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ON A NONEXISTENCE OF POSITIVE SOLUTION OF LAPLACE EQUATION IN UPPER HALF-SPACE

1. Introduction

Consider the following Laplace equation in the upper half-space

$$\Delta u = 0, \quad (x,y,z) \in R^3_+ = \{(x,y,z) \in R^3 : z > 0\} \tag{1.1}$$

with a nonlinear boundary condition of the form

$$-u_z(x,y,0) = f(x,y,u(x,y,0)), \quad (x,y) \in R^2. \tag{1.2}$$

In [1] there was studied the Laplace equation of axial symmetry form

$$u_{rr} + \frac{1}{r}u_r + u_{zz} = 0, \quad r > 0,\ z > 0, \tag{1.3}$$

with a nonlinear boundary condition

$$-u_z(r,0) = I_0 exp(-r^2/r_0^2) + u^\alpha(r,0), \quad r \geq 0, \tag{1.4}$$

where I_0, r_0, α are given positive constants. The problem (1.3), (1.4) is a stationary case of the problem relative to ignition by radiation. In [1] it was proved that the problem (1.3), (1.4) in the case $0 < \alpha \leq 2$ has no positive solution. Afterwards, this result has been extended in [2] for more general nonlinear boundary condition

$$-u_z(r,0) = g(r,u(r,0)), r \geq 0. \tag{1.5}$$

In this paper we consider the problem (1.1), (1.2) with a given function f which is continuous, nondecreasing and bounded below by the power function of order α with respect to the third variable. By constructing a suitable functional sequence, we prove that for $0 < \alpha \leq 2$ the problem (1.1), (1.2) has no positive solution. This result is a relative extension of that from [1], [2].

2. Hypotheses and statement of the main result

We omit the definition of a usual function space. Put

$$R_+ = \{x \in R : x \geq 0\}.$$

Assume that the function $f : R^2 \times R_+ \to R$ satisfies the following conditions:

(H_1) f is continuous,

(H_2) f is nondecreasing with respect to third variable, i.e.,

$$(f(x,y,u) - f(x,y,v))(u - v) \geq 0, \quad \forall x, y \in R, \quad \forall u, v \in R_+,$$

(H_3) there exist two positive constants α and M such that

$$f(x,y,u) \geq M u^\alpha, \quad \forall x, y \in R, \quad \forall u \in R_+,$$

(H_4) the intergal $\int\int_{R^2} \frac{f(x,y,0)dxdy}{1+\sqrt{x^2+y^2}}$ exists and is positive.

Besides, the solution of the problem (1.1), (1.2) is supposed to satisfy the following conditions:

(S_1) $u \in C^2(R^3_+) \cap C(\overline{R^3_+}), u_z \in C(\overline{R^3_+})$, where

$$\overline{R^3_+} = \{(x,y,z) \in R^3 : z \geq 0\},$$

(S_2) u is regular at infinity.,i.e.,

(i) $\max_{x^2+y^2+z^2=R^2} |u(x,y,z)| \to 0$, as $R \to +\infty$,

(ii) there exists a constant $C > 0$ such that

$$|\text{grad} u(x,y,z)| \leq \frac{C}{x^2+y^2+z^2}$$

as $x^2 + y^2 + z^2$ is sufficiently large.

The solution u of the problem (1.1), (1.2) satisfying the conditions (S_1), (S_2) can be represented by (see [3])

$$u(x,y,z) = A[f(\xi,\eta,u(\xi,\eta,0))](x,y,z), \quad \forall (x,y,z) \in R^3_+, \tag{2.1}$$

where A is the linear operator defined by

$$A[v(\xi,\eta)](x,y,z) = \frac{1}{2\pi} \int\int_{R^2} \frac{v(\xi,\eta)d\xi d\eta}{\sqrt{(x-\xi)^2+(y-\eta)^2+z^2}}. \tag{2.2}$$

We state the main result of this paper as follows.

THEOREM 1. *Suppose that f satisfies the hypotheses* (H_1)-(H_4) *with* $0 < \alpha \leq 2$. *Then the problem* (1.1), (1.2) *has no positive solution satisfying* (S_1), (S_2).

3. Proof of Theorem 1

Suppose, by contradiction, that the problem (1.1), (1.2) has a positive solution $u(x,y,z)$ satisfying (S_1), (S_2). Let $z \to 0_+$ in the integral equation (2.1) and put $u(x,y,0) = u(x,y)$. Then we obtain

$$u(x,y) = A[f(\xi,\eta,u(\xi,\eta))](x,y), \tag{3.1}$$

where A is a linear operator defined by

$$A[v(\xi,\eta)](x,y) = \frac{1}{2\pi}\int\int_{R^2} \frac{v(\xi,\eta)d\xi d\eta}{\sqrt{(x-\xi)^2+(y-\eta)^2}}. \tag{3.2}$$

Construct a recurrent functional sequence $\{u_n(x,y)\}$ defined by

$$u_1(x,y) = \frac{m_1}{1+\sqrt{x^2+y^2}}, \quad m_1 = \frac{1}{2\pi}\int\int_{R^2} \frac{f(\xi,\eta,0)d\xi d\eta}{1+\sqrt{\xi^2+\eta^2}}, \tag{3.3}$$

$$u_{n+1}(x,y) = A[f(\xi,\eta,u_n(\xi,\eta))](x,y), \quad n \geq 1. \tag{3.4}$$

Then, we have two following lemmas.

LEMMA 1.

$$A[f(\xi,\eta,0))](x,y) \geq u_1(x,y), \quad \forall x,y \in R. \tag{3.5}$$

Proof. From the inequalities

$$\begin{aligned} \frac{1}{\sqrt{(x-\xi)^2+(y-\eta)^2}} &\geq \frac{1}{\sqrt{x^2+y^2}+\sqrt{\xi^2+\eta^2}} \\ &\geq \frac{1}{1+\sqrt{x^2+y^2}}\cdot\frac{1}{1+\sqrt{\xi^2+\eta^2}}, \\ &\qquad \forall x,y,\xi,\ \eta \in R, \end{aligned} \tag{3.6}$$

by (3.3), we obtain (3.5).

LEMMA 2. *The functional sequence* $\{u_n(x,y)\}$ *defined by* (3.3), (3.4) *satisfies the following conditions*:

(i) $\{u_n\}$ *is nondecreasing.,i,e.*,

$$u_n(x,y) \leq u_{n+1}(x,y), \quad \forall n \in N, \quad \forall x,y \in R. \tag{3.7}$$

(ii) $\{u_n\}$ *is bounded above by* $u(x,y)$.,*i.e.*,

$$u_n(x,y) \leq u(x,y), \quad \forall n \in N, \quad \forall x,y \in R. \tag{3.8}$$

(iii) $u_n(x,y)$ *converges pointwise and satisfies*

$$\lim_{n\to+\infty} u_n(x,y) \leq u(x,y), \quad \forall x,y \in R. \tag{3.9}$$

Proof. Runs easily by recurrence on n.

By means of (3.8), (3.9), the Theorem 1 will be proved, if we can show that

(a) there exists $n \in N$ such that $u_n(x,y) = +\infty, \quad \forall x, y \in R$, or

(b) there exists (x,y) such that $\lim_{n\to+\infty} u_n(x,y) = +\infty$.

Further, we consider three cases of different values of α. (see (H_3)).

Case 1. $0 < \alpha \leq 1$.

LEMMA 3. *For $0 < \alpha \leq 1$ we have*

$$A[(1+\sqrt{\xi^2+\eta^2})^{-\alpha}](x,y) = +\infty, \quad \forall x, y \in R. \tag{3.10}$$

Proof. Using the first inequality in (3.6) and then changing to polar coordinates, we obtain

$$\begin{aligned} A[(1+\sqrt{\xi^2+\eta^2})^{-\alpha}](x,y) &\geq \frac{1}{2\pi} \iint_{R^2} \frac{d\xi d\eta}{(1+\sqrt{\xi^2+\eta^2})^{\alpha}(\sqrt{\xi^2+\eta^2}+\sqrt{x^2+y^2})} \\ &= \int_0^{+\infty} \frac{r dr}{(1+r)^{\alpha}(r+\sqrt{x^2+y^2})} \\ &= +\infty. \end{aligned} \tag{3.11}$$

From Lemma 3 and the hypothesis (H_3) we have

$$u_2(x,y) \geq M m_1^{\alpha} A[(1+\sqrt{\xi^2+\eta^2})^{-\alpha}](x,y) = +\infty.$$

Hence, the Theorem 1 is proved in Case 1.

Case 2. $1 < \alpha < 2$.

LEMMA 4. *For $\alpha > 1$ we have*

$$A[(1+\sqrt{\xi^2+\eta^2})^{-\alpha}](x,y) \geq \frac{1}{2(\alpha-1)(1+\sqrt{x^2+y^2})^{\alpha-1}}. \tag{3.12}$$

Proof. Similarly as in (3.11) we have

$$A[(1+\sqrt{\xi^2+\eta^2})^{-\alpha}](x,y) \geq \int_{\sqrt{x^2+y^2}}^{+\infty} \frac{r dr}{(1+r)^{\alpha}(r+\sqrt{x^2+y^2})}. \tag{3.13}$$

From the inequality

$$(3.14)\qquad \frac{r}{r+\sqrt{x^2+y^2}} \geq \frac{1}{2}, \quad \forall r \geq \sqrt{x^2+y^2}$$

we obtain (3.12) and Lemma 4 is proved.

Using now the recurrent relation (3.3), (3.4), hypothesis (H_3) and Lemma 4, we obtain

$$(3.15)\qquad u_2(x,y) \geq A[Mu_1^\alpha(\xi,\eta)](x,y) \geq m_2(1+\sqrt{x^2+y^2})^{-\lambda_2}$$

where

$$(3.16)\qquad \lambda_2 = \alpha - 1, \quad m_2 = Mm_1^\alpha/2\lambda_2$$

By recurrence, we can prove that

$$(3.17)\qquad u_n(x,y) \geq m_n(1+\sqrt{x^2+y^2})^{-\lambda_n}$$

with

$$(3.18)\qquad \lambda_n = \frac{1-(2-\alpha)\alpha^{n-1}}{\alpha-1}, \quad m_n = \frac{Mm_{n-1}^\alpha}{2\lambda_n}, \quad \forall n \geq 2.$$

Since $1 < \alpha < 2$, we can choose a natural number N (depending on α), such that

$$\frac{-\ln(2-\alpha)}{\ln\alpha} \leq N < 1 - \frac{\ln(2-\alpha)}{\ln\alpha},$$

namely we choose N such that $0 < \alpha\lambda_N \leq 1$. By hypothesis ($H_3$) and Lemma 3, we have

$$(3.19)\qquad u_{N+1}(x,y) \geq Mm_N^\alpha A[(1+\sqrt{\xi^2+\eta^2})^{-\alpha\lambda_N}](x,y) = +\infty.$$

Therefore, Theorem 1 is proved in Case 2.

Case 3. $\alpha = 2$.

LEMMA 5. *We have*

$$(3.20)\qquad A[(1+\sqrt{\xi^2+\eta^2})^{-2}](x,y) \geq \frac{\ln(1+\sqrt{x^2+y^2})}{4\sqrt{x^2+y^2}}.$$

Proof. Similarly as in (3.11) we have

$$(3.21)\qquad A[(1+\sqrt{\xi^2+\eta^2})^{-2}](x,y) \geq \int_1^{+\infty} \frac{rdr}{(1+r)^2(r+\sqrt{x^2+y^2})}.$$

Using the inequality

$$\frac{r}{(1+r)^2} \geq \frac{1}{4r}, \quad \forall r \geq 1, \tag{3.22}$$

we have (3.20).

LEMMA 6. *Putting*

$$v_n(x,y) = \begin{cases} 0, & x^2+y^2 \leq 1, \\ \dfrac{C_n}{\sqrt{x^2+y^2}}(\ln\sqrt{x^2+y^2})^{2^n-2}, & x^2+y^2 \geq 1, \end{cases} \tag{3.23}$$

where

$$C_n = M^{2^{n-1}-1}(\frac{1}{2}m_1\sqrt{\ln 2})^{2^{n-1}}\frac{1}{\ln 2}, \tag{3.24}$$

we have

$$u_n(x,y) \geq v_n(x,y), \quad \forall x,y \in R, \quad \forall n \geq 2. \tag{3.25}$$

Proof. First, observe that the inequality (3.25) holds for $n = 2$. Indeed, using Lemma 5, we have

$$\begin{aligned} u_2(x,y) &\geq Mm_1^2A[(1+\sqrt{\xi^2+\eta^2})^{-2}](x,y) \\ &\geq \frac{C_2}{\sqrt{x^2+y^2}}\ln(1+\sqrt{x^2+y^2}). \end{aligned} \tag{3.26}$$

Hence, from (3.23), (3.26) we have

$$u_2(x,y) \geq v_2(x,y), \quad \forall x,y \in R. \tag{3.27}$$

Suppose that (3.25) holds for a fixed n. It is easy to see that

$$u_{n+1}(x,y) \geq 0, \quad \forall x,y \in R. \tag{3.28}$$

With $x^2+y^2 \geq 1$, we have

$$\begin{aligned} u_{n+1}(x,y) &\geq MA[v_n^2(\xi,\eta)](x,y) \\ &\geq MC_n^2 \int\limits_{\sqrt{x^2+y^2}}^{+\infty} \frac{(\ln r)^{2^n-1}dr}{r(r+\sqrt{x^2+y^2})} \\ &\geq MC_n^2(\ln\sqrt{x^2+y^2})^{2^n-1} \int\limits_{\sqrt{x^2+y^2}}^{+\infty} \frac{dr}{r(r+\sqrt{x^2+y^2})} \end{aligned} \tag{3.29}$$

$$= \frac{MC_n^2 \ln 2}{\sqrt{x^2+y^2}} (\ln \sqrt{x^2+y^2})^{2^{n-1}}$$

$$= \frac{C_{n+1}}{\sqrt{x^2+y^2}} (\ln \sqrt{x^2+y^2})^{2^{n-1}}, \quad \forall x, y \in R,$$

because $MC_n^2 \ln 2 = C_{n+1}$, by (3.24).

From (3.23), (3.28) we obtain

$$u_{n+1}(x,y) \geq v_{n+1}(x,y), \quad \forall x, y \in R.$$

This fact shows by induction that Lemma 6 is true.

By (3.23), (3.24), we rewrite $v_n(x,y)$ for $x^2+y^2 \geq 1$ in the form

$$v_n(x,y) = \frac{1}{M\sqrt{x^2+y^2} \ln 2} \left(\frac{1}{2} M m_1 \sqrt{\ln 2 \ln \sqrt{x^2+y^2}}\,\right)^{2^{n-1}}. \tag{3.30}$$

Choose x, y such that $\frac{1}{2} M m_1 \sqrt{\ln 2 \ln \sqrt{x^2+y^2}} > 1$, or

$$x^2 + y^2 > exp(8/M^2 m_1^2 \ln 2) = r_0^2.$$

Then we have

$$\lim_{n \to +\infty} u_n(x,y) \geq \lim_{n \to +\infty} v_n(x,y) = +\infty, x^2+y^2 > r_0^2.$$

Hence Theorem 1 is proved in Case 3.

Combining Cases 1-3 we see that Theorem 1 holds for $0 < \alpha \leq 2$.

Remarks: (i) In [1], the function $v_n(x,y)$ is given in the form of a functional series and is more complicated than (3.23).

(ii) The conclusion does not hold for $\alpha > 2$. For example let $\alpha = 3$ and $f(x,y,u) = ku^3$, where k is a given positive constant. Of course, f does not satisfy the hypothesis (H_4). The function $v(x,y,z) = (x^2+y^2+(z+k)^2)^{-1/2}$ is a positive solution of the problem (1.1), (1.2) satisfying (S_1), (S_2).

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References

[1] F.V. Bunkin, V.A. Galaktionov, N.A. Kirichenko, S.P. Kurdyumov, A.A. Samarsky, *On a nonlinear boundary value problem of ignition by radiation*, J. Comp. Math. Phys. 28 (1988), 549–559. (in Russian).

[2] Nguyen Thanh Long, Dinh Van Ruy, *On a nonexistence of positive solution of Laplace equation in upper half-space with Cauchy data*, Demonstratio Math. 28 (1995), 921–927.

[3] V.S. Vladimirov, *Equations of Mathematical Physics*, Mir Publishers Moscow, 1984.

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