2} + \frac{4(1-c)}{(n+1)(n+2)}\right] r^2 + \left[\frac{4(1-c)^2}{(n+1)^2(n+2)^2} + \frac{c^2(1-c)}{2(n+1)^3(n+2)}\right] r^4 \right\}^{\frac{1}{2}} \tag{4.19}$$

with eouality for

$$\theta = \pm \cos^{-1}\left(\frac{2c(1-c)r^2 - c(n+1)(n+2)}{8(1-c)(n+1)r}\right). \tag{4.20}$$

The proof of Lemma 4 is given in the same way as Lemma 2.

By Lemma 3 and Lemma 4 we have immediately:

THEOREM 4. *Let the function* $f(z)$ *be in the class* $R^*_{n,c}$. *Then for* $0 \le r < 1$ *we have*

$$|f'(re^{i\theta})| \ge 1 - \frac{c}{(n+1)}r - \frac{2(1-c)}{(n+1)(n+2)}r^2 \tag{4.21}$$

with eouality for $f_3'(z)$ *at* $z = r$, *and*

$$|f'(re^{i\theta})| \le \text{Max}\{\underset{\theta}{\text{Max}}\,|f_3'(re^{i\theta}), f_4'(-r)\} \tag{4.22}$$

where $\text{Max}_\theta\,|f_3'(re^{i\theta})|$ *is given by Lemma* 4.

Remark. Putting $c = 1$ in Theorem 4 we obtain the following result due to Owa [2]:

THEOREM 5. *Let the function $f(z)$ defined by (1.8) be in the class R_n^*. Then for $|z| = r < 1$, we have*

$$1 - \frac{r}{(n+1)} \leq |f'(z)| \leq 1 + \frac{r}{(n+1)}. \tag{4.23}$$

5. Radii of starlikeness and convexity

THEOREM 6. *Assume that the function $f(z)$ be in the class $R_{n,c}^*$. Then $f(z)$ is starlike of order ϱ $(0 \leq \varrho < 1)$ in the disc $|z| < r_1(n, c, \varrho)$, where $r_1(n, c, \varrho)$ is the largest value for which*

$$\frac{c(2-\varrho)}{2(n+1)} r + \frac{(1-c)(k-\varrho)}{k\delta(n,k)} r^{k-1} \leq 1 - \varrho, \tag{5.1}$$

for $k \geq 3$. The ineauality is sharp with the extremal function

$$f_k(z) = z - \frac{c}{2(n+1)} z^2 - \frac{(1-c)}{k\delta(n,k)} z^k \quad \textit{for some } k. \tag{5.2}$$

Proof. It suffices to show that

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| \leq 1 - \varrho \quad (0 \leq \varrho < 1) \quad \text{for } |z| < r_1(n, c, \varrho).$$

Note that

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| \leq \frac{\frac{c}{2(n+1)} r + \sum_{k=3}^{\infty} (k-1) a_k r^{k-1}}{1 - \frac{c}{2(n+1)} r - \sum_{k=3}^{\infty} a_k r^{k-1}} \leq 1 - \varrho \tag{5.3}$$

for $|z| \leq r$ if and only if

$$\frac{c(2-\varrho)}{2(n+1)} r + \sum_{k=3}^{\infty} (k-\varrho) a_k r^{k-1} \leq 1 - \varrho. \tag{5.4}$$

Since $f(z)$ is in $R_{n,c}^*$, by (2.1) we may take

$$a_k = \frac{(1-c)\lambda_k}{k\delta(n,k)}, \quad k \geq 3, \tag{5.5}$$

where $\lambda_k \geq 0, k \geq 3$ and

$$\sum_{k=3}^{\infty} \lambda_k \leq 1. \tag{5.6}$$

For each fixed r, choose the positive integer $k_0 = k_0(r)$ for which the expression $\frac{(k_0-\varrho)}{k_0\delta(n,k_0)} r^{k_0-1}$ is maximal. Then it follows that

$$\sum_{k=3}^{\infty} (k-\varrho) a_k r^{k-1} \leq \frac{(1-c)(k_0-\varrho)}{k_0\delta(n,k_0)} r^{k_0-1}. \tag{5.7}$$

Hence we conclude that $f(z)$ is starlike of order ϱ in $|z| < r_1(n,c,\varrho)$ provided that

$$\frac{c(2-\varrho)}{2(n+1)}r + \frac{(1-c)(k_0-\varrho)}{k_\delta(n,k_0)}r^{k_0-1} \leq 1-\varrho. \tag{5.8}$$

We find the value $r_0 = r_0(n,c,\varrho)$ and the corresponding integer $k_0(t_0)$ so that

$$\frac{c(2-\varrho)}{2(n+1)}r_0 + \frac{(1-c)(k_0-\varrho)}{k_0\delta(n,k_0)}r_0^{k_0-1} = 1-\varrho. \tag{5.9}$$

Then this value r_0 is the radius of starlikeness of order ϱ for functions $f(z)$ belonging to the class $R^*_{n,c}$.

In a similar manner, we can prove the following theorem concerning the radius of convexity of order ϱ for functions from the class $R^*_{n,c}$.

THEOREM 7. *Let the function $f(z)$ be in the class $R^*_{n,c}$. Then $f(z)$ is convex of order ϱ $(0 \leq \varrho < 1)$ in the disc $|z| < r_2(n,c,\varrho)$, where $r_2(n,c,\varrho)$ is the largest value for which*

$$\frac{c(2-\varrho)}{(n+1)}r + \frac{(1-c)(k-\varrho)}{\delta(n,k)}r^{k-1} \leq 1-\varrho, \tag{5.10}$$

for $k \geq 3$. The result is sharp for the function $f(z)$ given by (5.2).

6. The class $R^*_{n,c_k,N}$

Instead of fixing just the second coefficient, we can fix finitely many coefficients. Let $R^*_{n,c_k,N}$ denote the class of functions in $R^*_{n,c}$ of the form

$$(z) = z - \sum_{k=2}^{N} \frac{c_k}{2(n+1)}z^k - \sum_{k=N+1}^{\infty} a_k z^k, \tag{6.1}$$

where $c_k \geq 0, 0 \leq \sum_{k=2}^{N} c_k = c \leq 1$. Note that $R^*_{n,c_k,2} = R^*_{n,c}$.

THEOREM 8. *The extreme points of $R^*_{n,c_k,N}$ are of the form*

$$z - \sum_{k=2}^{N} \frac{c_k}{k\delta(n,k)}z^k$$

and

$$z - \sum_{k=2}^{N} \frac{c_k}{k\delta(n,k)}z^k - \frac{(1-c)}{k\delta(n,k)}z^k \quad \textit{for } k = N+1, N+2, \ldots$$

The details of the proof are omitted.

The characterization of the extreme points enables us to solve the standard extremal problems in the same manner as it was done for $R^*_{n,c}$. We omit the details.

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