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Katarzyna Litewska, Jerzy Muszyński

TIME-SPACE FINITE ELEMENTS METHOD FOR MIXED HYPERBOLIC PROBLEM

In many technical applications (see e.g. [2]) the space – time finite elements method is used, neverless without any theoretical explanations of its convergence. The aim of this work is to prove that it can be applied to some mixed hyperbolic problem. The results of this paper generalize in some sense the results obtained in [3]–[5] for the ordinary differential equation initial problem.

1. The problem

Let $T > 0$ and $L > 0$ be fixed constants. Let

$$f : \langle 0,T\rangle \times \langle 0,L\rangle \times R \longrightarrow R$$

be a continuous function and let it satisfy the Lipschitz conditions in the last argument

$$(1)\qquad \begin{gathered}(\exists S>0)(\forall (t,x)\in \langle 0,T\rangle \times \langle 0,L\rangle)(\forall u,v\in R)\\ \mid f(t,x,u)-f(t,x,v)\mid \leq S\mid u-v\mid .\end{gathered}$$

For any $u_0, u_1 \in C(\langle 0,L\rangle)$ consider the mixed problem (for $t \in \langle 0,T\rangle$ and $x \in \langle 0,L\rangle$)

$$(2)\qquad u_{tt} - u_{xx} = f(t,x,u),$$

$$(3)\qquad u(t,0) = 0, u(t,L) = 0$$

and

$$(4)\qquad u(0,x) = u_0(x), u_t(0,x) = u_1(x).$$

Assume that the solution of the given problem exists and is of the class $C^2(\langle 0,T\rangle \times \langle 0,L\rangle)$. Introduce the finite elements method by the standard procedure. Choose any natural integers M, N, and let $\tau = \frac{T}{M}$, $h = \frac{L}{N}$. Define functions $\Phi_p^\tau, p = 0,1,2,\ldots,M$, by the formula

$$(5) \qquad \Phi_p^\tau = \begin{cases} \frac{t}{\tau} - p + 1 & \text{for } t \in (p\tau - \tau, p\tau > \cap \langle 0, T\rangle \\ -\frac{t}{\tau} + p + 1 & \text{for } t \in (p\tau, p\tau + \tau) \cap \langle 0, T\rangle \\ 0 & \text{otherwise} \end{cases}$$

and similar functions

$$(6) \qquad \Psi_r^h = \begin{cases} \frac{x}{h} - r + 1 & \text{for } x \in (rh - h, rh > \cap \langle 0, L\rangle \\ -\frac{x}{h} + r + 1 & \text{for } x \in (rh, rh + h) \cap \langle 0, L\rangle \\ 0 & \text{otherwise.} \end{cases}$$

The function $u^{\tau,h}$ will be called the approximate solution if it satisfies the boundary conditions

$$(7) \qquad u^{\tau,h}(t,0) = 0, u^{\tau,h}(t,L) = 0$$

for $t = k\tau,\ k = 0,1,\ldots,M$, the initial conditions

$$(8) \qquad u^{\tau,h}(0,x) = u_0(x), u_t^{\tau,h}(0,x) = u_1(x),$$

for the points $x = lh, l = 0,\ldots,N$ and the "Galerkin rule"

$$(9) \qquad \int_0^T \int_0^L \Big\{ -\frac{\partial}{\partial t}\Phi_k^\tau(t)\Psi_l^h(x)u_t(t,x) + \Phi_k^\tau(t)\frac{\partial}{\partial x}\Psi_l^h(x)u_x(t,x) - \\ - \Phi_k^\tau(t)\Psi_l^h(x)f(t,x,u(t,x))\Big\}dxdt = 0$$

for $k = 1,\ldots,M-1$ and $l = 1,\ldots,N-1$.

We look for an approximate solution $u^{\tau,h}$ of the form

$$(10) \qquad u^{\tau,h}(t,x) = \sum_{k=0}^{M-1}\sum_{l=1}^{N-1} \alpha_{k,l}^{\tau,h}\Phi_k^\tau(t)\Psi_l^h(x),$$

where $\alpha_{k,l}^{\tau,h}$ are unknown constants for $k=0,1,2,\ldots,M$ and $l=0,1,2,\ldots,N$. In the sequel we will write $\alpha_{k,l}$ instead of $\alpha_{k,l}^{\tau,h}$. Notice that

$$(11) \qquad \alpha_{k,l} = u^{\tau,h}(k\tau, lh).$$

Let

$$(12) \qquad \tau < \frac{h}{2}$$

and

$$(13) \qquad \tau \le \frac{1}{6108^{\frac{1}{4}}S^{\frac{1}{2}}},\ h \le \frac{1}{2S^{\frac{1}{2}}}.$$

Under these conditions we show that the the approximate solutions $u^{\tau,h}$ tend to the solution u of the problem (2)-(4) when h and τ tend to zero in a some norm which will be given later.

2. The approximate scheme

It is easy to see that the function $u^{\tau,h}$ satisfies the boundary conditions (7) if and only if

$$\alpha_{k,0} = 0,\ \alpha_{k,N} = 0,\ k = 0,\dots,M. \tag{14}$$

From the initial conditions (8) we have

$$\alpha_{0,l} = u_0(lh),\ \frac{1}{\tau}(\alpha_{1,l} - \alpha_{0,l}) = u_1(lh),\ l = 0,\dots,N. \tag{15}$$

Now let us examine the condition (9). The functions Φ_l^h have the support on $\langle lh-h, lh+h\rangle$, the function Ψ_k^τ on $\langle k\tau-\tau, k\tau+\tau\rangle$ and thus

$$\int\limits_{lh-h}^{lh+h}\int\limits_{k\tau-\tau}^{k\tau+\tau}\Big\{-\frac{\partial}{\partial t}\Phi_k^\tau(t)\Psi_l^h(x)u_t^{\tau,h}(t,x)+\Phi_k^\tau(t)\frac{\partial}{\partial x}\Psi_l^h(x)u_x^{\tau,h}(t,x)- -\Phi_k^\tau(t)\Psi_l^h(x)f(t,x,u^{\tau,h}(t,x))\Big\}dtdx = 0. \tag{16}$$

Inserting $u^{\tau,h}$ from (10) into (16), taking into account that in the interval $\langle lh-h, lh+h\rangle$ only three functions: $\Psi_{l-1}^h, \Psi_l^h, \Psi_{l-1}^h$ and in the interval $\langle k\tau-\tau, k\tau+\tau\rangle$ only the functions $\Phi_{k-1}^\tau, \Phi_k^\tau, \Phi_{k+1}^\tau$ aren't identically equal to zero, after some transformations we obtain

$$\begin{aligned}&\frac{1}{\tau^2}(\alpha_{k-1,l-1}-2\alpha_{k,l-1}+\alpha_{k+1,l-1})+\frac{4}{\tau^2}(\alpha_{k-1,l}-2\alpha_{k,l}+\alpha_{k+1,l})+\\&+\tfrac{1}{\tau^2}(\alpha_{k-1,l+1}-2\alpha_{k,l+1}+\alpha_{k+1,l+1})--\tfrac{1}{h^2}(\alpha_{k-1,l-1}-2\alpha_{k-1,l}+\alpha_{k-1,l+1})-\\&-\frac{4}{h^2}(\alpha_{k,l-1}-2\alpha_{k,l}+\alpha_{k,l+1})-\frac{1}{h^2}(\alpha_{k+1,l-1}-2\alpha_{k+1,l}+\alpha_{k+1,l+1})=\\&=\frac{6}{\tau h}\int\limits_{lh-h}^{lh+h}\int\limits_{k\tau-\tau}^{k\tau+\tau}\Phi_k^\tau(t)\Psi_l^h(x)f(t,x,\sum_{p=k-1}^{k+1}\sum_{r=l-1}^{l+1}\alpha_{p,r}\Phi_p^\tau(t)\Psi_r^h(x))dtdx,\\&k=1,2,\dots,M-1,\ l=1,2,\dots,N-1.\end{aligned} \tag{17}$$

The resulting system (17) with conditions (14),(15) will be now treated as a difference scheme for the problem (2)–(4) and thus for the proof of "legality" of the application of time–space finite elements method it is sufficient to proof that this scheme approximates the problem (2)–(4) and is stable ([1]).

3. The approximation

THEOREM 1. *Under given conditions the scheme (14),(15),(17) approximates the mixed problem (2)-(4).*

Proof. Let us denote by u the exact solution of the problem (2),(3),(4). Notice that u is of the class $C^2(\langle 0,T\rangle \times \langle 0,L\rangle)$. To prove that the scheme (14),(15),(17) approximates the problem (2),(3),(4) we must proof that the given below expressions δ_0, δ_1 and δ_2 tend to zero as τ and h tend to zero. These expressions are equal to

$$\delta_0 = u(0,x) - u_0(x), x = h, \ldots, (N-1)h, \tag{18}$$

$$\delta_1 = \frac{1}{\tau}(u(\tau,x) - u(0,x)) - u_1(x), x = h, \ldots, (N-1)h, \tag{19}$$

$$\delta_2 = \delta_{21} - \delta_{22}, \tag{20}$$

where

$$\begin{aligned} \delta_{21} = {} & \frac{1}{\tau^2}(u(t-\tau,x-h) - 2u(t,x-h) + u(t+\tau,x-h)) + \\ & + \frac{4}{\tau^2}(u(t-\tau,x) - 2u(t,x) + u(t+\tau,x)) + \\ & + \frac{1}{\tau^2}(u(t-\tau,x+h) - 2u(t,x+h) + u(t+\tau,x+h)) - \\ & - \frac{1}{h^2}(u(t-\tau,x-h) - 2u(t-\tau,x) + u(t-\tau,x+h)) - \\ & - \frac{4}{h^2}(u(t,x-h) - 2u(t,x) + u(t,x+h)) - \\ & - \frac{1}{h^2}(u(t+\tau,x-h) - 2u(t+\tau,x) + u(t+\tau,x+h)) \end{aligned} \tag{21}$$

and

$$\begin{cases} \delta_{22} = \dfrac{6}{\tau h} \displaystyle\int\limits_{x-h}^{x+h} \int\limits_{t-\tau}^{t+\tau} \Phi_k^\tau(s)\Psi_l^h(y), \\ f(s,y, \displaystyle\sum_{p=-1}^{1} \sum_{r=-1}^{1} u(t+p\tau, x+rh)\Phi_p^\tau(s)\Psi_r^h(y))dsdy, \end{cases} \tag{22}$$

where $x = h, \ldots, (N-1)h, t = \tau, \ldots, (M-1)\tau$.

Observe, that $\delta_0 = 0$. From (19) for some $\theta \in (0,1)$ we have

$$\delta_1 = u_t(\theta\tau, x) - u_t(0,x) \to 0, \tag{23}$$

as $\tau \to 0$, because u is of the class C^2 on $\langle 0,T\rangle \times \langle 0,L\rangle$.

If τ and h tend to zero then the terms in (21) tends to u_{tt} and u_{xx} correspondingly, and δ_{21} tends to

$$6(u_{tt}(t,x) - u_{xx}(t,x)),$$

which is equal to $6f(t,x,u(t,x))$. Thus we have to prove that

$$(24)\qquad \delta_3 = \left|\frac{1}{6}\delta_{22} - f(t,x,u(t,x))\right|$$

tends to zero.

It is easy to see that

$$(25)\qquad f(t,x,u(t,x)) = \frac{1}{\tau h}\int_{x-h}^{x+h}\int_{t-\tau}^{t+\tau} \Phi_k^\tau(s)\Psi_l^h(y) f(t,x,u(t,x))dsdy,$$

and that for (s,y) belonging to the region of integration

$$(26)\qquad \sum_{p=k-1}^{k+1}\sum_{r=l-1}^{l+1} \Phi_p^\tau(s)\Psi_r^h(y) = 1,$$

and thus that

$$(27)\qquad \delta_3 \le \frac{1}{\tau h}\int_{x-h}^{x+h}\int_{t-\tau}^{t+\tau} \mid f(s,y,\sum_{p=-1}^{1}\sum_{r=-1}^{1} u(t+p\tau, x+rh)\Phi_p^\tau(s)\Psi_r^h(y)) -$$

$$- f(t,x,\sum_{p=-1}^{1}\sum_{r=-1}^{1} u(t,x)\Phi_p^\tau(s)\Psi_r^h(y)) \mid dsdy =$$

$$= 4 \mid f(\tilde{s},\tilde{y},\sum_{p=-1}^{1}\sum_{r=-1}^{1} u(t+p\tau, x+rh)\Phi_p^\tau(\tilde{s})\Psi_r^h(\tilde{y})) -$$

$$- f(t,x,\sum_{p=-1}^{1}\sum_{r=-1}^{1} u(t,x)\Phi_p^\tau(\tilde{s})\Psi_r^h(\tilde{y})) \mid$$

for some $\tilde{y} \in \langle x-h, x+h\rangle, \tilde{s} \in \langle t-\tau, t+\tau\rangle$. The functions f and u are continuous. If $h \to 0$ and $\tau \to 0$, then $\tilde{s} \to t$ and $\tilde{y} \to x$, and the last righthand part of the inequality (27), and thus also δ_3 tends to zero.

It has been proved that the scheme (14),(15), (17) approximate the problem (2)-(4).

4. Transformations of the scheme

Let

$$(28)\qquad \alpha_m = (\alpha_{m,1}, \alpha_{m,2}, \ldots, \alpha_{m,N-1})^T$$

for $m = 0, \ldots, M$, let

$$A = \begin{pmatrix} a_1 & a_2 & 0 & 0 & \dots & 0 & 0 \\ a_2 & a_1 & a_2 & 0 & \dots & 0 & 0 \\ 0 & a_2 & a_1 & a_2 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & a_1 & a_2 \\ 0 & 0 & 0 & 0 & \dots & a_2 & a_1 \end{pmatrix},$$

(29)

$$B = \begin{pmatrix} b_1 & b_2 & 0 & 0 & \dots & 0 & 0 \\ b_2 & b_1 & b_2 & 0 & \dots & 0 & 0 \\ 0 & b_2 & b_1 & b_2 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & b_1 & b_2 \\ 0 & 0 & 0 & 0 & \dots & b_2 & b_1 \end{pmatrix}$$

where

$$(30) \qquad a_1 = \frac{4}{\tau^2} + \frac{2}{h^2}, a_2 = \frac{1}{\tau^2} - \frac{1}{h^2}, b_1 = \frac{8}{\tau^2} - \frac{8}{h^2}, b_2 = \frac{2}{\tau^2} + \frac{4}{h^2},$$

be $(N-1, N-1)$ matrices and let $F_m = F_m(\alpha_{m-1}, \alpha_m, \alpha_{m+1})$ be a vector of the form

$$(31) \qquad F_m = (f_{m,1}, f_{m,2}, \dots, f_{m,N-1})^T,$$

where $f_{m,l} = f_{m,l}(\alpha_{m-1}, \alpha_m, \alpha_{m+1})$ are given by the formula
(32)

$$f_{m,l} = \int\limits_{lh-h}^{lh+h} \int\limits_{m\tau-\tau}^{m\tau+\tau} \Phi_m^\tau(t)\Psi_l^h(x) f\Big(t, x, \sum_{p=m-1}^{m+1} \sum_{r=l-1}^{l+1} \alpha_{p,r}\Phi_p^\tau(t)\Psi_r^h(x)\Big) dtdx.$$

Let

$$(33) \qquad U_j = (u_j(h), \dots, u_j(Nh-h))^T, \; j = 0, 1.$$

Now the system (14),(15),(17) can be written in the form

$$(34) \qquad \alpha_0 = U_0, \frac{\alpha_1 - \alpha_0}{\tau} = U_1,$$

$$(35) \qquad A\alpha_{m+1} - B\alpha_m + A\alpha_{m-1} = \frac{6}{\tau h} F_m(\alpha_{m-1}, \alpha_m, \alpha_{m+1})$$

for $m = 1, \dots, M-1$.

5. The solvability of the scheme

The proof of the solvability of the system (34)-(35) will be presented in a more general case. Take

$$(36) \qquad \epsilon_m \in R^{N-1}, \; m = 0, 1, \dots M.$$

and consider the perturbed system

$$(37)\quad \begin{cases} \beta_0 = U_0 + \epsilon_0, \dfrac{\beta_1 - \beta_0}{\tau} = U_1 + \epsilon_1, \\ A\beta_{m+1} - B\beta_m + A\beta_{m-1} = \dfrac{6}{\tau h} F_m(\beta_{m-1}, \beta_m, \beta_{m+1}) + \epsilon_{m+1}, \\ m = 1, \ldots, M-1. \end{cases}$$

THEOREM 2. *Under our assumptions the system (37) is uniquely solvable.*

Proof. In the appendix to this work it is proved that under the given assumptions the matrix A is invertible, so the last equations in (37) can be written in the form

$$(38)\quad \beta_{m+1} - A^{-1}B\beta_m + \beta_{m-1} = \frac{6}{\tau h} A^{-1} F_m(\beta_{m-1}, \beta_m, \beta_{m+1}) + A^{-1}\epsilon_{m+1}, \quad m = 1, \ldots, M-1.$$

Observe that β_0 and β_1 are given by the first two formulas in (37). Suppose that we have already found $\beta_0, \beta_1, \ldots, \beta_m$. For the vector β_{m+1} we have the equation (38). Consider the operator $T : R^{N-1} \to R^{N-1}$ given by

$$(39)\quad T\omega = \frac{6}{\tau h} A^{-1} F_m(\beta_{m-1}, \beta_m, \omega) + A^{-1}B\beta_m - \beta_{m-1} + A^{-1}\epsilon_{m+1}.$$

Let us denote by $\| \ \|$ the Euclidean norm in R^{N-1} and the spectral norm of the matrix (induced by this vector norm). For two given vectors $w, \tilde{w} \in R^{N-1}$ using (1) we have

$$(40)\quad \|Tw - T\tilde{w}\| = \left\| \frac{6}{\tau h} A^{-1}(F_m(\beta_{m-1}, \beta_m, w) - F_m(\beta_{m-1}, \beta_m, \tilde{w})) \right\| \le \le 72S\|A^{-1}\|\|w - \tilde{w}\|.$$

In the case of $\tau < h$ the absolute values of the eigenvalues of the matrix A^{-1} are not greater than $\frac{\tau^2 h^2}{2h^2 + 4\tau^2}$ (Appendix, Theorem 3), so the norm $\|A^{-1}\|$ is bounded from above by this value. Under our assumptions (12), (13) the operator T is contractive, hence from the Banach Principle there exists a unique solution of the equation

$$(41)\quad T\omega = \omega,$$

i.e. a unique solution - vector β_{m+1} of the equation (38).

The proof is completed.

6. The stability

THEOREM 3. *Under given assumptions the scheme (37) is stable.*

Proof. The eigenvalues of the matrix A are simple, equal to

$$\lambda_{Ak} = a_1 - 2a_2 \cos \frac{k\pi}{n+1},\ k = 1, \dots, n, \tag{42}$$

and similarly for B we have

$$\lambda_{Bk} = b_1 - 2b_2 \cos \frac{k\pi}{n+1},\ k = 1, \dots, n. \tag{43}$$

The eigenvectors for the matrices A and B that correspond to λ_{Ak} and λ_{Bk} (with the same k) are the same and equal to

$$X_{k,p} = \nu(-1)^p \sin p\frac{k\pi}{N},\ k, p = 1, \dots, N-1, \tag{44}$$

where ν is an arbitrary constant. To make these ν unitary take

$$\nu = \sqrt{\frac{2}{N}}. \tag{45}$$

Let Λ_A, Λ_B be diagonal matrices with the eigenvalues of the matrices A and B on their diagonals, correspondingly. Let W be a matrix with vectors $X_1, \dots, X_{N-1}$ as a columns.

The matrix W is orthonormal, $W^{-1} = W^T$ and thus

$$W^T A = \Lambda_A W^T, W^T B = \Lambda_B W^T. \tag{46}$$

Denote

$$\gamma_{m+1} = \frac{6}{\tau h} F_m(\alpha_{m-1}, \alpha_m, \alpha_{m-1}), \kappa_m = W^T \beta_m, \delta_m = W^T \gamma_m. \tag{47}$$

After the multiplication of the last equation in (37) by the matrix W^T we obtain the following vector difference equation

$$\Lambda_A \kappa_{m+1} - \Lambda_B \kappa_m + \Lambda_A \kappa_{m-1} = \delta_{m+1}, \kappa_m, \delta_m \in R^{N-1}. \tag{48}$$

Introduce now the complex numbers

$$\rho_k = \cos\phi_k + i \sin\phi_k,\ k = 1, \dots, N-1$$

where

$$\cos\phi_k = \frac{\lambda_{Bk}}{2\lambda_{Ak}}, \sin\phi_k = \frac{\sqrt{4\lambda_{Ak}^2 - \lambda_{Bk}^2}}{2\lambda_{Ak}}, \tag{49}$$

and the diagonal matrices P and $\tilde{P}$ such that

$$P = diag(\rho_1, \dots, \rho_{N-1}), \tilde{P} = diag(\tilde{\rho}_1, \dots, \tilde{\rho}_{N-1}). \tag{50}$$

From the first two equations in (37) we can find the vectors β_0 and β_1, thus we can treat as given the vectors

$$\kappa_0 = W^T \beta_0, \kappa_1 = W^T \beta_1. \tag{51}$$

The solution of the vector difference equation (48) with initial conditions given by (51) is equal

$$\kappa_m = (ImP)^{-1}\Big(Im(P^m(\kappa_1 - P\kappa_0)) + \Lambda_A^{-1}\sum_{r=1}^{m-1} Im(P^{m-r})\delta_{r+1}\Big). \tag{52}$$

By the orthonormality of the matrix W we have

$$\beta_m = W\kappa_m, \gamma_m = W\delta_m. \tag{53}$$

Then (the matrix $P\tilde{P}$ is unit in the considered case)

$$\beta_m = (W(ImP)^{-1}Im(P^m))W^T\beta_1 - W((ImP)^{-1}Im(P^{m-1}))W^T\beta_0 + \\ + \sum_{r=1}^{m-1} W((ImP)^{-1}\Lambda_A^{-1}Im(P^{m-r}))W^T\gamma_{r+1}. \tag{54}$$

Analogously

$$\alpha_m = (W(ImP)^{-1}Im(P^m))W^T\alpha_1 - W((ImP)^{-1}Im(P^{m-1}))W^T\alpha_0 + \\ + \sum_{r=1}^{m-1} W((ImP)^{-1}\Lambda_A^{-1}Im(P^{m-r}))W^T\sigma_{r+1}. \tag{55}$$

Note that

$$\beta_0 - \alpha_0 = \epsilon_0, \beta_1 - \alpha_1 = \epsilon_0 + \tau\epsilon_1$$

and then

$$\begin{aligned} \beta_m - \alpha_m &= \\ &= \tau W((ImP)^{-1}Im(P^m))W^T\epsilon_1 + \\ &+ W((ImP)^{-1}(Im(P^m - Im(P^{m-1})))W^T\epsilon_0 + \\ &+ \sum_{r=1}^{m-1} W((ImP)^{-1}\Lambda_A^{-1}Im(P^{m-r}))W^T(\gamma_{r+1} - \sigma_{r+1}). \end{aligned} \tag{56}$$

Introduce in the space R^{N-1} a norm of the type

$$\|x\| = (\sum_{p=1}^{N-1} hx_p^2)^{\frac{1}{2}}. \tag{57}$$

Then

$$\begin{aligned} \|\beta_m - \alpha_m\| &\le \tau\|W((ImP)^{-1}Im(P^m))W^T\|\ \|\epsilon_1\| + \\ &+ \|W((ImP)^{-1}(Im(P^m - Im(P^{m-1})))W^T\|\ \|\epsilon_0\| + \\ &+ \sum_{r=1}^{m-1} \|W((ImP)^{-1}\Lambda_A^{-1}Im(P^{m-r}))W^T\|\ \|(\gamma_{r+1} - \sigma_{r+1})\|, \end{aligned} \tag{58}$$

where the norm of the matrix is the spectral one.

For any matrix C the matrices C and $WCW^T = WCW^{-1}$ are similar, and then they have the same eigenvalues. Thus

$$\begin{aligned}(59)\qquad &\|\beta_m - \alpha_m\| \leq \tau\|((ImP)^{-1}Im(P^m))\| \, \|\epsilon_1\| + \\ &+ \|((ImP)^{-1}(Im(P^m - Im(P^{m-1})))\| \, \|\epsilon_0\| + \\ &+ \sum_{r=1}^{m-1} \|((ImP)^{-1}\Lambda_A^{-1}Im(P^{m-r}))\| \, \|(\gamma_{r+1} - \sigma_{r+1})\|.\end{aligned}$$

The quadratic matrices P and Λ_A are diagonal and ρ_k and λ_k are their eigenvalues. Then the diagonal matrices

$(ImP)^{-1}Im(P^m), (ImP)^{-1}[Im(P^m) - Im(P^{m-1})], (ImP)^{-1}\Lambda_A^{-1}Im(P^{m-r})$

have on their diagonals the elements

$(Im(\rho_k))^{-1}Im(\rho_k^m), (Im(\rho_k))^{-1}Im(\rho_k^m - \rho_k^{m-1}), \Lambda_{A,k}^{-1}(Im(\rho_k))^{-1}Im(\rho_k^{m-r}),$

$k = 1, \ldots, N-1$ correspondingly. In this case

$$(Im(\rho_k))^{-1}(Im(\rho_k^s) - Im(\rho_k^{s-1})) = \frac{\cos(s - \frac{1}{2}\Phi_k)}{\cos\frac{1}{2}\Phi_k}$$

and

$$(Im(\rho_k))^{-1}Im((\rho_k)^{m-r}) = \frac{\sin(m-r)\Phi_k}{\sin\Phi_k}.$$

Taking into account (30),(42),(43), (49) and the assumption that $h > 2\tau$ we have

$$\left(\cos\frac{1}{2}\Phi_k\right)^2 > \frac{h^2 - \tau^2}{h^2 + 2\tau^2} > \frac{1}{2}$$

and $|(Im(\rho_k))^{-1}(Im(\rho_k^s) - Im(\rho_k^{s-1}))| < \sqrt{2}$.

Observe that

$$\left|\frac{\sin s\Phi_k}{\sin\Phi_k}\right| = \left|\sum_{k=0}^{s-1}\cos(2k - s + 1)\Phi_k\right| \leq s.$$

From Theorem 3 in the appendix we have

$$\max_k |\lambda_{A,k}^{-1}| \leq \frac{\tau^2 h^2}{2(2h^2 + \tau^2)}.$$

Thus

$$\begin{gathered}\|(ImP)^{-1}Im(P^m)\| \leq m, \\ \|(ImP)^{-1}(Im(P^m) - IM(P^{m-1}))\| \leq \sqrt{2}, \\ \|(ImP)^{-1}\Lambda_A^{-1}Im(P^{m-r})\| < \max_k |\lambda_{A,k}^{-1}|(m-r) \leq \frac{\tau^2h^2}{2(2h^2+\tau^2)}(m-r).\end{gathered}$$

From (59) we have ($\tau m < \tau M = T$)

$$\|\beta_m - \alpha_m\| \leq T\|\epsilon_1\| + \sqrt{2}\|\epsilon_0\| + + \frac{\tau^2 h^2}{2(2h^2+\tau^2)} \sum_{r=1}^{m-1} (m-r)\|(\gamma_{r+1} - \sigma_{r+1})\| \tag{60}$$

where

$$\gamma_{r+1} - \sigma_{r+1} = -\epsilon_{r+1} + \frac{6}{\tau h}(F_r(\beta_{r-1}, \beta_r, \beta_{r+1}) - F_r(\alpha_{r-1}, \alpha_r, \alpha_{r+1})). \tag{61}$$

Thus

$$\|\beta_m - \alpha_m\| \leq T\|\epsilon_1\| + \sqrt{2}\|\epsilon_0\| + \frac{\tau^2 h^2}{2(2h^2+\tau^2)} \sum_{r=1}^{m-1} (m-r)\|\epsilon_{r+1}\| + + \frac{3\tau h}{2h^2+\tau^2} \sum_{r=1}^{m-1} (m-r)\|(F_r(\beta_{r-1}, \beta_r, \beta_{r+1}) - F_r(\alpha_{r-1}, \alpha_r, \alpha_{r+1}))\|. \tag{62}$$

Observe that

$$|(f_{m,l}(\beta_{m-1}, \beta_m, \beta_{m+1}) - f_{m,l}(\alpha_{m-1}, \alpha_m, \alpha_{m+1}))| \leq \leq 4S\tau h \sum_{p=m-1}^{m+1} \sum_{r=l-1}^{l+1} |\beta_{p,r} - \alpha_{p,r}| \tag{63}$$

and then

$$\|(F_m(\beta_{m-1}, \beta_m, \beta_{m+1}) - F_r(\alpha_{m-1}, \alpha_m, \alpha_{m+1}))\| \leq \leq 12\sqrt{3} S\tau h \sum_{p=m-1}^{m+1} \|\beta_p - \alpha_p\|, \tag{64}$$

so

$$\|\beta_m - \alpha_m\| \leq T\|\epsilon_1\| + \sqrt{2}\|\epsilon_0\| + + \frac{T\tau}{4} \sum_{r=1}^{m-1} \|\epsilon_{r+1}\| + \frac{108\sqrt{3} S\tau^2 h^2}{2h^2+\tau^2} \sum_{r=1}^{m} (m-r+1)\|\beta_r - \alpha_r\|. \tag{65}$$

From (13) it follows that $\frac{108\sqrt{3}S\tau^2h^2}{2h^2+\tau^2} \leq \frac{1}{2}$, hence

$$\|\beta_m - \alpha_m\| \leq 2T\|\epsilon_1\| + 2\sqrt{2}\|\epsilon_0\| + \frac{T\tau}{2} \sum_{r=1}^{m-1} \|\epsilon_{r+1}\| + + \frac{216\sqrt{3} S\tau^2 h^2}{2h^2+\tau^2} \sum_{r=1}^{m-1} (m-r+1)\|\beta_r - \alpha_r\|. \tag{66}$$

Since $\frac{216\sqrt{3}S\tau^2h^2}{2h^2+\tau^2} \le 108\sqrt{3}S\tau^2$ and $m-r-1 \le m$, the last term in the inequality (66) is not greater than $108\sqrt{3}S\tau T \sum_{r=1}^{m-1} \|\beta_r - \alpha_r\|$, and thus

$$\|\beta_m - \alpha_m\| \le 2T\|\epsilon_1\| + 2\sqrt{2}\|\epsilon_0\| + \frac{T\tau}{2}\sum_{r=1}^{m-1}\|\epsilon_{r+1}\| + \tag{67}$$
$$+108\sqrt{3}ST\tau\sum_{r=1}^{m-1}\|\beta_r - \alpha_r\|.$$

Let $c = 216\sqrt{3}ST^2$. Introduce the next norm in $R^{N-1} \times R^{M-1}$: if $\beta = (\beta_1, \ldots, \beta_{M-1})$, where $\beta_k \in R^{N-1}$, then let

$$|||\beta||| = \max_k(\eta(k)\|\beta_k\|) \tag{68}$$

where

$$\eta(k) = \exp\Big(-c\frac{k}{M}\Big). \tag{69}$$

Then

$$\eta(m)\|\beta_m - \alpha_m\| \le 2T\|\epsilon_1\| + 2\sqrt{2}|||\epsilon||| + \tag{70}$$
$$+\frac{T\tau}{2}|||\epsilon|||\sum_{r=1}^{m-1}\frac{\eta(m)}{\eta(r+1)} + 108\sqrt{3}ST\tau|||\beta-\alpha|||\sum_{r=1}^{m-1}\frac{\eta(m)}{\eta(r)}$$

and (because $e^{kx} - 1 \ge kx$ for $k \le 0$)

$$|||\beta - \alpha||| \le 2T\|\epsilon_1\| + 2\sqrt{2}|||\epsilon||| + \frac{1}{432\sqrt{3}S}|||\epsilon||| + \frac{1}{2}|||\beta - \alpha|||. \tag{71}$$

Thus

$$|||\beta - \alpha||| \le 4T\|\epsilon_1\| + K|||\epsilon||| \tag{72}$$

where $K = 4\sqrt{2} + \frac{1}{216\sqrt{3}S}$. The scheme is stable.

This concludes the fact that the solutions of the considered scheme converge to the solution of the problem (2)-(4).

Appendix

THEOREM 1. *Let B_n be the (n,n) matrix of the form*

$$B_n = \begin{pmatrix} d & 1 & 0 & 0 & \ldots & 0 \\ 1 & d & 1 & 0 & \ldots & 0 \\ 0 & 1 & d & 1 & \ldots & 0 \\ .. & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & 0 & \ldots & d \end{pmatrix},$$

for some $n \in N$ and $d \in R$. If $|d| \neq 2$ then $detB_n \neq 0$. The eigenvalues of B_n are simple, real,and are given by the formula

$$\lambda_k = d - 2\cos\frac{k\pi}{N},\ k = 1, \ldots, N-1.$$

Proof. For the matrix B_n in the case where $|d| > 2$ we have

$$detB_n = \frac{1}{\sqrt{d^2-4}}(\rho_2^{n+1} - \rho_1^{n+1})$$

where

$$\rho_{1,2} = \frac{d \pm (d^2-4)^{\frac{1}{2}}}{2}$$

and

$$detB_n = (\frac{d}{2})^n(1+n)$$

if $|d| = 2$. In the first case we have $\rho_1 \neq \rho_2$, so $detB_n \neq 0$. The same is in the second case.

If $|d| < 2$ then

$$detB_n = \frac{\sin(n+1)\phi}{\sin\phi},$$

where

$$\sin\phi = \frac{1}{2}(4-d^2)^{\frac{1}{2}},$$

The equality $detB_n = 0$ takes place only if $\sin N\phi = 0$. In this case $(n+1)\phi = k\pi, k \in Z$, and $d = 2\cos\phi = 2cos\frac{k\pi}{n+1}, k \in Z$. From the periodicity of the function cos it follows that it is possible to restrict ourselves to $k = 1, \ldots, N-1$.

The eigenvalues of the matrix B_n are the solution of the equation

$$det(B_n - \lambda E) = \begin{vmatrix} d-\lambda & 1 & 0 & \ldots & 0 \\ 1 & d-\lambda & 1 & \ldots & 0 \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots & \ldots\ldots \\ 0 & 0 & 0 & \ldots & d-\lambda \end{vmatrix} = 0,$$

By the given analysis it is possible only if

$$d - \lambda = 2\cos\frac{k\pi}{n+1},\ k = 1, \ldots, n.$$

Hence the eigenvalues of B_n are equal to

$$\lambda_k = d - 2\cos\frac{k\pi}{n+1},\ k = 1, \ldots, n.$$

THEOREM 2. *In the case $\tau < h$ the matrix A is convertible.*

Proof. If $\tau \neq h$ we have $detA = (\frac{1}{\tau^2} - \frac{1}{h^2})^n detB_n$, where B_n is the same as in Theorem 1 with $d = \frac{4h^2+2\tau^2}{h^2-\tau^2}$. In the considered case $\tau < h$ and then $d > 2$. Thus $|d| > 2$ and from theorem 1 we obtain that $detB_n \neq 0$, thus $detA \neq 0$, the matrix A is convertible.

THEOREM 3. *If $\tau < h$ then the eigenvalues of the matrix A^{-1} are positive not greater than $\frac{\tau^2 h^2}{2h^2+4\tau^2}$.*

Proof. The matrix A is real and symmetric, so its eigenvalues $\tilde{\lambda}$ are real. In the case where $\tau \neq h$ we have

$$det(A - \tilde{\lambda}I) = (\frac{1}{\tau^2} - \frac{1}{h^2})^n det B_n,$$

where $d = \frac{4h^2+2\tau^2}{h^2-\tau^2} - \tilde{\lambda}\frac{h^2\tau^2}{h^2-\tau^2}$. If $\tau < h$ and $\tilde{\lambda} < \frac{2h^2+4\tau^2}{\tau^2 h^2}$,

$$d > \frac{4h^2 + 2\tau^2}{h^2 - \tau^2} - \frac{2h^2 + \tau^2}{h^2 - \tau^2} = 2.$$

In this case we have $|d| > 2$; from Theorem 1 it follows that $det B_n \neq 0$ and thus $det(A - \tilde{\lambda}I) \neq 0$. The values $\tilde{\lambda} < \frac{2h^2+4\tau^2}{\tau^2 h^2}$ in the case of $\tau < h$ can't be the eigenvalues of the matrix A, so all of them are positive and greater than $\frac{2h^2+4\tau^2}{\tau^2 h^2}$. Thus for $\tau < h$ the eigenvalues $\tilde{\lambda}_s$ of the matrix A satisfy:

$$\tilde{\lambda}_s \geq \frac{2h^2 + 4\tau^2}{\tau^2 h^2}.$$

If $(\tilde{\lambda}_1, \ldots, \tilde{\lambda}_k)$ is the collection of the eigenvalues of the matrix A then the collection $(\lambda_1, \ldots, \lambda_k)$, where $\lambda_l = (\tilde{\lambda}_l)^{-1}$, is the collection of the eigenvalues of the matrix A^{-1}. Thus we have: if $\tau < h$ then λ_s are positive and

$$\lambda_s \leq \frac{h^2\tau^2}{2\tau^2 + 4h^2}.$$

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INSTITUTE OF MATHEMATICS
WARSAW UNIVERSITY OF TECHNOLOGY
00-661 WARSAW, POLAND

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