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COINCIDENCE AND FIXED POINT THEOREMS ON PRODUCT SPACES

In this paper we establish a coincidence theorem for systems of multivalued and single-valued maps on the finite product of metric spaces. Fixed point theorems for multivalued maps are also derived. Our results include fixed point theorems of Czerwik [6], Kaneko-Seesa [16], Ray [21], Reddy-Subrahmanyam [22], Reich [23] and others.

1. Introduction and preliminaries

Let (Y, d) be a metric space, $T : Y \to Y$ and $P : Y \to CL(Y)$, the set of (nonempty) closed subsets of Y. Consider the following conditions on P for x, y in Y and some positive number $k < 1$,

$$H(Px, Py) \leq kd(Tx, Ty) \tag{1.1}$$

and

$$H(Px, Py) \leq k \max\{d(Tx, Ty), D(Ty, Py), [D(Tx, Py)] + D(Ty, Px)]/2\}, \tag{1.2}$$

where H is the generalized Hausdorff metric induced by d (see below).

The above conditions are generally termed as byhrid contractions (see, for instance, [1], [20]). Note that (1.1) implies (1.2).

We say that a point z in Y is:

(i) a coincidence point of P and T iff $Tz \in Pz$;

(ii) a fixed point of P and T iff $z = Tz \in Pz$

and

(iii) a hybrid fixed point iff $Tz \in PTz$.

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We have to emphasize that P and T satisfying (1.1) with $P(Y) \subset T(Y)$ need not have a common fixed point in complete Y even if T and P are continuous and commuting, i.e., $TPx \subset PTx, x \in Y$ (cf. Itoh-Takahashi [12]). We refer [16], [20], [27] and [31] for counterexamples and a good discussion on this aspect. Further, the condition (1.1) with $Tx = x, (x \in Y)$ contains the Nadler's (now classic) multivalued contraction [19]. Interesting generalizations of Nadler's contraction [op. cit.] due to Ćirić [4], Iséki [11], Ray [21] and Reich [23], are the special cases of (1.2). For example, the condition (1.2) with $Tx = x(x \in Y)$ was first studied by Ćirić [op. cit.]. The condition (1.2) with $P : X \to X$ and $Tx = x(x \in Y)$ is the condition (21′) of Rhoades [24, P. 267] (see also [10, p. 22], [26], [31], and [33]).

The study of hybrid contractions was essentially initiated, during 1981–1983, independently by Hadžić [8]. Singh-Kulshrestha [32] and Bhaskaran-Subrahmanyam [3] (see also [27, Remark 2.3]). These and similar conditions were further studied, among others, by Beg-Azam [2], Hadžić [9], Kaneko-Seesa [15]–[16], Khan et al. [17], Naimpally et al. [20], Rhoades et al. [25], Sastry et al. [27] and Singh-Pant [34].

The theory of multivalued maps has wide applications to game theory, mathematical economics, optimization theory, multifunctional equations, etc. A substantial place in the theory of multivalued maps, mainly due to its applications to functional equations, is claimed by the study of fixed points of nonlinear multivalued contractions (see, for instance, Wegrzyk [36]). Recent investigations of Corley (see, for instance, [5]) give a good relationship between hybrid fixed points and optimization problems.

In particular, he has shown that a Pareto type of maximization problem is equivalent to a hybrid fixed point problem. So, this is additional motivation for the results of this paper.

The main result of [32] contains the following coincidence theorem (see also [10, Th. 13]).

THEOREM 1.1. *Let Y be a metric space and P a multivalued map from Y to $CL(Y)$. If there exists a map T from Y to Y such that $P(Y) \subset T(Y), T(Y)$ is a complete subspace of Y and* (1.2) *holds, then P and T have a coincidence point, i.e., there exists $z \in Y$ such that $Tz \in Pz$.*

Following Jungck [13]–[14]; Kaneko-Seesa [16] and Beg-Azam [2] introduced the concept of compatibility (cf. Def. 1.5 below) of single-valued and multivalued maps. Kaneko-Sessa [op. cit.] established the following, wherein Y and (so) $CB(Y)$ are complete:

THEOREM 1.2. *Let Y be a complete metric space, $T : Y \to Y$ and $P : Y \to CB(Y)$, the set of nonempty closed bounded subsets of Y, be compatible* *continuous maps such that $P(Y) \subset T(Y)$ and (1.2) holds. Then there exists a point $z \in Y$ such that $Tz \in Pz$.*

Recently Baillon-Singh [1], motivated by the work of [5]–[9], [18], [22], [28], [30] and [31], have introduced coordinatewise weakly commuting systems of single- and multi-valued maps on the product of $n(\geq 2)$ metric spaces and established coincidence and hybrid fixed point theorems for such systems of maps. In this paper we introduce the concept of coordinatewise asymptotically commuting systems of single- and multi-valued maps on the product of $n(\geq 2)$ metric spaces (cf. Definition 1.4) and give a coincidence theorem for such a system of multivalued maps and two systems of single-valued maps on the product of n metric spaces (cf. Theorem 2.1). Fixed point theorems for multivalued maps are also obtained (cf. Corollaries 2.1–2.3). Several coincidence and fixed point theorems including Theorem 1.2 (above) may be obtained as special cases (see Remarks 2.1–2.4 and corollaries of this paper).

Throughout this paper we shall use the following notations and definitions:

Let (Y, d) be a metric space.

$CL(Y) = \{A : A$ is a nonempty closed subset of $Y\}$;

$CB(Y) = \{A : A$ is a nonempty closed and bounded subset of $Y\}$;

$N(\varepsilon, A) = \{x \in Y : d(x, a) < \varepsilon$ for some $a \in A, \varepsilon > 0, A \in CL(Y)\}$

and for $A, B \in CL(Y)$,

$$H(A,B) = \begin{cases} \inf \varepsilon > 0 : A \subset N(\varepsilon, B) \& B \subset N(\varepsilon, A), & \text{if the infimum exists} \\ \infty, & \text{otherwise.} \end{cases}$$

H is called the generalized Hausdorff metric induced by d. $D(x, A)$ will denote the ordinary distance between $x \in X$ and a nonempty subset $A \subset X$.

Let (a_{ik}) be an $n \times n$ square matrix with nonnegative entries. Define (cf. [6], [18])

$$c^1_{ik} = \begin{cases} a_{ik}, & i \neq k \\ 1 - a_{ik}, & i = k \end{cases} \quad i, k = 1, \ldots, n; \tag{1.3}$$

$$c^{t+1}_{ik} = \begin{cases} c^t_{11} c^t_{i+1,k+1} + c^t_{i+1,1} c_{1,k+1}, & i \neq k \\ c^t_{11} c^t_{i+1,k+1} - c^t_{i+1,1} c_{1,k+1}, & i = k, \end{cases} \tag{1.4}$$
$$t = 1, \ldots, n-1, \quad i, k = 1, \ldots, n-t.$$

Let

$$c^t_{ii} > 0, \quad t = 1, \ldots, n, \quad i = 1, \ldots, n-t+1. \tag{1.5}$$

Throughout this paper we shall assume that (X_i, d_i) are metric spaces $(CL(X_i), H_i)$ the generalized Hausdorff metric spaces induced by d_i. Further, let P_i and Q_i stand for multivalued maps from $X := X_1 \times \ldots \times X_n =$

$(X_1, \ldots, X_n)$ to $CL(X_i)$, and $T_i : X \to X_i, i = 1, \ldots, n$. For $X \supset A = (A_1, \ldots, A_n)$, we (as in [1]) use the notation $T(A) = (T_1 A_1, \ldots, T_n A_n)$.

DEFINITION 1.1 [2]. Two systems of maps $\{T_1, \ldots, T_n\}$ and $\{P_1, \ldots, P_n\}$ are coordinatewise commuting (or simply commuting) at a point $x \in X$ if and only if

$$T_i(P_1 x, \ldots, P_n x) \subseteq P_i(T_1 x, \ldots, T_n x), \quad i = 1, 2, \ldots, n.$$

For $n = 1$, this definition is that of Itoh and Takahashi [12]. For $n = 1$, the following definition is investigated in [15] and [31].

DEFINITION 1.2 [1]. Two systems of maps $\{T_1, \ldots, T_n\}$ and $\{P_1, \ldots, P_n\}$ are coordinatewise weakly commuting (or simply weakly commuting) at a point $x \in X$ if and only if

$$H_i(T_i(P_1 x, \ldots, P_n x), P_i(T_1 x, \ldots, T_n x)) \leq D_i(P_i x, T_i x), \quad i = 1, \ldots, n.$$

Two systems are coordinatewise weakly commuting on X if and only if they are coordinatewise weakly commuting at every point of X.

An equivalent formulation of Definition 1.2 for two systems of single-valued maps on X appears in [7].

We should remark that, in general, coordinatewise weakly commuting systems of maps need not to be coordinatewise commuting. However, the commuting systems are necessarily weakly commuting (see [1], [7], [30]).

DEFINITION 1.3. Two systems of maps $\{T_1, \ldots, T_n\}$ and $\{P_1, \ldots, P_n\}$ are coordinatewise asymptotically commuting (or simply asymptotically commuting) if and only if

$$H_i(P_i(T_1 x^m, \ldots, T_n x^m), T_i(P_1 x^m, \ldots, P_n x^m)) \to 0 \text{ (as } m \to \infty),$$

whenever $\{x^m\}$ is a sequence in X such that

$$P_i x^m \to M_i \in CL(X_i) \text{ and } T_i x^m \to x_i \in M_i.$$

As a special case of the above definition ($n = 1$) we have the following:

DEFINITION 1.4. The mappings $T_1 : X_1 \to X_1$ and $P_1 : X_1 \to CL(X_1)$ are asymptotically commuting (called compatible in [2] and [16] for $T_1 : X_1 \to X_1$ and $P_1 : X_1 \to CB(X_1)$) if and only if $H_1(P_1 T_1 x^m, T_1 P_1 x^m) \to 0$ (as $m \to \infty$) whenever $\{x^m\}$ is a sequence in X_1 such that $P_1 x^m \to M_1 \in CL(X_1)$ and $T_1 x^m \to u_1 \in M_1$.

If the map P_1 in this definition is single-valued then M_1 has just a single element u_1, and we get the definition of asymptotically commuting (or compatible) single-valued maps independently introduced by Tivari-Singh [35] and Jungck [13]. Since a sequence in the limiting tone is the main aspect in Definitions 1.3–1.4, the name „asymptotically commuting maps" seems to slightly better fit to the situation than „compatible maps". So, following [35], we shall henceforth prefer the name „asymptotically commuting".

Remark 1.1. The class of asymptotically commuting maps includes commuting and weakly commuting maps. Commuting maps are necessarily weakly and asymptotically commuting both (see, for instance, [1]–[2], [13]–[16], [28]–[31] and the following example).

EXAMPLE. Let $X_1 = [1, \infty)$ and $X_2 = [0, \infty)$ be metric spaces with the absolute value metric. Let $x := (x_1, x_2)$, $P_1 x = [1, x_1^2]$, $P_2 x = [x^2/4, x^2/2]$, $T_1 x = 2x^3 - 1$ and $T_2 x = x^2/5$. It can easily be verified that the systems of maps $\{P_1, P_2\}$ and $\{T_1, T_2\}$ are not coordinatewise weakly commuting but coordinatewise asymptotically commuting on $X := X_1 \times X_2$. Note that the above two systems are coordinatewise commuting at $x = (1, 0)$.

Remark 1.2. At any point of coincidence of two (or two systems of) maps, their commutativity, weak commutativity and asymptotic commutativity are equivalent at that point (see [1], [14] and [15]).

2. Coincidences and fixed points

THEOREM 2.1. *Let (X_i, d_i), $i = 1, \ldots, n$, be complete metric spaces and assume that $P_i : X \to CL(X_i)$, $S_i, T_i : X \to X_i$, $i = 1, \ldots, n$, are continuous maps such that*

(2.1) $P_i(X) \subset S_i(X) \cap T_i(X), i = 1, \ldots, n;$

(2.2) *the system $\{P_1, \ldots, P_n\}$ is asymptotically commuting with both the systems $\{S_1, \ldots, S_n\}$ and $\{T_1, \ldots, T_n\}$.*

If there exist non-negative numbers $b < 1$ and a_{ik} defined in (1.3) *and* (1.4) *such that* (1.5) *and the following hold:*

$$H_i(P_i x, P_i y) \le \max\Big\{ \sum_{k=1}^{n} a_{ik} d_k(S_k x, T_k y), \; b \max\{D_i(S_i x, P_i x), D_i(T_i y, P_i y), [D_i(S_i x, P_i y) + D_i(T_i y, P_i x)]/2\}\Big\} \tag{2.3}$$

for all $x := (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n) \in X$, then there exists a point $u \in X$ such that

$$S_i u \in P_i u \text{ and } T_i u \in P_i u, \; i = 1, \ldots, n. \tag{2.4}$$

Proof. First we note that the system (1.5) and

$$\sum_{k=1}^{n} a_{ik} r_k < r_i, \quad i = 1, \ldots, n,$$

are equivalent for some positive numbers $r_i, i = 1, \dots, n$. Further, if we put

$$h = \max_i \left(r_i^{-1} \sum_{k=1}^{n} a_{ik} r_k \right)$$

then $h \in (0,1)$ and so we may choose a system of positive numbers $r_1, \dots, r_n$ such that

$$\sum_{k=1}^{n} a_{ik} r_k \le h r_i, \quad i = 1, \dots, n; \tag{2.5}$$

(see [6], [18]).

Now we shall construct sequences $\{x_i^m\}$ and $\{z_i^m\}$, $i = 1, \dots, n$, in X_i. Pick x_i^0 in X_i. By (2.1), we can find an element x^1 in X such that $z_1^1 := S_i x^1 \in P_i x^0$. If $P_i x^0 = P_i x^1$ then x^1 becomes a coincidence point of P_i and $S_i, i = 1, \dots, n$. So let $P_i x^0 \neq P_i x^1$. Then we can choose an element x^3 in X such that $z_i^2 := T_i x^2 \in P_i x^2$ and

$$d_i(z_i^1, z_i^2) \le c^{-1/2} H_i(P_i x^0, P_i x^1) \le r_i, \quad i = 1, \dots, n,$$

where $c = \max\{h, b\}$.

If $P_i x^2 = P_i x^1$ then x^2 is coincidence point of P_i and T_i. So let $P_i x^2 \neq P_i x^1$. Then we can choose an element x^3 in X such that $z_i^3 := S_i x^3 \in P_i x^2$ and

$$d_i(z_i^2, z_i^3) \le c^{-1/2} H_i(P_i x^1, P_i x^2), \quad i = 1, \dots, n.$$

Continuing in this manner we choose sequences $\{x_i^m\}$ and $\{z_i^m\}$ such that

$$z_i^{2m+1} := S_i x^{2m+1} \in P_i x^{2m},$$
$$z_i^{2m+2} := T_i x^{2m+2} \in P_i x^{2m+1},$$

and

$$d_i(z_i^{2m}, z_i^{2m+1}) \le c^{-1/2} H_i(P_i x^{2m-1}, P_i x^{2m}),$$
$$d_i(z_i^{2m+1}, z_i^{2m+2}) \le c^{-1/2} H_i(P_i x^{2m}, P_i x^{2m+1}),$$
$$i = 1, \dots, n; \quad m = 1, 2, \dots.$$

We may assume without any loss of generality, that

$$d_i(z_i^1, z_i^2) \le r_i, \quad i = 1, \dots, n.$$

From (2.3) and (2.5), we have

$$\begin{aligned} d_i(z_i^2, z_i^3) &\le c^{-1/2} H_i(P_i x^1, P_i x^2) \\ &\le c^{-1/2} \max \Big\{ \sum_{k=1}^{n} a_{ik} d_k(S_k x^1, T_k x^2), b \max\{D_i(S_i x^1, P_i x^1), \\ &\quad D_i(T_i x^2, P_i x^2), [D_i(S_i x^1, P_i x^2) + D_i(T_i x^2, P_i x^1)]/2\} \Big\} \end{aligned} \tag{2.6}$$

$$\leq c^{-1/2} \max\Big\{ \sum_{k=1}^{n} a_{ik} d_k(z_k^1, z_k^2), b \max\{d_i(z_i^1, z_i^2), d_i(z_i^2, z_i^3), [d_i(z_i^1, z_i^3)]/2\}\Big\}$$
$$\leq c^{-1/2} \max\{hr_i, bd_i(z_i^1, z_i^2), bd_i(z_i^2, z_i^3)\}.$$

Now if $d_i(z_i^1, z_i^2) < d_i(z_i^2, z_i^3)$, then

$$d_i(z_i^2, z_i^3) \leq c^{-1/2} \max\{hr_i, bd_i(z_i^2, z_i^3)\} \leq c^{-1/2} hr_i \leq c^{1/2} r_i,$$

since otherwise we get a contradiction.

If $d_i(z_i^1, z_i^2) \geq d_i(z_i^2, z_i^3)$ then $d_i(z_i^2, z_i^3) \leq c^{-1/2} \max\{hr_i, br_i)\} \leq c^{1/2} r_i$. Similarly from (2.3), we have

$$\text{(2.7)} \quad d_i(z_i^3, z_i^4) \leq c^{-1/2} H_i(P_i x^2, P_i x^3)$$
$$= c^{-1/2} H_i(P_i x^3, P_i x^2)$$
$$\leq c^{-1/2} \max\Big\{ \sum_{k=1}^{n} a_{ik} d_k(z_k^3, z_k^2), bd_i(z_i^3, z_i^4), d_i(z_i^2, z_i^3)\Big\}, \quad i = 1, \ldots, n.$$

This, as before, gives us

$$d_i(z_i^3, z_i^4) \leq c^{1/2}(c^{1/2} r_i) = c^{2/2} r_i, \quad i = 1, \ldots, n.$$

By the induction argument we conclude that

$$d_i(z_i^{m+1}, z_i^{m+2}) \leq c^{m/2} r_i, \quad m = 1, 2, 3, \ldots; \quad i = 1, \ldots, n.$$

So each $\{z_i^m\}$ is a Cauchy sequence and therefore it converges to some point

$$u_i \in X_i, \quad i = 1, \ldots, n.$$

From the above relations (see also (2.6)–(2.7)), it follows that $\{P_i x^m\}$ is also a Cauchy sequence in $CL(X_i), i = 1, \ldots, n$. So there exist M_i in $CL(X_i)$ such that $P_i x^m \to M_i$, $i = 1, \ldots, n$. Thus

$$D_i(u_i, M_i) \leq d_i(u_i, S_i(x^{2m+1})) + D_i(S_i(x^{2m+1})), M_i)$$
$$\leq d_i(u_i, S_i(x^{2m+1})) + H_i(P_i(x^{2m}), M_i) \to 0 \text{ as } m \to \infty.$$

This yields $u_i \in M_i$ for $i = 1, \ldots, n$.

By (2.2) and the continuity of the systems of maps $\{S_1, \ldots, S_n\}$ and $\{P_1, \ldots, P_n\}$ we get $H_i(P_i(S_1(x^{2m+1}), \ldots, S_n(x^{2m+1}))$, $S_i(P_1(x^{2m+1}), \ldots, P_n(x^{2m+1}))) \to 0$ as m $\to \infty$. So $P_i(u) = S_i(M_1, \ldots, M_n)$. Consequently

$$S_i(u_1, \ldots, u_n) \in P_i(u_1, \ldots, u_n), \quad i = 1, \ldots, n.$$

Similarly

$$P_i(u_1, \ldots, u_n) = T_i(M_1, \ldots, M_n)$$

and

$$T_i(u_1, \ldots, u_n) \in P_i(u_1, \ldots, u_n), \quad i = 1, \ldots, n.$$

This completes the proof.

Remark 2.1. It is worth noting (see the above proof) that $S_i(u)$ need not to be equal to $T_i(u)$, $u = (u_1, \ldots, u_n)$, $i = 1, \ldots, n$. Further, if $S_i(u) = u_i$ {respectively $T_i(u) = u_i$}, $i = 1, \ldots, n$, then evidently $u_i \in P_i u$ and

$$(u_1, \ldots, u_n) = (S_1 u, \ldots, S_n u) \in (P_1 u, \ldots, P_n u)$$

{respectively $(u_1, \ldots, u_n) = (T_1 u, \ldots, T_n u) \in (P_1 u, \ldots, P_n u)$}.

Remark 2.2. Theorem 2.1 extends and generalizes several coincidence and fixed point results on metric spaces (cf. Hadžić [8], Jungck [13], Kaneko [15], Kaneko-Sessa [16] and others).

COROLLARY 2.1. *Theorem* 1.2 (*above*).

Proof. Its slightly improved version follows from Theorem 2.1 with $(Y, d) = (X_i, d_i)$, $P = P_i$, $T = S_i = T_i$, $i = 1, \ldots, n$, and $n = 1$, wherein $k = \max\{a_{11}, b\}$. Its derivation is akin to [30, p. 799].

Remark 2.3. If $S_i(x) = T_i(x) = x_i$, $i = 1, \ldots, n$, in Theorem 2.1 then the continuity conditions on P_i, $i = 1, \ldots, n$, are not needed. So the following fixed point theorem is an immediate consequence from Theorem 2.1 when $S_i(x) = T_i(x) = x_i$, $i = 1, \ldots, n$.

COROLLARY 2.2. *Let* (X_i, d_i), $i = 1, \ldots, n$, *be complete metric spaces. If maps* $P_i : X \to CL(X_i)$, $i = 1, \ldots, n$, *satisfy* (1.3), (1.4), (1.5) *and*

$$H_i(P_i x, P_i y) \leq \max\Big\{ \sum_{k=1}^{n} a_{ik} d_k(x_k, y_k), \quad b \max\{D_i(x_i, P_i x), D_i(y_i, P_i y), [D_i(x_i, P_i y) + D_i(y_i, P_i x)]/2\} \Big\} \tag{2.8}$$

for all $x := (x_1, \ldots, x_n), y := (y_1, \ldots, y_n) \in X$; *then the system of inclusions* $u_i \in P_i u$, $u = (u_1, \ldots, u_n)$, $i = 1, \ldots, n$, *has a solution.*

COROLLARY 2.3. *Let* Y *be a complete metric space and take a mapping* $P : Y \to CL(Y)$. *If there exists a constant* k, $0 \leq k < 1$, *such that for all* $x, y \in Y$,

$$H(Px, Py) \leq k \max\{d(x, y), D(x, Px), D(y, Py), [D(x, Py) + D(y, Px)]/2\},$$

then P *has a fixed point in* Y.

Proof. It follows from Corollary 2.2 when $(Y, d) = (X_i, d_i)$, $P = P_i$, $i = 1, \ldots, n$, and $n = 1$, wherein $k = \max\{a_{11}, b\}$.

Remark 2.4. Fixed point theorems from Beg-Azam [2], Ćirić [4], Iseki [11], Nadler [19], Ray [21], Reich [23] and Rus [26] for multivalued maps on metric spaces (see Corollary 2.3), and those from Czerwik [6], Matkowski [18], and Singh et al. [30], [33] on product of metric spaces may be obtained as special cases from Corollary 2.2.

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