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THE POLAR GEODESIC COORDINATE SYSTEM ON SURFACES IN THE EUCLIDEAN 3-DIMENSIONAL SPACE

1. Introduction

Let (M^2, g) denotes a differentiable, i.e. of class C^∞, complete 2-dimensional manifold provided with a Riemannian metric g. We suppose that M^2 admits a (global) polar, geodesic coordinate system (u, v) in the following sense. The metric g can be written in the form

$$(1.1)\qquad ds^2 = du^2 + B^2(u,v)dv^2, \quad 0 \le u < \infty, \quad 0 \le v < 2\pi,$$

where (see [1], p. 79)

$$(1.2)\qquad B(0,v) = 0, \quad B'(0,v) = 1, \quad 0 \le v < 2\pi.$$

The prime denotes differentiation with respect to u. We extend in (1.1) the variable u from $0 \le u < \infty$ to $-\infty < u < \infty$ by means of the equivalence relation defined on pairs (u, v) by

$$(1.3)\qquad (-u, v) \approx (u, (v+\pi)\mathrm{mod}2\pi), \quad u \ne 0, \quad 0 \le v < 2\pi,$$

and

$$(1.4)\qquad (0, v_1) \approx (0, v_2), \quad 0 \le v_1, v_2 < 2\pi.$$

More generally, if

$$(1.5)\qquad B(u_0, v) = 0, \quad -\infty < u_0 < \infty, \quad u_0 = \mathrm{const}, \quad 0 \le v < 2\pi,$$

then we extend the equivalence relation defined by (1.3), (1.4) setting

$$(1.6)\qquad (u_0 - u, v) \approx (u_0 + u, (v+\pi)\mathrm{mod}2\pi), \quad u \ne u_0, \quad 0 \le v < 2\pi,$$

and

$$(1.7)\qquad (u_0, v_1) \approx (u_0, v_2), \quad 0 \le v_1, v_2 \le 2\pi.$$

We suppose that the function $B(u,v)$ takes the same value at equivalent pairs $(u,v), -\infty < u < \infty, 0 \leq v < 2\pi$.

If we consider the ratio of the length of an arbitrary subarc $[v_1, v_2] \subset [0, 2\pi), v_1 < v_2$, of an infinitesimal geodesic circle $u + u_0 = u_1, u_1 \to u_0$, with center $(u_0, v) \leq M^2$, $0 \leq v < 2\pi$, such that (1.5) holds and of the length of the same subarc of an infinitesimal circle with center at the origin of the tangent plane to M^2 at $(u_0, v) \leq M^2$, $0 \leq v < 2\pi$, with the same radius $u, u \to 0$, we get

$$(1.8) \qquad |B'(u_0, v)| = 1, \quad 0 \leq v < 2\pi.$$

Thus, (1.5) implies (1.8).

Every equivalence class of pairs (u,v) defines a point of M^2 and only these equivalence classes define points of M^2. An equivalence class of pairs (u_0, v), $-\infty < u_0 < \infty$, $0 \leq v < 2\pi$, such that (1.5) holds is called a pole of the polar, geodesic coordinate system (u,v). Thus, $(0,v)$, $0 \leq v < 2\pi$, and pairs equivalent with $(0,v)$ by means of (1.6) define a pole of (u,v). It does not lead to confusion, if we denote a point of M^2 by its representative (u_1, v_1) and write $(u_1, v_1) \in M^2$. In particular, in the following $(0,v) \in M^2$, $0 \leq v < 2\pi$, is a pole of (u,v).

In the following we suppose that the poles of (u,v) are the only points of M^2 at which $B(u,v)$ is zero.

Every pole $(u_0, v) \in M^2$, $0 \leq v < 2\pi$, is isolated. Indeed, let us assume indirect that there exists a sequence $(u_k, v) \in M^2$, $0 \leq v < 2\pi, k = 1, 2, \ldots$, of poles of (u,v) such that $\lim_{k\to\infty} u_k = u_0, u_k \neq u_0$. We have

$$\lim_{k\to\infty} \frac{B(u_k, v) - B(u_0, v)}{u_k - u_0} = 0$$

contrary to (1.8).

For every point different from a pole we have $B(u,v) \neq 0$, and therefore from (1.2) it follows that there exists a number $u_0 > 0$ such that

$$(1.9) \qquad B(u,v) > 0, \quad 0 < u < u_0, \quad 0 \leq v < 2\pi.$$

If $(u_0, v) \in M^2$, $0 \leq v < 2\pi$, is a pole of (u,v), then from (1.9) it follows

$$(1.10) \qquad \frac{B(u,v) - B(u_0, v)}{u - u_0} < 0, \quad 0 < u < u_0, \quad 0 \leq v < 2\pi.$$

From (1.8) and (1.10) it follows

$$(1.11) \qquad B'(u_0, v) = -1, \quad 0 \leq v < 2\pi.$$

The curves $v = v_0$, $0 \leq v_0 < 2\pi$, are geodesic lines on M^2. The curves $u = u_1$, $0 < u_1 < u_0$, are geodesic circles and u_1 is the radius of such a circle. By $D^2(u_1) \subset M^2$ we denote a (geodesic) disk bounded by the circle $u = u_1$. We denote

$$D(u_1) = D^2(u_1) \setminus \{(0,v)\}, \quad 0 \le v < 2\pi. \tag{1.12}$$

In Section 2 we characterize complete, 2-dimensional manifolds which admit a polar geodesic coordinate system (Theorem 2.1). In Section 3 by means of the curvature and torsion functions of the geodesic circles $u =$ const and geodesic lines $v =$ const, $0 \le v < 2\pi$, we derive a system of partial differential equations (3.11) which define isometric immersions of $D(u_1) \subset M^2$, $0 < u_1 < u_0$, in the Euclidean 3-dimensional space E^3 (Theorem 3.1). In Section 4 we investigate the system of partial differential equations (3.11) at the pole $(0,v) \in M^2, 0 \le v < 2\pi$. We prove that the torsion function of an infinitesimal geodesic circle $u = u_1, u_1 \to 0$, is defined by $w(0,v) = \frac{3}{2}(\varkappa_1 - \varkappa_2)\sin 2v$, $0 \le v < 2\pi$, where $\varkappa_1, \varkappa_2$ are principal curvatures of a surface $x(D^2(u_1)) \subset E^3$ at the pole $(0,v) \in M^2$, $0 \le v < 2\pi$, for $v = 0$ and $v = \frac{\pi}{2}$ respectively (Theorem 4.1); x denotes an isometric immersion of $D^2(u_1)$ in E^3. In Section 5 we investigate surfaces of negative Gauss curvature in E^3 by means of the system (3.11). We get an unexpected result which asserts that every solution of (3.11) defined on the whole of $M^2 \setminus \{(0,v)\}, 0 \le v < 2\pi$, has a singulatiry at $(0,v) \in M^2$ (Theorem 5.1). This implies e.g. that there does not exist a proof of the theorem of Hilbert [3] by means of (3.11). In Section 6 examples are presented.

2. Complete 2-dimensional manifolds with a polar, geodesic coordinate system

We have the following

THEOREM 2.1. *Let (M^2, g) denote a complete Riemannian manifold referred to polar, geodesic coordinates (u,v) such that* (1.3) *and* (1.4) *hold and g is defined by* (1.1). *Then the coordinate system (u,v) has at most* 2 *different poles. If (u,v) has one pole $(0,v) \in M^2$, $0 \le v < 2\pi$, then M^2 is diffeomorphic with the Euclidean plane E^2. If (u,v) has 2 poles $(0,v), (u_0,v) \in M^2$, $0 \le v < 2\pi$, then M^2 is diffeomorphic with the 2-dimensional sphere S^2 and (u,v) satisfies (1.6), (1.7), where $(u_0,v) \in M^2$, $0 \le v < 2\pi$, denotes the representative of the second pole with the smallest $u_0 > 0$.*

Proof. Let us suppose that $(0,v) \in M^2$, $0 \le v < 2\pi$, is the only pole of (u,v). By E^2 we denote the Euclidean plane referred to polar coordinates (u,v) with the identifications (1.3), (1.4). We define $M^2 \to E^2$ setting that corresponding points have the same coordinates. This is a diffeomorphism, analytic if $B(u,v)$ is analytic; otherwise M^2 cannot be a complete manifold.

Let us suppose that besides $(0, v) \in M^2$, $0 \leq v < 2\pi$, there exists a further pole $(u_0, v) \in M^2$, $0 \leq v < 2\pi$, $u_0 \neq 0$, of (u, v). Setting in (1.6) $u_0 = 0$, $u = u_0$, we get

$$(2.1) \qquad (-u_0, v) \approx (u_0, (v + \pi)\mathrm{mod}2\pi), \quad 0 \leq v < 2\pi.$$

If v varies from 0 to 2π, then $(v + \pi)\mathrm{mod}2\pi$ also varies from 0 to 2π and from (1.7) it follows

$$(2.2) \qquad (u_0, v) \approx (u_0, (v + \pi)\mathrm{mod}2\pi), \quad 0 \leq v < 2\pi.$$

From (2.1) and (2.2) it follows

$$(2.3) \qquad (-u_0, v) \approx (u_0, v), \quad 0 \leq v < 2\pi.$$

We can set therefore $u_0 > 0$ and in the following u_0 denotes the smallest positive number such that (1.7) holds. Let us suppose inductively

$$(2.4) \qquad (0, v) \approx (2ku_0, v), \quad 0 \leq v < 2\pi, \quad k = 0, \pm 1, \pm 2, \ldots,$$

$$(2.5) \qquad (u_0, v) \approx ((2k + 1)u_0, v).$$

Setting in (1.6) $u = (2k + 1)u_0$ we get

$$(2.6) \qquad (-2ku_0, v) \approx ((2k + 2)u_0, (v + \pi)\mathrm{mod}2\pi), \quad 0 \leq v < 2\pi.$$

From (2.3) applied to $2ku_0$ and (1.7) it follows by (2.6).

$$(2.7) \qquad (2ku_0, v) \approx ((2k + 2)u_0, v), \quad 0 \leq v < 2\pi.$$

This proves (2.4), and (2.5) is similarly proved. Let us suppose indirect that there exists a further pole $(u_1, v) \in M^2$, $0 \leq v < 2\pi$, of (u, v) such that u_1 is the smallest number with the property $u_1 > u_0 > 0$. There exists a non negative integer k such that $(2k + 1)u_0 < u_1 < (2k + 2)u_0$ or $(2k + 2)u_0 < u_1 < (2k + 3)u_0$. In the first case we have $-u_0 < u_2 < 0$, where $u_2 = (2k + 1)u_0 - u_1$. By (1.6) we get

$$(2.8) \quad ((2k+1)u_0 - u_2, v) \approx ((2k+1)u_0 + u_2, (v+\pi)\mathrm{mod}2\pi), \quad 0 \leq v < 2\pi.$$

Hence

$$(2.9) \qquad (u_1, v) \approx (2(2k + 1)u_0 - u_1, (v + \pi)\mathrm{mod}2\pi), \quad 0 \leq v < 2\pi,$$

From (1.7) applied to the left side of (2.9) (see (2.2)) and (2.9) it follows

$$(2.10) \quad \begin{aligned} (2(2k + 1)u_0 - u_1, (v + \pi)\mathrm{mod}2\pi) &\approx (2(2k + 1)u_0 - u_1, v) \\ &= ((2k + 1)u_0 + u_2, v), \quad 0 \leq v < 2\pi, \end{aligned}$$

and $2ku_0 < u_3 < (2k+1)u_0$, where $u_3 = (2k+1)u_0 + u_2$. From (2.9) and (2.10) it follows

$$(u_1, v) \approx (u_3, v), \quad 0 \leq v < 2\pi. \tag{2.11}$$

Therefore, $(u_3, v) \in M^2$, $0 \leq v < 2\pi$, is a further representative of the pole defined by $(u_1, v) \in M^2$, $0 \leq v < 2\pi$, and $u_1 > u_3 > 0$. This contradicts the definition of u_1, and therefore $(0, v), (u_0, v) \in M^2$, $0 \leq v < 2\pi$, are the only poles of (u, v). The proof in the second case is the same.

By

$$d\sigma_2^2 = du^2 + \frac{u_0^2}{\pi^2} \sin^2 \frac{\pi}{u_0} u dv^2, \quad -\infty < u < \infty, \quad 0 \leq v < 2\pi, \tag{2.12}$$

we denote the Riemannian metric of the sphere $S^2(\frac{u_0}{\pi})$ with radius $\frac{u_0}{\pi}$ written in a polar, geodesic coordinate system (u, v) with poles $(0, v), (u_0, v) \in S^2(\frac{u_0}{\pi})$, $0 \leq v < 2\pi$, such that the identifications (1.3), (1.4), (1.6), (1.7) hold. We define $M^2 \to S^2(\frac{u_0}{\pi})$ setting that corresponding points are defined by the same equivalence classes of coordinates of (u, v). This is a diffeomorphism onto the sphere, analytic if $B(u, v)$ is analytic. This ends the proof.

The Gauss curvature of (1.1) has the form

$$K(u, v) = -\frac{B''(u, v)}{B(u, v)}. \tag{2.13}$$

We have the following

THEOREM 2.2. *By the assumptions of Theorem* 2.1, *if*

$$-\frac{B''}{B} \leq -c^2 \textit{ respectively } -\frac{B''}{B} \geq c^2, \quad c > 0, \tag{2.14}$$

then

$$B(u, v) \geq \frac{1}{c} shcu \tag{2.15}$$

and M^2 is diffeomorphic with E^2 respectively M^2 is diffeomorphic with the sphere S^2.

Proof. From the first inequality of (2.14) it follows

$$B'' - c^2 B = \gamma, \quad \gamma(u, v) \geq 0. \tag{2.16}$$

The solution of (2.16) has the form

$$B = \frac{1}{c} shcu + \frac{1}{c} shcu \int_0^u \gamma ch c\eta d\eta - \frac{1}{c} chcu \int_0^u sh c\eta d\eta \tag{2.17}$$

and

$$B' = chcu + chcu \int_0^u \gamma chc\eta d\eta - shcu \int_0^u \gamma shc\eta d\eta. \tag{2.18}$$

Since $chcu > shcu$, it follows

$$f = chcu \int_0^u \gamma chc\eta d\eta - shcu \int_0^u \gamma shc\eta d\eta \geq 0. \tag{2.19}$$

From (2.18) and (2.19) follows

$$B = \int_0^u B' d\eta = \frac{1}{c} shcu + \int_0^u f d\eta \geq \frac{1}{c} shcu. \tag{2.20}$$

Hence, (2.15) is proved, and therefore (u, v) has a single pole $(0, v) \in M^2$, $0 \leq v < 2\pi$. Now, the first part of Theorem 2.2 follows from the first part of Theorem 2.1.

From the second inequality of (2.14) it follows

$$B'' + c^2 B = \gamma, \quad \gamma(u, v) \leq 0. \tag{2.21}$$

The solution of (2.21) has the form

$$B = \frac{1}{c} \sin cu + \frac{1}{c} \sin cu \int_0^u \gamma \cos c\eta d\eta - \frac{1}{c} \cos cu \int_0^u \gamma \sin c\eta d\eta. \tag{2.22}$$

From (2.22) and $\gamma \leq 0$, it follows

$$B(\frac{\pi}{c}, v) = \frac{1}{c} \int_0^{\frac{\pi}{c}} \gamma \sin cudu \leq 0. \tag{2.23}$$

From (1.9) and (2.23) it follows that there exist numbers $0 < u_0 \leq \frac{\pi}{c}$, $0 \leq v_0 < 2\pi$, such that $B(u_0, v_0) = 0$, and therefore by our assumption $B(u_0, v) = 0$ for every v, $c \leq v < 2\pi$. Hence, it follows that $(u_0, v) \in M^2$, $0 \leq v < 2\pi$, is a second pole of the coordinate system (u, v). Now the second part of Theorem 2.2 follows from the second part of Theorem 2.1. This ends the proof.

The assertion of the second part of Theorem 2.2 is the same as in the theorems of Berger, Klingenberg and Toponogov (see [2], §§7.3, 7.8) in the case $n = 2$. However the assumptions in these theorems are entirely different from the ones in Theorem 2.2.

PROPOSITION 2.1. *By the assumptions of Theorem* 2.1, *if the Gauss curvature of* (1.1) *satisfies*

$$K(u,v) < 0, \tag{2.24}$$

then M^2 *is diffeomorpgic with* E^2,

$$\lim_{u\to\infty} (B(u,v) - u) = \infty \tag{2.25}$$

and $B(u,v)$ *is a concave function for every* v, $0 \le v < 2\pi$. *If*

$$K(u,v) > 0, \quad \lim_{u\to\infty} K(u,v) = 0 \tag{2.26}$$

then M^2 *is diffeomorphic with* E^2,

$$\lim_{u\to\infty} (u - B(u,v)) = \infty \tag{2.27}$$

and $B(u,v)$ *is a convex function for every* v, $0 \le v < 2\pi$.

Proof. In the case (2.24) we prove at first that

$$B''(u,v) > 0 \quad \text{for} \quad u > 0, \quad B''(0,v) = 0, \quad 0 \le v < 2\pi. \tag{2.28}$$

From (1.2), (1.9), (2.13) and (2.24) it follows that there exists a number $u_0 > 0$ such that

$$B''(u,v) > 0 \quad \text{for} \quad 0 < u < u_0, \quad B''(0,v) = 0, \quad 0 \le v < 2\pi. \tag{2.29}$$

Let us suppose indirect that $u_0 < \infty$ is the greatest number such that (2.29) holds. From (2.29) it follows that $B(u,v)$ is a concave function for $0 < u < u_0$ and fixed v, $0 \le v < 2\pi$, and since $B(0,v) = 0$ it follows

$$B(u_0,v) > 0, \quad 0 \le v < 2\pi. \tag{2.30}$$

From (2.13), (2.24) and (2.30) it follows

$$B''(u_0,v) > 0, \quad 0 \le v < 2\pi, \tag{2.31}$$

contrary to the definition of u_0. This proves (2.28). From (1.2) and (2.28) it follows

$$f(u) = B(u,v) - u > 0 \text{ for } u > 0, \ f(0) = 0, \ v = \text{const}, \ 0 \le v < 2\pi. \tag{2.32}$$

Since $f''(u) = B''(u,v) > 0$ for $u > 0$, it follows that $f(u)$ is a positive concave function for $u > 0$. Hence, $f(u)$ tends to infinity with $u \to \infty$, and (2.25) follows. From (2.30), where $u_0 > 0$ is arbitrary, it follows that the coordinate system (u,v) has a single pole $(0,v) \in M^2$, $0 \le v < 2\pi$, and by Theorem 2.1 it follows that M^2 is diffeomorphic with E^2.

In the case (2.26) we have

$$B(u,v) > 0 \quad \text{for every} \quad u > 0, \quad 0 \le v < 2\pi. \tag{2.33}$$

Let us suppose indirect that there exists a number $u_0 > 0$ such that $B(u_0, v) = 0$, $0 \le v < 2\pi$, then (u, v) has 2 different poles, and by Theorem 2.1 M^2 is diffeomorphic with the sphere S^2. Since S^2 is compact it follows from the inequality (2.26) that there exists a constant $A > 0$ such that $K(u,v) \ge A$ for every point $(u,v) \in M^2$ contrary to the second condition in (2.26). From (2.13), (2.26) and (2.33) it follows

$$B''(u,v) < 0 \quad \text{for} \quad u > 0, \quad B''(0,v) = 0, \quad 0 < v < 2\pi. \tag{2.34}$$

From (1.2) and (2.34) it follows

$$g(u) = u - B(u,v) > 0 \text{ for } u > 0,\ g(0) = 0,\ v = \text{const},\ 0 \le v < 2\pi. \tag{2.35}$$

Since $g''(u) = -B''(u,v)$ for $u > 0$, it follows that $g(u)$ is a positive concave function for $u > 0$, and therefore $g(u)$ tends to infinity with $u \to \infty$. Hence, (2.27) follows, and from (2.33) and Theorem 2.1 it follows that M^2 is diffeomorphic with E^2. This ends the proof.

3. A system of partial differential equations

The differentiation with respect to $B\partial v$ we mark by a dot. We assume $K(u,v) \neq 0$ for every point $(u,v) \in M^2$, where $K(u,v)$ is the Gauss curvature (2.13).

Let

$$x : D(u_0) \to E^3, \quad D(u_0) \subset M^2, \quad u_0 > 0, \tag{3.1}$$

denotes an isometric immersion, where $D(u_0)$ is defined by (1.12). By e_1, e_2, e_3 we denote the unit vectors of the Frenet frame of a curve $x(u,v), u =$ const. The unit tangent vector t to the curve $x(u,v), v =$ const, can be written in the form

$$x' = t = -e_2 \cos\varphi - e_3 \sin\varphi, \quad 0 \le \varphi < 2\pi. \tag{3.2}$$

By the Frenet formulas we have

$$(x')^{\cdot} = t^{\cdot} = k \cos\varphi e_1 + (w + \dot\varphi)(e_2 \sin\varphi - e_3 \cos\varphi), \tag{3.3}$$

where $k(u,v), w(u,v)$ denote the curvature and torsion function of the curve $x(u,v), u =$ const, respectively. On the other hand we have

$$\dot x = e_1 \quad \text{and it follows} \quad e_1' = -\frac{B'}{B} e_1 + (x')^{\cdot}. \tag{3.4}$$

From (3.3) and (3.4) it follows

$$k\cos\varphi = \frac{B'}{B}, \quad e_1' = (w+\dot\varphi)(e_2\sin\varphi - e_3\cos\varphi). \tag{3.5}$$

We have $k(u,v) > 0$. Indeed, we have $k(u,v) \geq 0$. Let us suppose indirect that there exists a point $(u,v) \in D(u_0)$ such that $k(u,v) = 0$. Then from (2.13) and the first formula of (3.5) we get by differentiation with respect to u

$$k'\cos\varphi = \frac{B'}{B} \neq 0. \tag{3.6}$$

From (3.6) it follows $k'(u,v) \neq 0$, and therefore there exists such a number $\Delta u \neq 0$ that $k(u+\Delta u, v) < 0$. This contradiction proves that $k(u,v) > 0$. Now from (1.9) and the first formula of (3.5) it follows $0 \leq \varphi < \frac{\pi}{2}$.

Since e_2, e_3 are unit vectors, it follows

$$e_2' = \alpha e_3 + \beta_1 e_1, \quad e_3' = \gamma e_2 + \delta_1 e_1. \tag{3.7}$$

We have $\alpha + \gamma = (e_2\cdot e_3)' = 0$. From the second formula of (3.5) and (3.7) it follows

$$\beta_1 + (w+\dot\varphi)\sin\varphi = (e_1\cdot e_2)' = 0, \quad \delta_1 - (w+\dot\varphi)\cos\varphi = (e_1\cdot e_2)' = 0. \tag{3.8}$$

Thus, we have

$$\alpha = -\gamma, \quad \beta_1 = \beta\sin\gamma, \quad \delta_1 = -\beta\cos\varphi, \quad w + \dot\varphi + \beta = 0, \tag{3.9}$$

and (3.7) can be written in the form

$$e_2' = \alpha e_3 + \beta\sin\varphi e_1, \quad e_3' = -\alpha e_2 - \beta\cos\varphi e_1. \tag{3.10}$$

THEOREM 3.1. *For every isometric immersion* (3.1) *the functions* k, w, φ, α, β *satisfy the following system of partial differential equations*

$$\begin{cases} -\dot\beta\sin\varphi + \beta^2\cos\varphi = k' + k\dfrac{B'}{B}, \\ \dot\beta\cos\varphi + \beta^2\sin\varphi = \alpha k, \\ \dot\alpha + \beta\dfrac{B'}{B} = w' + w\dfrac{B'}{B}, \\ w + \dot\varphi + \beta = 0, k\cos\varphi = \dfrac{B'}{B}. \end{cases} \tag{3.11}$$

Proof. We have

$$(\dot e_1)' = -\frac{B'}{B}\dot e_1 + (e_1')^{\cdot}. \tag{3.12}$$

From (3.10), (3.12) and the Frenet formulas it follows

$$(e_1')^{\cdot} = k\frac{B'}{B}e_2 + k'e_2 + \alpha k e_3 + \beta \sin\varphi e_1. \tag{3.13}$$

Differentiating the second formula of (3.5) with respect to $B\partial v$ and using Frenet formulas we get

$$\begin{aligned}(e_1')^{\cdot} &= (w+\dot\varphi)^{\cdot}(e_2\sin\varphi - e_3\cos\varphi)\\ &+(w+\dot\varphi)[(-ke_1+we_3)\sin\varphi + e_2\dot\varphi\cos\varphi + w\cos\varphi e_2 + \dot\varphi\sin\varphi e_3].\end{aligned} \tag{3.14}$$

The comparison of coefficients by e_1, e_2, e_3 of (3.13) and (3.14) leads to the first two equations of (3.11) and to the first equation in the last row of (3.11).

We have

$$(\dot e_2)' = -\frac{B'}{B}\dot e_2 + (e_2')^{\cdot}. \tag{3.15}$$

Applying Frenet formulas to the left and right side of (3.15) we get

$$(\dot e_2)' = -k'e_1 - ke_1' + w'e_3 + we_3', \tag{3.16}$$

$$-\frac{B'}{B}\dot e_2 + (e_2')^{\cdot} = k\frac{B'}{B}e_1 - w\frac{B'}{B}e_3 + (e_2')^{\cdot}. \tag{3.17}$$

From (3.10), (3.15), (3.16), (3.17) and the Frenet formulas, it follows

$$\begin{aligned}&\dot\alpha e_3 - awe_2 + \dot\beta\sin\varphi e_1 + \beta\dot\varphi\cos\varphi e_1 + \beta k\sin\varphi e_2 + k\frac{B'}{B}e_1 - w\frac{B'}{B}e_3\\ &= -k'e_1 - k(w+\dot\varphi)(e_2\sin\varphi - e_3\cos\varphi) + w'e_3 - w(\alpha e_2 + \beta\cos\varphi e_1).\end{aligned} \tag{3.18}$$

The comparison of coefficients by e_1, e_2, e_3 in (3.18) leads to the first and third equations of (3.11). The second equation of the last row of (3.11) is defined by the first formula of (3.5). This ends the proof.

From (3.2) and (3.10) it follows

$$t' = (\varphi' + \alpha)(e_2\sin\varphi - e_3\cos\varphi) = \varkappa n, \tag{3.19}$$

where $\varkappa = \varphi' + \alpha$ is the curvature function of the curve $x(u,v), v = \text{const}$, and n denotes the principal normal of this curve. The principal normal n coincide with the normal vector to the surface $x(D(u_0))$ at $x(u,v)$. We have

$$n' = (\varphi' + \alpha)(e_2\cos\varphi + e_3\sin\varphi) + \beta e_1 = -\varkappa t + \beta e_1. \tag{3.20}$$

From (3.20) it follows that β is the torsion function of the curve $x(u,v), v =$ const, and e_1 is the binormal vector of this curve. Thus, (3.2), (3.19), (3.20) and the second formula of (3.5) are the Frenet formulas of the curve $x(u,v), v = \text{const}, 0 \le v < 2\pi$.

From (3.2), (3.4) and (3.10) it follows that the second quadratic form of (3.1) has the form

$$n.d^2x = (\varphi' + \alpha)du^2 + 2B(w + \dot{\varphi})dudv + B^2 k \sin\varphi dv^2. \tag{3.21}$$

4. The system (3.11) at the pole $(0, v) \in M^2$, $0 \le v < 2\pi$

In this section we assume that (3.1) is regular also at the pole $(0, v) \in M^2$, $0 \le v < 2\pi$, i.e. that there exists an isometric immersion

$$x : D^2(u_0) \to E^3, \quad D^2(u_0) \subset M^2. \tag{4.1}$$

From (1.2), (1.9) and the second equation of the last row of (3.11) it follows

$$\lim_{u\to 0} k(u, v) = \infty, \quad 0 \le v < 2\pi. \tag{4.2}$$

The function $k\cos\varphi$ is the geodesic curvature of $x(u, v), u =$ const, and $k\sin\varphi$ is the (first) normal curvature of $x(D^2(u_0))$ in the direction tangent to this curve at $(u, v) \in D^2(u_0)$. By $\varkappa_1, \varkappa_2$ we denote the principal curvatures of the surface $x(D^2(u_0))$ at $(0, v) \in M^2$, $0 \le v < 2\pi$, and $v = 0, v = \frac{\pi}{2}$ define the corresponding principal directions at $(0, v) \in M^2$, $0 \le v < 2\pi$. We have

$$\lim_{u\to 0} k\sin\varphi = \varkappa_1 \cos^2 v + \varkappa x_2 \sin^2 v, \quad 0 \le v < 2\pi. \tag{4.3}$$

From (4.2) and (4.3) it follows

$$\lim_{u\to 0} \varphi(u, v) = 0, \quad 0 \le v < 2\pi. \tag{4.4}$$

From (1.2), (1.9) and (4.4) it follows

$$\lim_{u\to 0} k\sin\varphi = \lim_{u\to 0} \frac{\sin\varphi}{B}\frac{B'}{\cos\varphi} = \varphi'(0, v), \quad 0 \le v < 2\pi. \tag{4.5}$$

From (4.3) and (4.5) it follows

$$\varphi'(0, v) = \varkappa_1 \cos^2 v + \varkappa_2 \sin^2 v, \quad 0 \le v < 2\pi. \tag{4.6}$$

By (3.19) is $\varkappa = \varphi' + \alpha$ the (second) normal curvature of $x(D^2(u_0)) \subset E^3$ in the direction tangent to $x(u, v), v =$ const, at the point $(u, v) \in M^2$. Hence, we have

$$\varphi'(0, v) + \alpha(0, v) = \varkappa_1 \sin^2 v + \varkappa_2 \cos^2 v, \quad 0 \le v < 2\pi. \tag{4.7}$$

From (4.6) and (4.7) it follows

$$\alpha(0, v) = -(\varkappa_1 - \varkappa_2)\cos 2v, \quad 0 \le v < 2\pi. \tag{4.8}$$

We have the following

THEOREM 4.1. *For every isometric immersion* (4.1) *we have*

$$w(0,v) = \frac{3}{2}(\varkappa_1 - \varkappa_2)\sin 2v, \quad 0 \le v < 2\pi, \tag{4.9}$$

$$\beta(0,v) = -\frac{1}{2}(\varkappa_1 - \varkappa_2)\sin 2v. \tag{4.10}$$

Proof. We have by (4.6)

$$\lim_{u\to 0}\dot\varphi(u,v_0) = \lim_{u\to 0}\frac{\partial\varphi(u,v)}{B(u,v_0)\partial v}\Big|_{v=v_0} = \frac{\partial}{\partial v}\left(\lim_{u\to 0}\frac{\varphi(u,v)}{B(u,v_0)}\right)\Big|_{v=v_0} \tag{4.11}$$
$$= \frac{\partial\varphi'(0,v)}{\partial v}\Big|_{v=v_0} = -(\varkappa_1 - \varkappa_2)\sin 2v_0, \quad 0 \le v_0 < 2\pi.$$

Since $\beta(u,v)$ is the torsion function of $x(u,v)$, $v=$ const, at the point $(u,v)\in M^2$ it follows that $\beta(u,v)$ together with their derivatives with respect to u exists at $(0,v)\in M^2$ for every v, $0\le v<2\pi$. Hence, by (4.11) from the first equation of the last row of (3.11), it follows

$$\lim_{u\to 0}(w + \dot\varphi + \beta) = w(0,v) - (\varkappa_1 - \varkappa_2)\sin 2v + \beta(0,v) = 0. \tag{4.12}$$

From (4.12) it follows that a finite limit of $w(u,v)$ at $(0,v)\in M^2, 0\le v<2\pi$, exists.

The third equation of (3.11) can be written in the form

$$\beta' + 2\beta\frac{B'}{B} + (\varphi' + \alpha)^{\cdot} = 0. \tag{4.13}$$

From (4.13) we have

$$\lim_{u\to 0}\left(B\beta' + \beta B' + \frac{\partial}{\partial v}(\varphi' + \alpha)\right) = 0. \tag{4.14}$$

From (1.2), (4.7) and (4.14) it follows (4.10). From (4.12) and (4.10) it follows (4.9). This ends the proof.

Remark 4.1. The formula (4.9) can be proved directly, applying the known formula of the torsion function to the curve $x(u,v)$, $u=u_0$, $u_0>0$, $u_0\to 0$. The formula (4.10) is a known theorem of O. Bonnet.

5. Complete surfaces of negative Gauss curvature in Euclidean 3-dimensional space

Let $k, w, \varphi, \alpha, \beta$ denote a solution of the system (3.11) in the disk $D^2(u_0)\subset M^2$. By the fundamental theorem of surfaces theory the first quadratic form (1.1) and the second quadratic form (3.21) define (up to motions of E^3) a unique surface $x(D^2(u_0))\subset E^3$ for sufficiently small $u_0>0$.

The equation of lines of curvature of $x(D^2(u_0)) \subset E^3$ has the form

$$\beta du^2 + B((\varphi' + \alpha) - k\sin\varphi)dudv - B^2\beta k\sin\varphi dv^2 = 0. \tag{5.1}$$

From the first two equations and the second equation in the last row of the system (3.11), it follows the Gauss equation

$$\beta^2 = \frac{B''}{B} + (\varphi' + \alpha)k\sin\varphi. \tag{5.2}$$

We have the following

LEMMA 5.1. *If the Gauss curvature* (2.13) *is negative for all* $(u, v) \in D^2(u_0)$, *then there exists a number* u_1, $0 < u_1 \leq u_0$, *such that the 2 geodesic lines* $x(u, v)$, $v = 0$, $v = \frac{\pi}{2}$, $-u_1 < u < u_1$, *are plane lines of curvature.*

Proof. From (4.10) it follows that there exists a number $u_1, 0 < u_1 \leq u_0$, such that for every $\widetilde{u}$, $0 < \widetilde{u} < u_1$, there exist 4 points $(\widetilde{u}, \widetilde{v}_i) \in D^2(u_1)$, $i = 1, 2, 3, 4$, such that $\beta(\widetilde{u}, \widetilde{v}_i) = 0$ and $\frac{\partial}{\partial v}(\widetilde{u}, \widetilde{v}_i) \neq 0$. From the implicit function theorem it follows that for every $\widetilde{u}$, $0 < \widetilde{u} < u_i$, there exist numbers $0 < u_3 < u_2 \leq u_1$ such that $u_3 < \widetilde{u} < u_2$ and differentiable functions $v_i = v_i(u)$, $u_3 < u < u_2$, $v_i(\widetilde{u}) = \widetilde{v}_i$, such that $\beta(u, v_i(u)) = 0$, $u_3 < u < u_2$, $i = 1, 2, 3, 4$. From (2.13) and (5.2) it follows that $\varphi' + \alpha$ and $k\sin\varphi$ are principal curvatures at points $(u, v) \in D^2(u_1) \subset M^2$ such that $\beta(u, v) = 0$, i.e. $((\varphi' + \alpha) - k\sin\varphi) \neq 0$. Hence, from (5.1) it follows $dv = 0$ for $du = \Delta u = u_2 - u_3 > 0$, and therefore $v_i(u) = \widetilde{v}_i$ for $u_3 < u < u_2$, $i = 1, 2, 3, 4$. This means that the line of curvature which passes through the point $x(u, \widetilde{v}_i) \in x(D^2(u_1)), u_3 < u < u_2$, is tangent to the geodesic line $x(u, v)$, $v = \widetilde{v}_i$, at the point $x(u, \widetilde{v}_i) \in x(D^2(u_1))$. Therefore the direction of the geodesic line $x(u, v)$, $v = \widetilde{v}_i$, at $x(u, \widetilde{v}_i)$, $u_3 < u < u_2$, is a principal direction, and it follows that the geodesic line $x(u, v)$, $v = \widetilde{v}_i$, $u_3 < u < u_2$, is a plane line of curvature $(\beta(u, \widetilde{v}_i) = 0)$ for $i = 1, 2, 3, 4$. By the indirect argument we extend the curve $x(u, v)$, $v = \widetilde{v}_i$, $u_3 < u < u_2$, to a plane line of curvature $x(u, v)$, $v = \widetilde{v}_i$, $0 \leq u < u_1$. Since $v = 0, v = \frac{\pi}{2}$ define the principal directions at $(0, v) \in D^2(u_1)$, $0 \leq u < 2\pi$, it follows $\widetilde{v}_i = 0$ or $\widetilde{v}_i = \frac{\pi}{2}$ or $\widetilde{v}_i = \pi$ or $\widetilde{v}_i = \frac{3}{2}\pi$, $i = 1, 2, 3, 4$. Therefore we can set $\widetilde{v}_1 = 0$, $\widetilde{v}_2 = \frac{\pi}{2}$, $\widetilde{v}_3 = \pi$, $\widetilde{v}_4 = \frac{3}{2}\pi$. From this with the use of (1.3) Lemma 5.1 follows.

We have the following

THEOREM 5.1. *Let us suppose that the Gauss curvature* (2.13) *is negative for every point* $(u, v) \in M^2$. *For every solution* $k, w, \varphi, \alpha, \beta$ *of the system* (3.11) *on* $M^2 \setminus \{(0, v)\}$, $0 \leq v < 2\pi$, *and every isometric immersion*

$$y : M \to E^3, \quad M \subset N^2 \setminus \{(0, v)\}, \quad 0 \leq v < 2\pi, \tag{5.3}$$

of an open, connected set M, *such that* (1.1) *and* (3.21) *are the first and second quadratic forms of* (5.3) *respectively, there does not exist an extension of* (5.3) *to an isometric immersion*

$$x : M^2 \to E^2, \tag{5.4}$$

such that (1.1) *and* (3.21) *are the first and second quadratic forms of* (5.4) *respectively.*

Proof. Let us suppose indirect that (5.4) exists. By Lemma 5.1 there exists a number $u_1 > 0$ such that $x(u,v)$, $v = 0$, $v = \frac{\pi}{2}$, $-u_1 \leq u \leq u_1$, are plane lines of curvature on $x(M^2) \subset E^2$. By the theorem of Hadamard ([2], §7.2) the point $(u_1, 0) \in M^2$ can be chosen as a new pole of a new polar, geodesic coordinate system of M^2. Applying Lemma 5.1 to the system (3.11) written in the new polar geodesic coordinate system, it follows that the plane line of curvature $x(u,v)$, $v = 0$, $0 \leq u \leq u_1$, can be prolonged to a plane line of curvature $x(u,v)$, $v = 0$, $0 \leq u \leq u_1 + u_2$, $u_2 > 0$. In this way, by the indirect argument it follows that the curves $x(u,v)$, $v = 0$, $v = \frac{\pi}{2}$, $-\infty < u < \infty$, are plane lines of curvature which together are geodesic lines on the surface $x(M^2) \subset E^3$. Since, by the theorem of Hadamard the pole of a polar, geodesic coordinate system can be chosen arbitrary on M^2, it follows that every line of curvature on $x(M^2)$ is a plane geodesic line. Therefore the Riemann metric of M^2 induced by the isometric immersion (5.4) can be written in the form $ds^2 = dw_1^2 + dw_2^2$, where (w_1, w_2) are parameters on the lines of curvature. Hence, $x(M^2)$ is a surface with Gauss curvature $K(w_1, w_2) = 0$ contrary to our assumption. This ends the proof.

COROLLARY 5.1. *If the Gauss curvature* (2.13) *is negative for every point* $(u,v) \in M^2$, *then every solution of the system* (3.11) *on* $M^2 \setminus \{(0,v)\}$, $0 \leq v < 2\pi$, *has a singularity at the pole* $(0,v) \in M^2$, $0 \leq v < 2\pi$, *in the following sense. For every* $u_0 > 0$ *an isometric immersion* (3.1) *such that* (1.1), (3.21) *are the first and second quadratic forms of* $x(D(u_0)) \subset E^3$ *cannot be prolonged to an isometric immersion* (4.1). *Indeed, otherwise applying the global version of the fundamental theorem of surface theory to* (1.1) *and* (3.21) *we get* (5.4).

Remark 5.1. From Theorem 5.1 it follows that an isometric immersion of a connected manifold M^2 provided with a complete Riemannian metric g defined by (1.1), of negative Gauss curvature at every point of M^2 cannot be a solution of (3.11). This implies that by means of (3.11) we cannot prove e.g. that the complete Riemannian metric induced from E^3 on the hyperbolic paraboloid $z = xy$ and written in polar geodesic coordinates can be a Riemannian metric of a complete surface in E^3 isometric with the hyperbolic paraboloid.

Moreover by the same reason, there does not exist a proof of the theorem of Hilbert [3] which asserts that the Lobachevski plane L^2, i.e. $B = shu$ in (1.1), cannot be isometrically immersed in E^3, discussing the system (3.11). More generally, discussing the system (3.11) we cannot get a proof of the theorem of Efimov [3], which asserts that there does not exist a complete surface in E^3 such that $K(u,v) \le -c^2$, $c \ne 0$, $(u,v) \in M^2$.

The attempt to get a proof of the theorems of Hilbert or Efimov by means of the system (3.11) fails because the theorema egregium in the polar, geodesic coordinate system (u,v) is reduced to the simple formula (2.13) and the Codazzi–Mainardi equations are identically satisfied by (3.11), they do not deliver a further condition in addition to (3.11). From Theorem 5.1 it follows that these difficulties cannot be avoided.

6. Examples

a) Let $B(u,v) = \sin u$, $0 \le u \le \pi$, $0 \le v < 2\pi$. We suppose $w(u,v) = 0$ identically and $\varphi(u,v) = \varphi(u)$. Thus from the first equation of (3.11) it follows

$$k(u) = \frac{D}{\sin u}, \quad d > 0, \quad D = \text{const.} \tag{6.1}$$

From $k\cos\varphi = ctgu$ and (1.9) we get

$$D\cos\varphi = \cos u. \tag{6.2}$$

If $D > 1$, then from (6.2) it follows

$$\arccos\frac{1}{D} \le \varphi \le \arccos\frac{1}{D} + \frac{\pi}{2}, \quad \varphi = \arccos\frac{\cos u}{D}, \quad 0 \le u \le \pi. \tag{6.3}$$

The formulas (6.3) characterize the spherical surfaces of revolution of the elliptic type.

If $0 < D < 1$, then from (6.2) we have $-D \le \cos u \le D$ and therefore

$$\arccos D \le u \le \arccos D + \frac{\pi}{2}, \varphi = \arccos\frac{\cos u}{D}. \tag{6.4}$$

The formulas (6.4) characterize the spherical surfaces of revolution of the hyperbolic type.

b) Let $K(u,v) \ge c^2$, $c \ne 0$, $w(u,v) = 0$ identically, $\varphi(u,v) = \varphi(u)$ on $D^2(u_0) \subset M^2, u_0 > 0$. We suppuse that $(u_0, v) \in M^2$, $0 \le v < 2\pi$, is the second pole of (u,v). Then $x(\overline{D}^2(u_0)) \subset E^3$ is a convex surface of revolution with the axis of revolution passing through the poles; $\overline{D}^2(u_0)$ denotes the closure of $D^2(u_0) \subset M^2$.

Indeed, from Theorem 2.1 it follows that M^2 is diffeomorphic with S^2. From (1.9) and (2.13) it follows that $B'(u,v)$ is a decreasing function of u, and from (1.2) and (1.11) it follows

$$-1 \leq B'(u,v) \leq 1 \quad \text{for} \quad 0 \leq u \leq u_0, \quad 0 \leq v \leq 2\pi. \tag{6.5}$$

The system (3.11) takes the form

$$k' + k\frac{B'}{B} = 0, \quad w = \alpha = \beta = 0, \quad k\cos\varphi = \frac{B'}{B}. \tag{6.6}$$

From (6.6) it follows

$$k(u,v) = \frac{C(v)}{B(u,v)}, \quad C(v)\cos\varphi(u) = B'(u,v). \tag{6.7}$$

From (1.2) and (4.4) it follows for $u = 0$ and consequently for every u, $0 \leq u \leq u_0$, $C(v) = 1$. Hence

$$k(u,v) = \frac{1}{B(u,v)}, \quad \cos\varphi(u) = B'(u,v). \tag{6.8}$$

From (6.5) and (6.8) it follows that $\varphi(u)$ is defined for $0 \leq u \leq u_0$ and

$$B'(u,v) = B'(u). \tag{6.9}$$

From (6.9) it follows

$$B(u,v) = \int_0^u B'(\eta)d\eta + D(v). \tag{6.10}$$

Since $B(0,v) = D(v) = 0$ it follows

$$B(u,v) = B(u), \quad k(u,v) = \frac{1}{B(u)}, \quad 0 < u < u_0. \tag{6.11}$$

Since $B'(u)$ is a decreasing function such that $B'(0) = 1, B'(u_0) = -1$, it follows that there exists such a number u_1, $0 < u_1 < u_0$, that $B'(u_1) = 0$. Therefore the function $B(u)$ is increasing for $0 \leq u \leq u_1$ and decreasing for $u_1 \leq u \leq u_0$. Hence, (6.11) and $K(u,v) \geq c^2$ define a convex surface of revolution of positive Gauss curvature.

c) Let $K(u,v) \leq -c^2, c \neq 0, w(u,v) = 0, \varphi(u,v) = \varphi(u)$. From (2.18) it follows

$$B'(u,v) \geq 1, \quad \lim_{u\to\infty} B'(u,v) = \infty. \tag{6.12}$$

The solution of the system (3.11) has the form (6.7). However because of (6.12) this solution has a singularity at $u = 0$. Indeed, from the second formula of (6.7) it follows

$$1 \leq B'(u,v) \leq C(v) \tag{6.13}$$

and for every v_0, $0 \leq v_0 < 2\pi$, there exists a number $u_0 = u(v_0)$ such that

$$B'(u_0, v_0) = C(v_0). \tag{6.14}$$

From (1.9) and (2.13) it follows that $B'(u,v)$ is an increasing function of u. Therefore from (6.14) it follows

$$1 \leq B'(u, v_0) < C(v_0) \quad \text{for} \quad 0 \leq u < u_0. \tag{6.15}$$

Hence $C(v_0) > 1$, and therefore

$$\cos\varphi(0) = \frac{1}{C(v_0)}, \quad \cos\varphi(u_0) = 1. \tag{6.16}$$

From the second formula of (6.7) it follows that $\varphi(u)$ is a decreasing function such that

$$\frac{1}{C(v_0)} < \cos\varphi(u) < 1 \quad \text{for} \quad 0 < u < u_0. \tag{6.17}$$

From (6.16) it follows that (4.4) is not satisfied, and therefore at $(0, v) \in M^2$, $0 \leq v < 2\pi$, the solution (6.7) has a singulatiry.

If we suppose $B(u,v) = B(u)$ and $u(v_0) = \text{const} = u_0 > 0$, $0 \leq v_0 < 2\pi$, then (6.7) define a surface of revolution. Let $L(u)$ denotes a plane curve $u = \text{const}$, $0 < u < u_0$. The length of $L(u)$ is

$$\int_0^{2\pi} B(u)dv = 2\pi B(u). \tag{6.18}$$

From the first formula of (6.7) it follows that the radius $r(u)$ of the circle $L(u)$ is equal $r(u) = \frac{B(u)}{C}$, where $C = C(v_0) > 1$, $0 \leq v_0 < 2\pi$. Hence $2\pi B(u) > 2\pi r(u)$. This implies that in the case $K(u,v) \leq -c^2, c \neq 0$, the solution (6.7) of (3.11) defines a surface of revolution which partly overlaps.

d) Let

$$x : D^2(u_0) \to E^3, \quad D^2(u_0) \subset L^2, u_0 > 0, \tag{6.19}$$

denotes an immersion of the geodesic disk $D^2(u_0)$ of the Lobachevski plane L^2 in E^3 with the property that the geodesic circle $u = u_0$ is isometrically mapped on a metric circle in E^3 (or more generally on a closed plane curve without selfintersections). Then (6.19) cannot be an isometric immersion. Indeed, the length of the circle $u = u_0$ is $2\pi sh u_0$. Since (6.19) is an isometric mapping for $u = u_0$, this is also the length of the metric circle (or another simple closed curve) $x(u_0, v)$, $0 \leq v < 2\pi$. For $\Delta u > 0$ sufficiently small there exists a tubular neighborhood of $x(u_0, v)$, $0 \leq v < 2\pi$ such that the disks with centers $x(u_0, v)$, $0 \leq v < 2\pi$, have radius Δu. The length of

$x(u_0 - \Delta u, v)$, $0 \leq v < 2\pi$, is $2\pi sh(u_0 - \Delta u)$. Let $e(v)$, $0 \leq v < 2\pi$, denotes a vector field of unit vectors along the curve $x(u_0, v)$, $0 \leq v < 2\pi$, such that $dx(u_0, v) \cdot e(v) = 0$. Then $x(u_0, v) + \Delta u e(v)$, $0 \leq v < 2\pi$, is a closed curve contained in the toroidal surface defined as the boundary of the tubular neighborhood of $x(u_0, v), 0 \leq v < 2\pi$. The length of the shortest closed surve of the form $x(u_0, v) + \Delta u e(v), 0 \leq v < 2\pi$, is $2\pi(shu_0 - \Delta u) > 2\pi sh(u_0 - \Delta u)$. Hence, (6.19) cannot be an isometric immersion.

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