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COMPLETENESS OF HERMITE SYSTEM IN THE BERGMAN SPACE OVER A STRIP

We prove that holomorphic Hermite system $h_k(z) = e^{z^2/2}(e^{-z^2})^{(k)}$ $k = 0, 1, \dots$ is linearly dense in the Bergman space over strip $S_c = \{z \in \mathbb{C}; |Imz| < c\}$, $c \in (0, \infty)$. The Genchev transform [4], [5], [14] is used to deduce this result fom classical Steklov theorem.

1. Introductory remarks

Functions $h_k(z)$ (see below) are related to classical Hermite polynomials given by the formula

$$H_k(z) := (-1)^k e^{z^2}(e^{-z^2})^{(k)} \qquad k = 0, 1, \dots \tag{1}$$

See Watson [22], Plancherel -Rotach [12], Hille [8], Akhiezer [1], Lebedev [10], Vilenkin [21], Rusev [13]. (Natanson [11] uses a slightly different definition in which the factor $(-1)^k$ is absent.) Hermite polynomials were introduced in 1859 by P.L.Tchebycheff [19] and in 1864 by Ch.Hermite [7]. In fact they occur already in Laplace's *Theorie analytique des probabilites.* By definition

$$h_k(z) = (-1)^k e^{-z^2/2} H_k(z) \qquad k = 0, 1, \dots \tag{2}$$

Steklov [17] (see also [11,p.472]) proved that Hermite polynomials $H_k(t)$, $t \in \mathbb{R}$ are linearly dense in $L^2(\mathbb{R}, e^{-t^2})$. An extensive study of Hermite polynomials in a complex domain was undertaken by Watson and (later) by Hille with an eye to represent a function holomorphic in a strip $S_c := \{z = x + iy; |y| < c\}$ by pointwise convergent Hermitian series. Nevertheless the (obvious) fact that for every $c \in (0, \infty)$ the functions (2) are square integrable over S_c hence belong to the Bergman space $L^2H(S_c)$has attracted no attention (comp. eg. [3]).In this context Theorem 2 indicates

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a link between classical orthogonal polynomials and holomorphic geometry [3], [15]. We find it convenient to use Fourier transform described by the formula

$$f^{\uparrow}(s) := (L^2) \lim_{A\to\infty} \frac{1}{\sqrt{2\pi}} \int_{-A}^{A} e^{ist} f(t)dt. \tag{3}$$

Up to the constant factor the same definition of Fourier transform appears in Vilenkin [21, p.91]. By Plancherel theorem (3) defines a unitary operator on $L^2(\mathbb{R})$ and the inverse transform is described by the formula

$$f^{\downarrow}(t) := (L^2) \lim_{A\to\infty} \frac{1}{\sqrt{2\pi}} \int_{-A}^{A} e^{-ist} f(s)ds. \tag{4}$$

For the Fourier transform (3) h_k is an eigenfunction which belongs to the eigenvalue i^k. See Akhiezer [1, p.39].

In the following we shall need the Genchev transform [4], [5], [14]. Let $D := \{z \in \mathbb{C}; Re\ z \in J\}$be a one dimensional tube over interval J. For $f \in L^2H(D)$ and fixed $x \in J$ denote $g(y) := f(x+iy)$. The function

$$g(y) := f(x+iy) \tag{5}$$

is independent of x and is called the Genchev transform for f. We recall the following

THEOREM 1 (Genchev,Dzhrbashyan [5], [4]). *The correspondence $f \to G_f$ defines a unitary mapping of $L^2H(D)$ onto $L^2(\mathbb{R}, w_J)$ where*

$$w_J(t) := \int_{e^{-2tx}} dx \tag{6}$$

is an elementary weight determined by J.

2. Reduction to a real variable problem

Our aim is to prove the following

THEOREM 2. *For every $c \in (0,\infty\}$ Hermite system $h_k(z)$ $k = 0,1,\ldots$ is linearly dense in the Bergman space $L^2H(S_c)$.*

Proof. Obviously, it suffices to prove that under some unitary mapping of $L^2H(S_c)$ the system h_k, $k = 0,1,\ldots$ is mapped onto a linearly dense subset.

We begin with the mapping of $L^2H(S_c)$ onto $L^2H(D)$ where $J = (-c,c)$. It maps $f(z)$, $z \in S_c$ onto $f(z/i)$, $z \in D$. It will be composed with Genchev transform which maps $L^2H(D)$ onto $L^2(\mathbb{R}, t^{-1}\sinh(2ct))$. Finally we shall map the latter space onto $L^2(\mathbb{R}, e^{-t^2}t^{-1}\sinh(2ct))$ by the mapping which takes $f(t), t \in \mathbb{R}$ onto $f(t)e^{t^2/2}$, $t \in \mathbb{R}$. One verifies easily that

(1) Under the rotation the function $h_k(z), z \in S_c$ is mapped onto $h_k^*(z) := h_k(z/i), z \in D$
(2) Under the Genchev transform the function h_k^* is mapped onto $i^k h_k(t), t \in \mathbb{R}$. Indeed (by taking x=0) we see that $G_{h_k^*}$ is the Fourier transform of $h_k^*(iy) = h_k(y), y \in \mathbb{R}$.The latter is an eigenfunction, hence the claim is established
(3) Under the third mapping the function $h_k(t) = (-1)^k e^{-t^2/2}H_k(t)$ is mapped onto$(-1)^k H_k(t)$.

Since (by Lemma 2 of the next section) Hermite polynomials are linearly dense in the space$L^2(\mathbb{R}, e^{-t^2}t^{-1}\sinh(2ct))$ the proof of the theorem is completed.

3. Relation with Steklov density theorem

We shall need an easy extension of Steklov theorem [16] stated in [11, p.472]. (The statement in [11] corresponds to the case $a = 1$.)

LEMMA 1. *For every $a \in (0,\infty)$ the set of all polynomials is dense in $L^2(\mathbb{R}, e^{-(at)^2})$.*

Proof. The set $C_c(\mathbb{R})$ of continuous functions with compact supports is dense in$L^2(\mathbb{R}, e^{-(at)^2})$ hence it suffices to show that every $f \in C_c(\mathbb{R})$ can be approximated in $L^2(\mathbb{R}, e^{-(at)^2})$ by polynomials. Take arbitrary $\epsilon > 0$; since $a\epsilon > 0$ and $f^*(t) := f(t/a) \in C_c(\mathbb{R})$ there exists (by Steklov theorem) a polynomial W(t) such that

$$\int_{\mathbb{R}} |f(t/a) - W(t)|^2 e^{-t^2} dt < a\varepsilon. \tag{7}$$

Substituting $t = as$ and dividing by a yields

$$\int_{\mathbb{R}} |f(s) - W(as)|^2 e^{-(as)^2} ds < \varepsilon \tag{8}$$

and we see that the polynomial $W^*(s) := W(as)$ gives the desired approximation of f, Q.E.D.

We can now easily prove the lemma referred to in section 2.

LEMMA 2. *Hermite polynomials $H_k(t)$, $k = 0, 1, \ldots$ are linearly dense in $L^2(\mathbb{R}, p(t))$, where $p(t) := e^{-t^2}t^{-1}\sinh(2ct)$.*

Proof. The linear span of Hermite polynomials is the set of all polynomials and we need to show that the latter is dense in $L^2(\mathbb{R}, p(t))$. As in

Lemma 1 it suffices to show that every $f \in C_c(\mathbb{R})$ can be approximated in $L^2(\mathbb{R}, p(t))$ by polynomials. Note that the ratio

$$\frac{p(t)}{e^{-t^2/2}} = e^{-t^2/2} \frac{e^{2ct} - e^{-2ct}}{2t} \tag{9}$$

is bounded since it is continuous and tends to 0 for $t \to \pm\infty$. Therefore it suffices to show that f can be approximated by polynomials in $L^2(\mathbb{R}, e^{-t^2/2})$. The latter follows by taking $a = 1/\sqrt{2}$ in Lemma 1, Q.E.D.

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