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ON MAXIMAL SUBALGEBRAS AND MAXIMAL IDEALS OF BOOLEAN ALGEBRAS

We prove that the existence of a maximal subalgebra and the existence of a maximal ideal are effectively equivalent in every Boolean algebra.

The following effectively equivalent statements are an easy consequence of the Axiom of Choice:

- every non trivial Boolean algebra possesses a maximal ideal,
- every proper ideal of a Boolean algebra is contained in a maximal ideal.

Both of them are known as the Prime Ideal Theorem. The Prime Ideal Theorem has many interesting mathematical equivalents (for example, the Stone Representation Theorem, see [2]) and being itself weaker then the Axiom of Choice it can replace it in proofs of many important statements (for example, Hahn–Banach Theorem, Stone–Čech Compactification Theorem, see [1]).

The following lemma is well-known, see [3].

LEMMA 1. *Let $\mathcal{I}$ be a proper ideal of a Boolean algebra $\mathcal{B}$. Then the following conditions are equivalent:*

(i) *$\mathcal{I}$ is a maximal ideal of $\mathcal{B}$,*
(ii) *for every $x \in \mathcal{B}$, $\{x, \neg x\} \cap \mathcal{I} \neq \emptyset$,*
(iii) *$\mathcal{I} \cup \neg\mathcal{I} = \mathcal{B}$ where $\neg\mathcal{I} = \{\neg x : x \in \mathcal{I}\}$.*

The next lemma is also well-known, see [4], [5].

LEMMA 2. *Let $\mathcal{A}$ be a subalgebra of a Boolean algebra $\mathcal{B}$ and let $b \in \mathcal{B} \setminus \mathcal{A}$. Let $\mathcal{C}$ be the subalgebra of $\mathcal{B}$ generated by the set $\mathcal{A} \cup \{b\}$. Then every element of $\mathcal{C}$ has a form $(b \wedge x) \vee (\neg b \wedge y)$ for some $x, y \in \mathcal{A}$, i.e. $\mathcal{C} = \{(b \wedge x) \vee (\neg b \wedge y) : x, y \in \mathcal{A}\}$.*

The six-variable tautology mentioned in the proof of the following lemma is surely the longest of all tautologies ever put to use by the present author.

LEMMA 3. *Let a, b, x, y, z, s be elements of a Boolean algebra. If $a = (b \wedge x) \vee (\neg b \wedge y)$ and $b = (a \wedge z) \vee (\neg a \wedge s)$ then $a \div b = (s - x) \vee (y - z)$.*

Instead of proving Lemma 2 by complicated calculations one can use a computer to check that the corresponding formula, i.e.

$$[[a \leftrightarrow ((b \wedge x) \vee (\neg b \wedge y))] \wedge [b \leftrightarrow ((a \wedge z) \vee (\neg a \wedge s))]] \\ \rightarrow [(a \div b) \leftrightarrow ((s \wedge \neg x) \vee (y \wedge \neg z))],$$

is a tautology. The symbol $\div$ denotes the operation of symmetric difference, i.e. $a \div b = (a - b) \vee (b - a)$.

LEMMA 4 [Maximal Subalgebra Lemma]. *Let $\mathcal{A}$ be a proper subalgebra of $\mathcal{B}$. Then the following conditions are equivalent:*

(i) *$\mathcal{A}$ is a maximal subalgebra of $\mathcal{B}$,*
(ii) *$\{a, b, a \div b\} \cap \mathcal{A} \neq \emptyset$ for every $a, b \in \mathcal{B}$.*

Proof. Suppose that $\mathcal{A}$ is a maximal subalgebra of $\mathcal{B}$ and that $a, b \in \mathcal{B} \setminus \mathcal{A}$. By the maximality of $\mathcal{A}$ and Lemma 2 it follows that for some $x, y, z, s \in \mathcal{A}$ we have that $b = (a \wedge x) \vee (\neg a \wedge y)$ and $a = (b \wedge z) \vee (\neg b \wedge s)$. Using Lemma 3 one gets that $a \div b = (s - x) \vee (y - z) \in \mathcal{A}$ which was to be proved.

Suppose now that for every $a, b \in \mathcal{B}$, $\{a, b, a \div b\} \cap \mathcal{A} \neq \emptyset$, and there exists a proper subalgebra of $\mathcal{B}$ exceeding $\mathcal{A}$. Then for some $c \in \mathcal{B} \setminus \mathcal{A}$ the subalgebra of $\mathcal{B}$ generated by $\mathcal{A} \cup \{c\}$ must be proper and consequently some element $d \in \mathcal{B}$ must lie outside its universe. The assumption about $\mathcal{A}$ yields that $c \div d$, $\neg(c \div d) \in \mathcal{A}$ but in every Boolean algebra we have $d = [c \wedge \neg(c \div d)] \vee [\neg c \wedge (c \div d)]$ which means that d belongs to the subalgebra of $\mathcal{B}$ generated by $\mathcal{A} \cup \{c\}$, — a contradiction. ■

LEMMA 5. *Let $\mathcal{A}$ be a maximal subalgebra of a Boolean algebra $\mathcal{B}$ and let $b \in \mathcal{B} \setminus \mathcal{A}$. Then for every $x \in \mathcal{A}$ at least one of the following is true:*

(i) *$\{b \wedge x, \neg b \wedge x\} \subseteq \mathcal{A}$, or*
(ii) *$\{b \wedge \neg x, \neg b \wedge \neg x\} \subseteq \mathcal{A}$.*

Proof. If $b \in \mathcal{B} \setminus \mathcal{A}$ and $x \in \mathcal{A}$ then it follows that:

(1) $\{b \wedge x, b \wedge \neg x\} \cap \mathcal{A} \neq \emptyset$ since otherwise, by Lemma 4, it follows that $\mathcal{A} \ni (b \wedge x) \div (b \wedge \neg x) = b$, - a contradiction;
(2) $\{\neg b \wedge x, \neg b \wedge \neg x\} \cap \mathcal{A} \neq \emptyset$ for similar reasons;
(3) $\{\neg b \wedge x, b \wedge \neg x\} \not\subseteq \mathcal{A}$ since otherwise $\mathcal{A} \ni (b - x) \vee (x - b) = b \div x$ and $b = (b \div x) \div x \in \mathcal{A}$, - a contradiction;
(4) $\{b \wedge x, \neg b \wedge \neg x\} \not\subseteq \mathcal{A}$ for similar reasons.

Combining (1), ..., (4) it is easy to prove that either (i) or (ii) must be true. ■

LEMMA 6 [Key Lemma]. *Let $\mathcal{A}$ be a subalgebra of a Boolean algebra $\mathcal{B}$ and let $b \in \mathcal{B} \setminus \mathcal{A}$. Let $\mathcal{I}_b$ be the ideal of $\mathcal{A}$ generated by the set $(\mathcal{A} \cap (b]_{\mathcal{B}}) \cup (\mathcal{A} \cap (\neg b]_{\mathcal{B}})$, i.e.*

$$\mathcal{I}_b = \{a \in \mathcal{A} : \exists_{x,y \in \mathcal{A}} a \leq x \vee y, x \leq b, y \leq \neg b\}.$$

Then $\mathcal{I}_b$ is a proper ideal of $\mathcal{A}$. Moreover,
(1) *if $\mathcal{A}$ is a maximal subalgebra of $\mathcal{B}$ then $\mathcal{I}_b$ is a maximal ideal of $\mathcal{A}$,*
(2) *if $\mathcal{I}_0$ is a maximal ideal of $\mathcal{A}$ containing $\mathcal{I}_b$, and $\mathcal{I}$ is the ideal of $\mathcal{B}$ generated by $\mathcal{I}_0$ then $\{b, \neg b\} \cap \mathcal{I} = \emptyset$.*

Proof. Suppose that $b \in \mathcal{B} \setminus \mathcal{A}$ and $\mathcal{I}_b = \{a \in \mathcal{A} : \exists_{x,y \in \mathcal{A}} a \leq x \vee y, x \leq b, y \leq \neg b\}$. To prove that $\mathcal{I}_b$ is a proper ideal, suppose that for some $x, y \in \mathcal{A}$, $x \leq b$, $y \leq \neg b$ and $x \vee y = \mathbf{1}$. Then $b = b \wedge \mathbf{1} = b \wedge (x \vee y) = (b \wedge x) \vee (b \wedge y)$ and $b \wedge y = \mathbf{0}$ because $y \leq \neg b$. Thus $b = b \wedge x$ which implies that $b \leq x$ and finally we get that $b = x$ which is not possible because $b \in \mathcal{B} \setminus \mathcal{A}$ and $x \in \mathcal{A}$.

To show (1) let us assume that $\mathcal{A}$ is a maximal subalgebra of $\mathcal{B}$. By Lemma 1, in order to prove that $\mathcal{I}_b$ is a maximal ideal we need only to show that for every $x \in \mathcal{A}$, $\{x, \neg x\} \cap \mathcal{I}_b \neq \emptyset$. Take any $x \in \mathcal{A}$. Then by Lemma 5 at least one of the following is true:

(i) $\{b \wedge x, \neg b \wedge x\} \subseteq \mathcal{A}$,
(ii) $\{b \wedge \neg x, \neg b \wedge \neg x\} \subseteq \mathcal{A}$.

It is clear that (i) implies that $x \in \mathcal{I}_b$ and (ii) implies that $\neg x \in \mathcal{I}_b$. Thus $\{x, \neg x\} \cap \mathcal{I}_b \neq \emptyset$.

To prove (2) suppose that $\mathcal{I}_0$ is a maximal ideal of $\mathcal{A}$ containing $\mathcal{I}_b$ and $\mathcal{I}$ is the ideal of $\mathcal{B}$ generated by $\mathcal{I}_0$ i.e.:

$$\mathcal{I} = \{a \in \mathcal{B} : \exists_{x \in \mathcal{I}_0} a \leq x\}.$$

Now we prove that $b \notin \mathcal{I}$. Indeed, if $b \in \mathcal{I}$ then there exists $x \in \mathcal{I}_0$ such that $b \leq x$ and consequently $\neg x \leq \neg b$. Hence $\neg x \in \mathcal{I}_b \subseteq \mathcal{I}_0$ and finally $\mathbf{1} = x \vee \neg x \in \mathcal{I}_0$ - a contradiction. The fact that $\neg b \notin \mathcal{I}$ can be proved similarly. ∎

THEOREM 1 [Effective]. *If $\mathcal{A}$ is a maximal subalgebra of a Boolean algebra $\mathcal{B}$ and $b \in \mathcal{B} \setminus \mathcal{A}$ then there exists a maximal ideal of $\mathcal{B}$ containing b.*

Proof. Suppose that $\mathcal{A}$ is a maximal subalgebra of $\mathcal{B}$ and $b \in \mathcal{B} \setminus \mathcal{A}$. Then by Key Lemma (1), $\mathcal{I}_b = \{a \in \mathcal{A} : \exists_{x,y \in \mathcal{A}} a \leq x \vee y, x \leq b, y \leq \neg b\}$ is a maximal ideal of $\mathcal{A}$. Let $\mathcal{I}$ be the ideal of $\mathcal{B}$ generated by the set $\mathcal{I}_b \cup \{b\}$. Then $\mathcal{I} = \{a \in \mathcal{B} : \exists_{y \in \mathcal{A}} a \leq b \vee y, y \leq \neg b\}$ and it is easy to see, that the ideal $\mathcal{I}$ must be proper. Indeed, if $\mathbf{1} = b \vee y$ for some $y \in \mathcal{A}$ such that $y \leq \neg b$, then $\neg b = \neg b \wedge \mathbf{1} = \neg b \wedge (b \vee y) = \neg b \wedge y$ which implies that $\neg b \leq y$, and finally that $\neg b = y$ which is not possible.

Suppose now that the ideal $\mathcal{I}$ is not maximal. Then, by Lemma 1, $\mathcal{I} \cup \neg \mathcal{I}$ is a proper subalgebra of $\mathcal{B}$ properly containing $\mathcal{I}_b \cup \neg \mathcal{I}_b$ ($b \in (\mathcal{I} \cup \neg \mathcal{I}) \setminus$

$(\mathcal{I}_b \cup \neg\mathcal{I}_b))$ and $\mathcal{I}_b \cup \neg\mathcal{I}_b = \mathcal{A}$ which contradicts the maximality of $\mathcal{A}$. Thus the ideal $\mathcal{I}$ is a maximal ideal of $\mathcal{B}$ containing the element b, as required. ∎

THEOREM 2 [Based on Prime Ideal Theorem]. *If $\mathcal{A}$ is a proper subalgebra of a Boolean algebra $\mathcal{B}$ then $\mathcal{A}$ is contained in a maximal subalgebra of $\mathcal{B}$.*

Proof. Suppose that $\mathcal{A}$ is a proper subalgebra of $\mathcal{B}$ and $b \in \mathcal{B} \setminus \mathcal{A}$. Then by Key Lemma, $\mathcal{I}_b$ is a proper ideal of $\mathcal{A}$ and by Prime Ideal Theorem it can be extended to a maximal ideal $\mathcal{I}_0$ of $\mathcal{A}$. Using Key Lemma (2) we get that the ideal $\mathcal{I}$ of $\mathcal{B}$ generated by $\mathcal{I}_0$ does not intersect $\{b, \neg b\}$. Using Prime Ideal Theorem again we can find two different maximal ideals $\mathcal{I}_1, \mathcal{I}_2$ of $\mathcal{B}$ such that $\mathcal{I}_1$ contains a proper ideal of $\mathcal{B}$ generated by $\mathcal{I} \cup \{b\}$ and $\mathcal{I}_2$ contains a proper ideal of $\mathcal{B}$ generated by $\mathcal{I} \cup \{\neg b\}$. It is known (see [6]) that $(\mathcal{I}_1 \cap \mathcal{I}_2) \cup \neg(\mathcal{I}_1 \cap \mathcal{I}_2)$ is a maximal subalgebra of $\mathcal{B}$. Since $\mathcal{I}_0 \subseteq (\mathcal{I}_1 \cap \mathcal{I}_2)$ then $(\mathcal{I}_0 \cup \neg\mathcal{I}_0) \subseteq (\mathcal{I}_1 \cap \mathcal{I}_2) \cup \neg(\mathcal{I}_1 \cap \mathcal{I}_2)$. Hence, by Lemma 1 we get that $\mathcal{A} \subseteq (\mathcal{I}_1 \cap \mathcal{I}_2) \cup \neg(\mathcal{I}_1 \cap \mathcal{I}_2)$. ∎

THEOREM 3. *The following sentences are effectively equivalent:*

(1) *every non-trivial Boolean algebra possesses a maximal subalgebra,*
(2) *every non-trivial Boolean algebra possesses a maximal ideal,*
(3) *every proper ideal of a Boolean algebra is contained in a maximal ideal,*
(4) *every proper subalgebra of a Boolean algebra is contained in a maximal subalgebra.*

Proof. (2) and (3) are the equivalent statements of Prime Ideal Theorem. The implication (3)⇒(4) follows from Theorem 2. The implication (4)⇒(1) is trivial and the implication (1)⇒(2) is a consequence of Theorem 1. ∎

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