DEMONSTRATIO MATHEMATICA
Vol. XXIX No 4 1996

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THE MEANING OF BASIC CATEGORY THEORETICAL NOTIONS IN SOME CATEGORIES OF PARTIAL ALGEBRAS. III

Introduction

This paper is a direct continuation of [BW92a] and of [BW92b], in this note referred to as Part I and Part II, respectively. And we continue here the discussion of the conditions on a similarity type, under which some category theoretical constructions like products, equalizers, pullbacks etc. and their dual notions exist in general in the categories of partial algebras of that type with all homomorphisms (yielding the category $\mathfrak{Hom}(\tau)$), all closed homomorphisms ($\mathfrak{C}$-$\mathfrak{Hom}(\tau)$), all quomorphisms ($\mathfrak{Quom}(\tau)$), all closed quomorphisms ($\mathfrak{C}$-$\mathfrak{Quom}(\tau)$) or all conformisms ($\mathfrak{Conf}(\tau)$), respectively as morphisms. And in the case of existence we give descriptions of their constructions. The numberings of definitions, lemmas, propositions and theorems here continue those in Part I and Part II. In particular "Theorem nd" concerns the category theoretical construction dual to the one treated in "Theorem n". However, we do not continue numbering the theorems according to their numbers in Table 1 of Part I – which gives a survey of the main results as far as existence is concerned –, since we postpone the more special treatments of inverse and direct limits to the end, and rearrange other theorems for proof technical reasons. At the end of this note we insert a new table (Table 2) extending Table 1 by existence results concerning four other types of morphisms not treated in this series of papers but in [M93] and [AMRS95], respectively. Moreover, in this table we have rearranged the lines in such a way that limit constructions and colimit constructions form different connected blocks. This may help in particular in connection with the existence and non-existence proofs with respect to Theorems 4 (Completeness) and 5 (Pullbacks) and to their duals.

With respect to the definitions of the basic concepts and to some preliminary results the reader is referred to Part I and [B86]. For concepts not defined here see among others the book [B86], the note [BW87] or the survey articles [B82], [B92] or [B93]. As far as category theoretical concepts and results are concerned see [HS73], [AdHS90] or [Ml71].

The authors are grateful to F. Rosselló for many helpful suggestions and remarks.

The Main Theorems (continuation)

THEOREM 3 (Equalizers). *Let $f, g : \mathbb{A} \rightharpoonup \mathbb{B}$ be any two morphisms of one of the five categories under consideration.*

- *Then the equalizer of f and g exists without any restrictions on τ in the categories $\mathfrak{Hom}(\tau)$, $\mathfrak{C}$-$\mathfrak{Hom}(\tau)$, and $\mathfrak{Quom}(\tau)$.*
- *In the category $\mathfrak{C}$-$\mathfrak{Quom}(\tau)$ an equalizer of f and g always exists, if and only if the type τ either specifies only nullary or only unary operations,*
- *and in $\mathfrak{Conf}(\tau)$ an equalizer of f and g exists in general if and only if all fundamental operations are unary.*

Whenever the equalizer $(\mathbb{E}_{fg}, m_{fg})$ exists, then m_{fg} is an injective monomorphism of the category under consideration, which therefore is also a homomorphism from $\mathbb{E}_{fg}$ into $\mathbb{A}$.

In each case, when the equalizer of f and g exists in general, it has a special representative, where $\mathbb{E}_{fg}$ is a relative subalgebra of $\mathbb{A}$ and m_{fg} : $\mathbb{E}_{fg} \to \mathbb{A}$ is its full and injective homomorphic embedding $\mathrm{id}_{E_{fg}A}$ into $\mathbb{A}$. Thus it still remains to describe the carrier set E_{fg} in each case:

- *In $\mathfrak{Hom}(\tau)$ and $\mathfrak{C}$-$\mathfrak{Hom}(\tau)$ we have*
$$E_{fg} := \{a \in A \mid f(a) = g(a)\},$$
and this is a closed subset of $\mathbb{A}$.
- *In $\mathfrak{Quom}(\tau)$ and $\mathfrak{C}$-$\mathfrak{Quom}((0)_{\varphi\in\Omega})$ one has*
$$E_{fg} := \{a \in A \mid a \in \operatorname{dom} f \cap \operatorname{dom} g \text{ and } f(a) = g(a)\} \cup A \setminus (\operatorname{dom} f \cup \operatorname{dom} g).$$
- *In $\mathfrak{C}$-$\mathfrak{Quom}((1)_{\varphi\in\Omega})$ and $\mathfrak{Conf}((1)_{\varphi\in\Omega})$ one has*
$$\begin{aligned} E_{fg} := \{a \in A \mid \; & \text{there is no unary term } t \in F(\{x\}, \mathrm{TAlg}(\tau)) \\ & \text{such that } a \in \operatorname{dom} t^{\mathbb{A}} \text{ and } (t^{\mathbb{A}}(a) \in \operatorname{dom} f \setminus \operatorname{dom} g \\ & \text{or } t^{\mathbb{A}}(a) \in \operatorname{dom} g \setminus \operatorname{dom} f \text{ or } f(t^{\mathbb{A}}(a)) \neq g(t^{\mathbb{A}}(a)))\}. \end{aligned}$$

Observe that in this latter case E_{fg} is a closed subset of $\mathbb{A}$. As a matter of fact it is the largest closed subset of $\mathbb{A}$ having empty intersection with $D_{fg} := (\operatorname{dom} f \setminus \operatorname{dom} g) \cup (\operatorname{dom} g \setminus \operatorname{dom} f) \cup \{a \in \operatorname{dom} f \cap \operatorname{dom} g \mid f(a) \neq g(a)\}$.

Remark. Observe that for each similarity type, for which in any of these categories equalizers exist for any two morphisms, multiple equalizers exist for any arbitrary large family of morphisms having the same source and the same target object, and it is characterized in the corresponding way.

Proof. It is well known from category theory that for each equalizer the morphism is a monomorphism, and therefore totally defined according to Proposition 2 (cf. Part I) and in particular an injective homomorphism in the categories $\mathfrak{Hom}(\tau)$, $\mathfrak{C}$-$\mathfrak{Hom}(\tau)$, $\mathfrak{Quom}(\tau)$ and $\mathfrak{C}$-$\mathfrak{Quom}(\tau)$, whenever it exists.

Since $\{a \in A \mid f(a) = g(a)\}$ is always a closed subset of $\mathbb{A}$, if f and g are (closed) homomorphisms, the statement about $\mathfrak{Hom}(\tau)$ and $\mathfrak{C}$-$\mathfrak{Hom}(\tau)$ is easily seen to be true.

For *quomorphisms* one only has to observe, that, for all $a \in (\operatorname{dom} f \setminus \operatorname{dom} g) \cup (\operatorname{dom} g \setminus \operatorname{dom} f) = \operatorname{dom} f \cup \operatorname{dom} g \setminus (\operatorname{dom} f \cap \operatorname{dom} g)$, a cannot be the value of a quomorphism equalizing f and g, while there are no further restrictions (i.e. the proposed morphism really equalizes f and g).

If $m : \mathbb{E} \rightharpoonup \mathbb{A}$ is an equalizer of $f, g : \mathbb{A} \rightharpoonup \mathbb{B}$ in one of the categories $\mathfrak{C}$-$\mathfrak{Quom}(\tau)$ or $\mathfrak{Conf}(\tau)$, then $m(E)$ has to be a closed subset of $\mathbb{A}$, in particular $m(E)$ has to contain all nullary fundamental constants existing in $\mathbb{A}$. However, if $\tau = (\mathbf{0})_{\varphi\in\Omega}$, then – w.r.t. the constants – any closed quomorphism is defined on exactly those nullary fundamental constants defined in $\mathbb{A}$ which are also defined in $\mathbb{B}$ (and if they are defined in $\mathbb{B}$, then they have to be defined in $\mathbb{A}$). Moreover, if $\psi^{\mathbb{A}}$ exists and does not belong to, say, $\operatorname{dom} f$, for some $\psi \in \Omega$, then one also has $\psi^{\mathbb{A}} \notin \operatorname{dom} g$ and therefore $\psi^{\mathbb{E}_{fg}}$ exists, and one has $m(\psi^{\mathbb{E}}) = \psi^{\mathbb{A}}$. Hence f and g have to be defined on exactly the same constants of $\mathbb{A}$.

From this observation it easily follows that equalizers exist in $\mathfrak{C}$-$\mathfrak{Quom}((\mathbf{0})_{\varphi\in\Omega})$ and that they have the form described in the theorem (up to isomorphism).

Now consider $\mathfrak{C}$-$\mathfrak{Quom}((\mathbf{1})_{\varphi\in\Omega})$ or $\mathfrak{Conf}((\mathbf{1})_{\varphi\in\Omega})$, and let E_{fg} be defined as in the above theorem. Since a variable x is a term, it is easily seen that for all $a \in E_{fg}$ either ($a \in \operatorname{dom} f \cap \operatorname{dom} g$ and $f(a) = g(a)$) or $a \in A \setminus (\operatorname{dom} f \cup \operatorname{dom} g)$. Hence $f \circ m_{fg} = g \circ m_{fg}$. In addition, if there are $a \in A$ and a term $t \in F(\{x\}, \mathrm{TAlg}(\tau))$ such that $t^{\mathbb{A}}(a)$ exists and either $t^{\mathbb{A}}(a) \in (\operatorname{dom} f \setminus \operatorname{dom} g) \cup (\operatorname{dom} g \setminus \operatorname{dom} f)$ or $f(t^{\mathbb{A}}(a)) \neq g(t^{\mathbb{A}}(a))$, then a must not belong to the equalizer of f and g. Moreover, E_{fg} is a closed subset of $\mathbb{A}$, since E_{fg} is the complement of the initial segment of $\mathbb{A}$ generated by D_{fg} $(= (\operatorname{dom} f \cup \operatorname{dom} g) \setminus \{a \mid a \in \operatorname{dom} f \cap \operatorname{dom} g \text{ and } f(a) = g(a)\})$. Since an initial segment contains with every value of a fundamental operation also the corresponding arguments, its complement has to be a closed subset.

Thus m_{fg} is a closed homomorphism and therefore a closed quomorphism as well as a conformism. Now, if $h : \mathbb{C} \rightharpoonup \mathbb{A}$ is any closed quomorphism or conformism with $h \circ f = h \circ g$, then $h(C)$ has to be a closed subset of $\mathbb{A}$ disjoint from D_{fg}, and therefore contained in E_{fg} (which is the greatest such set). Thus $(\mathbb{E}_{fg}, m_{fg})$ $(= (\mathbb{E}_{fg}, \mathrm{id}_{E_{fg}A}))$ is an equalizer of f and g.

Finally let us consider the counterexamples in the remaining cases:

- In $\mathfrak{Conf}((0))$ consider Figure 9 for $f, g : \mathbb{A} \rightharpoonup \mathbb{B}$. If $h : \mathbb{C} \rightharpoonup \mathbb{A}$ were an equalizer, then $\varphi^{\mathbb{C}}$ would have to exist and one would have to have $h(\varphi^{\mathbb{C}}) = \varphi^{\mathbb{A}}$, and therefore $f \circ h \neq g \circ h$ contradicting the condition for h to be an equalizer of f and g.

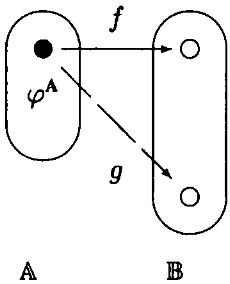

Figure 9: In $\mathfrak{Conf}((0))$ f and g have no equalizer.

- In connection with $\mathfrak{C}$-$\mathfrak{Quom}((0,1))$ consider Figure 10 for $f, g : \mathbb{A} \rightharpoonup \mathbb{B}$. If $h : \mathbb{C} \rightharpoonup \mathbb{A}$ were an equalizer of f and g, then necessarily $\varphi^{\mathbb{A}} \in h(C)$, and therefore also $a = \psi^{\mathbb{A}}(\varphi^{\mathbb{A}}) \in h(C)$, since h is closed. But then $f \circ h \neq g \circ h$ contradicting that h should equalize f and g.

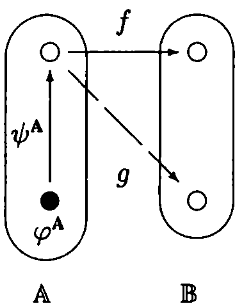

Figure 10: In $\mathfrak{C}$-$\mathfrak{Quom}((0,1))$ f and g have no equalizer.

- With respect to $\mathfrak{C}$-$\mathfrak{Quom}((2))$ or $\mathfrak{Conf}((2))$ consider the closed quomorphisms and conformisms $f, g : \mathbb{A} \rightharpoonup \mathbb{B}$ depicted in Figure 11. Let

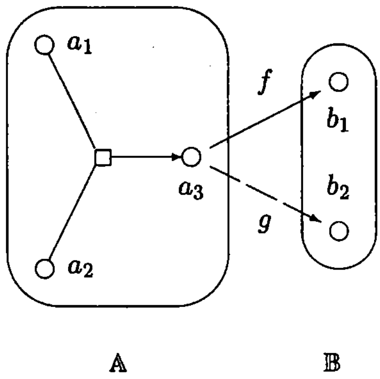

Figure 11: In $\mathfrak{C}$-$\mathfrak{Quom}((2))$ and $\mathfrak{Conf}((2))$ f and g have no equalizer.

$\mathbb{C} = (\{c\}, \emptyset)$ and consider the closed quomorphisms respectively conformisms $h_i : \mathbb{C} \rightharpoonup \mathbb{A}$, $c \mapsto a_i$, $i \in \{1,2\}$. Then one has $f \circ h_i = g \circ h_i$ for $i \in \{1,2\}$. Therefore, if there existed an equalizer $h : \mathbb{E} \rightharpoonup \mathbb{A}$ of f and g in $\mathfrak{C}$-$\mathfrak{Quom}((2))$ or $\mathfrak{Conf}((2))$, then E had to have at least two elements $e_1, e_2 \in \operatorname{dom} h$, $e_1 \neq e_2$, such that $h(e_i) = a_i$ $(i \in \{1,2\})$. But since h had to be a closed quomorphism or a conformism, $\varphi^{\mathbb{C}}(e_1, e_2)$ had to exist and to be mapped by h onto a_3. However then h could no longer equalize f and g. ∎

As a preparation for the result on coequalizers we prove some lemmas, but let us first give a definition.

DEFINITION 4. Let $f, g : A \rightharpoonup B$ be two partial mappings from a set A into a set B. Then we define

$$\Gamma_{fg} := \{(f(a), g(a)), (g(a), f(a)) \mid a \in \operatorname{dom} f \cap \operatorname{dom} g\}$$

to be the *coequalizing relation* of f and g. Moreover, define θ'_{fg} to be the equivalence relation on B generated by Γ_{fg}, and set

$$B_{fg} := \{b \in B \mid f^{-1}([b]_{\theta'_{fg}}) \cup g^{-1}([b]_{\theta'_{fg}}) \subseteq \operatorname{dom} f \cap \operatorname{dom} g\}$$

to be the *coequalizing domain* of f and g.

Finally, let θ_{fg} be the restriction of θ'_{fg} to B_{fg}, which we shall call the *coequalizing equivalence* of f and g. And let $\mathbb{B}_{fg}$, be the relative subalgebra of $\mathbb{B}$ with carrier set B_{fg}.

Remark. Obviously B_{fg} is a union of θ'_{fg}-classes, and, for any $b \in B_{fg}$, $[b]_{\theta_{fg}}$ has more than one element only if $b \in B^0_{fg}$.

LEMMA 7. *Let $\tau = (n_\varphi)_{\varphi\in\Omega}$ be a similarity type satisfying $n_\varphi \le 1$ for each $\varphi \in \Omega$. Let $f, g : \mathbb{A} \rightharpoonup \mathbb{B}$ be any two closed quomorphisms or conformisms, respectively.*

(1) *Let both f and g be closed quomorphisms. Then the coequalizing equivalence θ_{fg} is indeed a closed congruence relation on $\mathbb{B}_{fg}$.*

(2) *Let f and g be conformisms, then, for all $\varphi \in \Omega$ and for all $b \in B_{fg}$ with $[b]_{\theta_{fg}} \subseteq \operatorname{dom}\varphi^{\mathbb{B}_{fg}}$, one has*

$$\{\varphi^{\mathbb{B}}(b') \mid b' \in [b]_{\theta_{fg}}\} \subseteq [\varphi^{\mathbb{B}}(b)]_{\theta_{fg}}.$$

Proof. Ad (1): Since the statement here is quite similar to the first one of Lemma 1 in Part I we just ask the reader to adopt the first part of the proof there to this situation.

Ad (2): Consider $\varphi \in \Omega$ of arity $n_\varphi = 1$ – for $n_\varphi = 0$ the statement is trivially true –, and $b \in B_{fg}$ with $[b]_{\theta_{fg}} \subseteq \operatorname{dom}\varphi^{\mathbb{B}_{fg}}$. Let $b' \in [b]_{\theta_{fg}}$. Then, if $b \neq b'$, we have sequences $b_0 = b, b_1, \ldots, b_n = b' \in \operatorname{dom} f \cap \operatorname{dom} g$ and $a_1, \ldots, a_n \in A$ such that $\{b_{i-1}, b_i\} = \{f(a_i), g(a_i)\}$ for $i \in \{1, \ldots, n\}$. Since f and g are conformisms and $[b]_{\theta_{fg}} \subseteq \operatorname{dom}\varphi^{\mathbb{B}_{fg}}$, we again get $a_i \in \operatorname{dom}\varphi^{\mathbb{A}}$ and $\{\varphi^{\mathbb{B}_{fg}}(b_{i-1}), \varphi^{\mathbb{B}_{fg}}(b_i)\} = \{f(\varphi^{\mathbb{A}}(a_i)), g(\varphi^{\mathbb{A}}(a_i))\}$. This shows that $\varphi^{\mathbb{B}_{fg}}(b') \in [\varphi^{\mathbb{B}_{fg}}(b)]_{\theta_{fg}}$. ∎

LEMMA 8. *Let $\mathfrak{K}$ be any one of the categories $\mathfrak{Hom}(\tau)$, $\mathfrak{C}$-$\mathfrak{Hom}(\tau)$, $\mathfrak{Quom}(\tau)$, $\mathfrak{C}$-$\mathfrak{Quom}(\tau)$, or $\mathfrak{Conf}(\tau)$, respectively, let $f, g : \mathbb{A} \rightharpoonup \mathbb{B}$ be two morphisms in $\mathfrak{K}$, and let $h : \mathbb{B} \rightharpoonup \mathbb{C}$ be any $\mathfrak{K}$-morphism satisfying $h \circ f = h \circ g$.*

Then $\operatorname{dom} h$ *is a union of θ_{fg}-classes and $b \in \operatorname{dom} h$, $b' \in [b]_{\theta_{fg}}$ imply $h(b') = h(b)$.*

Proof. Assume $b \in \operatorname{dom} h$. If $b \notin f[A] \cup g[A]$, then $b \in B_{fg}$ and $[b]_{\theta_{fg}} = \{b\}$. Thus, assume that $b \in f[A] \cup g[A]$, and let $b' \in [b]_{\theta'_{fg}}$, $b \neq b'$. Then there are $a_1, \ldots, a_n \in A$, $b_0 = b, b_1, \ldots, b_n = b' \in B$ such that $\{b_{i-1}, b_i\} = \{f(a_i), g(a_i)\}$ for $1 \le i \le n$. If, say, $b = b_0 = f(a_1)$, $b_1 = g(a_1) \in \{f(a_2), g(a_2)\}$, then $h \circ f = h \circ g$ implies $b_1 \in \operatorname{dom} h$, and $h(b_1) = h(b_0)$. By induction on n we realize that $b' \in \operatorname{dom} h$ and $h(b') = h(b)$.

If $b \notin B_{fg}$, then there were $b'' \in [b]_{\theta'_{fg}}$ and $a'' \in A$ such that, say, $b'' = f(a'')$ and $a'' \notin \operatorname{dom} g$. Since $b \in \operatorname{dom} h$ implies $b'' \in \operatorname{dom} h$, as seen above, we get that $h \circ f(a'')$ exists, while $h \circ g(a'')$ does not exist, contradicting $h \circ f = h \circ g$. Therefore b has to belong to B_{fg}. ∎

THEOREM 3d (Coequalizers). *Let $f, g : \mathbb{A} \rightharpoonup \mathbb{B}$ be any two morphisms of one of the five categories under consideration. Then a coequalizer of f and g exists without any restrictions on the similarity type τ in the category $\mathfrak{Hom}(\tau)$ and it is given by* $(\mathrm{nat}_\theta, \mathbb{B}/\theta)$, *where θ is the congruence relation on $\mathbb{B}$ generated by $\Gamma_{fg} := \{(f(a), g(a)) \mid a \in A\}$.*

In $\mathfrak{C}$-$\mathfrak{Hom}(\tau)$, $\mathfrak{C}$-$\mathfrak{Quom}(\tau)$ *and* $\mathfrak{Conf}(\tau)$ *the coequalizer of* f *and* g *exists in general if and only if* $n_\varphi \leq 1$ *for all* $\varphi \in \Omega$, *and it exists in* $\mathfrak{Quom}(\tau)$ *if and only if* $n_\varphi = 0$ *for all* $\varphi \in \Omega$.

In $\mathfrak{C}$-$\mathfrak{Hom}(\tau)$ *the coequalizer, if it exists in general, is defined in the same way as in* $\mathfrak{Hom}(\tau)$.

In $\mathfrak{Quom}((0)_{\varphi\in\Omega})$, *in* $\mathfrak{C}$-$\mathfrak{Quom}(\tau)$ *and in* $\mathfrak{Conf}(\tau)$ *with* $n_\varphi \leq 1$ *for each* $\varphi \in \Omega$, *the coequalizer is defined – using the notation of Definition 4 – as* $(\mathrm{nat}_{\theta_{fg}} : \mathbb{B} \rightharpoonup \mathbb{C}(f,g), \mathbb{C}(f,g))$, *where* $C(f,g) := B_{fg}/\theta_{fg}$, *and*

- *for* $\varphi \in \Omega$ *with* $n_\varphi = 0$: $\varphi^{\mathbb{C}(f,g)}$ *exists and is equal to* $[\varphi^{\mathbb{B}}]_{\theta_{fg}}$, *if* $\varphi^{\mathbb{B}}$ *exists and belongs to* B_{fg},
- *for* $\varphi \in \Omega$ *with* $n_\varphi = 1$ *we have:*
- *in* $\mathfrak{C}$-$\mathfrak{Quom}(\tau)$: $\varphi^{\mathbb{C}(f,g)} = \varphi^{\mathbb{B}_{fg}/\theta_{fg}}$,
- *in* $\mathfrak{Conf}(\tau)$: $\mathrm{dom}\,\varphi^{\mathbb{C}(f,g)} := \{[b]_{\theta_{fg}} \mid b \in B_{fg}$, *and* $[b]_{\theta_{fg}} \subseteq \mathrm{dom}\,\varphi^{\mathbb{B}_{fg}}\}$ *and if* $[b]_{\theta_{fg}} \in \mathrm{dom}\,\varphi^{\mathbb{C}(f,g)}$, *then* $\varphi^{\mathbb{C}(f,g)}([b]_{\theta_{fg}}) := [\varphi^{\mathbb{B}_{fg}}(b)]_{\theta_{fg}}$.

Remark. As in the case of equalizers this result extends to arbitrary non-empty families $\mathcal{F} := (f_i : \mathbb{A} \rightharpoonup \mathbb{B})_{i\in I}$ of morphisms (instead of (f,g)), when one sets in analogy to Definition 4:

$$\Gamma_{\mathcal{F}} := \{(f_i(a), f_j(a)) \mid i,j \in I,\ \ a \in \bigcap_{i\in I} \mathrm{dom}\, f_i\},$$

$$B_{\mathcal{F}} := \{b \in B \mid \bigcup_{i\in I} f_i^{-1}([b]_{\theta'_{\mathcal{F}}}) \subseteq \bigcap_{i\in I} \mathrm{dom}\, f_i\}.$$

Proof. Let us first consider the case of *homomorphisms*:
Let $h : \mathbb{B} \to \mathbb{C}$ be any homomorphism such that $h \circ f = h \circ g$; then, obviously, for any $a \in A$, $(f(a), g(a)) \in \ker h$. Therefore $\theta := \mathrm{Con}_{\mathbb{B}}\, \Gamma_{fg} \subseteq \ker h$, and the diagram completion theorem for full and surjective homomorphisms (here for nat_θ) tells us that there is a unique homomorphism $h_0 : \mathbb{B}/\theta \to \mathbb{C}$ such that $h = h_0 \circ \mathrm{nat}_\theta$, showing that $(\mathrm{nat}_\theta, \mathbb{B}/\theta)$ is a coequalizer of f and g (it is obvious from the construction that $\mathrm{nat}_\theta \circ f = \mathrm{nat}_\theta \circ g$).

The same is true for *closed homomorphisms*, if $n_\varphi \leq 1$ for each $\varphi \in \Omega$, because of Lemma 1 in Part I.

However, in $\mathfrak{C}$-$\mathfrak{Hom}((2))$ there are no coequalizers in general. In order to realize this, consider the closed homomorphisms $f, g : \mathbb{A} \to \mathbb{B}$ as depicted in Figure 12, where $f(a) = b_1$, $g(a) = b_3$, $\mathrm{graph}\,\varphi^{\mathbb{B}} := \{((b_1, b_2), b_4), ((b_2, b_1), b_4)\}$. There is no closed homomorphism $h : \mathbb{B} \to \mathbb{C}$ at all, which would satisfy $h \circ f = h \circ g$, since (b_1, b_3) is not contained in any closed congruence relation of $\mathbb{B}$.

Let us now consider $\mathfrak{Quom}(\tau)$. If $\tau = (0)_{\varphi\in\Omega}$, then θ_{fg} is obviously a congruence relation on B_{fg}, since nullary fundamental constants do not impose any restrictions on equivalence relations to be congruence relations

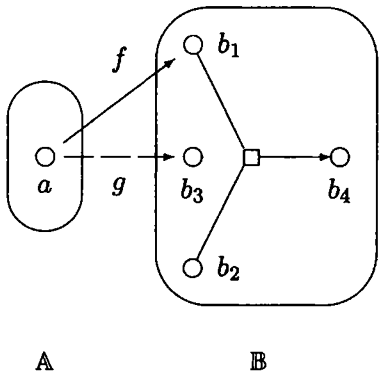

Figure 12: In $\mathfrak{C}$-$\mathfrak{H}$om ((2)) f and g have no coequalizer.

(observe also Remark 1.(iii) in Part II). From the definition of $\mathbb{C}(f,g)$ it follows that $\mathrm{nat}_{\theta_{fg}} : \mathbb{B} \rightharpoonup \mathbb{C}(f,g)$ is a quomorphism satisfying $\mathrm{nat}_{\theta_{fg}} \circ f = \mathrm{nat}_{\theta_{fg}} \circ g$. Let $h : \mathbb{B} \rightharpoonup \mathbb{D}$ be any quomorphism satisfying $h \circ f = h \circ g$. Then Lemma 8 tells us that $\mathrm{dom}\, h$ is a union of θ_{fg}-classes and that $\theta_{fg} \cap (\mathrm{dom}\, h)^2 \subseteq \ker h$. Therefore define for $b \in B_{fg}$:

$$h_0([b]_{\theta_{fg}}) := \begin{cases} h(b), & \text{if } b \in \mathrm{dom}\, h \\ \text{undefined}, & \text{else.} \end{cases}$$

Then h_0 is a partial mapping out of $C(f,g)$ into D. Assume that, for some nullary $\varphi \in \Omega$, $\varphi^{\mathbb{C}(f,g)}$ exists and belongs to $\mathrm{dom}\, h_0$. Then $\varphi^{\mathbb{B}}$ exists (since $\mathrm{nat}_{\theta_{fg}}$ is full) and belongs to $\mathrm{dom}\, h$. Since h is a quomorphism, $\varphi^{\mathbb{D}}$ has to exist, and one has to have that $\varphi^{\mathbb{D}} = h(\varphi^{\mathbb{B}}) = h_0(\varphi^{\mathbb{C}(f,g)})$. This shows that h_0 is indeed a quomorphism, which satisfies $h_0 \circ \mathrm{nat}_{\theta_{fg}} = h$.

In order to realize that there are no coequalizers in general in $\mathfrak{Quom}(\tau)$, if the type τ specifies at least one at least unary operation symbol, consider quomorphisms $f, g : \mathbb{A} \rightharpoonup \mathbb{B}$ as depicted in Figure 13. Obviously $\Gamma_{fg} = \{(b_1,b_2),(b_2,b_1),(b_3,b_4),(b_4,b_3)\}$, and $\theta_{fg} = \Gamma_{fg} \cup \Delta_B$, $B_{fg} = B$. Let $h_i : B \rightharpoonup C_i$ be defined in such a way that $\mathrm{dom}\, h_i = \bigcup\{c \mid c \in C_i\}$ and $h_i(b_j) = c$ iff $b_j \in c$ for all $i \in \{1,2\}$, $j \in \{1,2,\ldots,7\}$, and $c \in C_i$. It is easy to realize that h_1 and h_2 are quomorphisms, $h_i : \mathbb{B} \rightharpoonup \mathbb{C}_i$, satisfying $h_i \circ f = h_i \circ g$, $i \in \{1,2\}$. Now assume that $(h : \mathbb{B} \rightharpoonup \mathbb{D}, \mathbb{D})$ were an equalizer of f and g (i.e. e.g. $h \circ f = h \circ g$). Then there would be quomorphisms $g_i : \mathbb{D} \rightharpoonup \mathbb{C}_i$ such that $h_i = g_i \circ h$ ($i \in \{1,2\}$). Now, h_1 is total, and therefore h would have to be total and therefore a homomorphism with $\Gamma_{fg} \subseteq \ker h \subseteq \mathrm{Con}_{\mathbb{B}}\, \Gamma_{fg} = \ker h_1$. Yet this would imply $\ker h = \mathrm{Con}_{\mathbb{B}}\, \Gamma_{fg}$, inferring that g_1 were an isomorphism. However, there is no quomorphism $h_0 : \mathbb{C}_1 \rightharpoonup \mathbb{C}_2$ such that

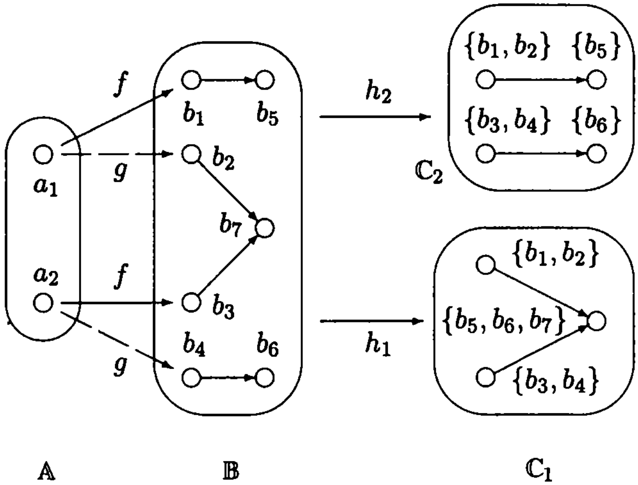

Figure 13: In $\mathfrak{Quom}((1))$ f and g have no coequalizer.

$h_0 \circ h_1 = h_2$, since $\{b_5, b_6, b_7\}$ would have to be mapped both onto $\{b_5\}$ and onto $\{b_6\}$, while h_2 is not defined on b_7. Hence there cannot exist a coequalizer of these particular quomorphisms f and g.

Next, consider $\mathfrak{C}$-$\mathfrak{Quom}(\tau)$. First let $n_\varphi \leq 1$ for all $\varphi \in \Omega$. From Lemma 7.(1) we know already that the coequalizer equivalence θ_{fg} is a closed congruence relation on $\mathbb{B}_{fg}$. We have to prove that

$$(\mathrm{nat}_{\theta_{fg}} : \mathbb{B} \rightharpoonup \mathbb{B}_{fg}/\theta_{fg}, \mathbb{B}_{fg}/\theta_{fg}) \text{ is a coequalizer of } f \text{ and } g.$$

By construction we have $\mathrm{nat}_{\theta_{fg}} \circ f = \mathrm{nat}_{\theta_{fg}} \circ g$ and that $\mathrm{nat}_{\theta_{fg}}$ is a closed quomorphism. Let $h : \mathbb{B} \rightharpoonup \mathbb{D}$ be any closed quomorphism coequalizing f and g: $h \circ f = h \circ g$. From Lemma 8 we already know that $\operatorname{dom} h$ is a union of θ_{fg}-classes, and that $\theta_{fg} \cap (\operatorname{dom} h)^2 \subseteq \ker h$. Since $\mathrm{nat}_{\theta_{fg}}|_{B_{fg}} : \mathbb{B}_{fg} \to \mathbb{B}_{fg}/\theta_{fg}$ is a closed and surjective homomorphism, since $\operatorname{dom} h \subseteq B_{fg}$, and since $\theta_{fg} \cap (B_{fg} \times \operatorname{dom} h) \subseteq \ker h$ (cf. Lemma 8), Lemma 6 from Part I implies that there exists a closed quomorphism $l : \mathbb{B}_{fg}/\theta_{fg} \rightharpoonup \mathbb{D}$ such that $l \circ \mathrm{nat}_{\theta_{fg}} = h$. Obviously, l is unique with this property. Thus the above statement has been proved.

In order to realize that there exist no coequalizers in general in $\mathfrak{C}$-$\mathfrak{Quom}(\tau)$ as well as in $\mathfrak{Conf}(\tau)$, if τ specifies at least one at least binary operation, consider Figure 14. The definitions of h_1, h_2 and h_3, and the argumentation about the non-existence of a coequalizer of f and g is carried through in a similar way as in connection with Figure 13 (we assume the binary operation to be commutative in this example). Therefore we leave the

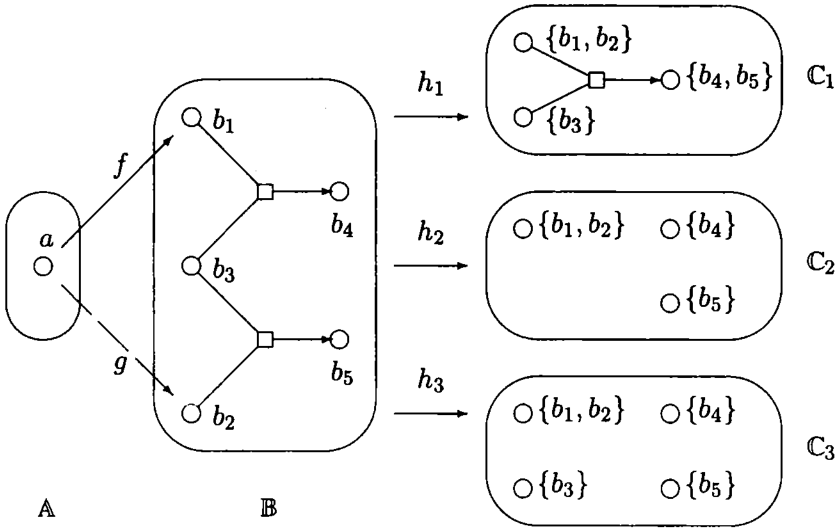

Figure 14: In $\mathfrak{C}$-$\mathfrak{Quom}((2))$ and $\mathfrak{Conf}((2))$ f and g have no coequalizer (h_3 and $\mathbb{C}_3$ are only needed for $\mathfrak{Conf}((2))$).

details to the reader. For the arguments concerning closed quomorphisms only h_1 and h_2 are needed. In connection with conformisms one has to exclude by h_3 and $\mathbb{C}_3$ the possibility that the coequalizer might have empty structure, and that $(h_3, \mathbb{C}_3)$ might serve as a coequalizer (but then there would not exist a conformism, say g from $\mathbb{C}_3$ into $\mathbb{C}_1$ satisfying $g \circ h_3 = h_1$, since the existence of g would enforce the binary operation on $\mathbb{C}_3$ to have a non-empty graph).

Finally consider the case of conformisms, i.e. $\mathfrak{Conf}(\tau)$, with $n_\varphi \leq 1$ for $\varphi \in \Omega$. The arguments concerning $\mathfrak{Quom}((0)_{\varphi\in\Omega})$ also show that $(\mathrm{nat}_{\theta_{fg}} : B \rightharpoonup C(f,g), C(f,g))$ is a coequalizer of f and g in the category $\mathfrak{Par}$ of all sets with all partial mappings as morphisms. Now, let $h : \mathbb{B} \rightharpoonup \mathbb{D}$ be a conformism satisfying $h \circ f = h \circ g$, then there exists a unique partial mapping $l : \mathbb{C}(f,g) \rightharpoonup \mathbb{D}$ such that $l \circ \mathrm{nat}_{\theta_{fg}} = h$. What remains to show is that l is indeed a conformism: Assume that $\varphi^{\mathbb{D}}(l([b]_{\theta_{fg}}))$ $(= \varphi^{\mathbb{D}}(h(b)))$ exists for some $b \in \mathrm{dom}\, h (\subseteq B_{fg})$, then $[b]_{\theta_{fg}} \subseteq [b]_{\ker h} \subseteq \mathrm{dom}\, \varphi^{\mathbb{B}_{fg}}$ and $\varphi^{\mathbb{B}_{fg}}([b]_{\theta_{fg}}) \subseteq [\varphi^{\mathbb{B}_{fg}}(b)]_{\theta_{fg}})$ (by Lemma 7.(2)), and therefore $[b]_{\theta_{fg}} \in \mathrm{dom}\, \varphi^{\mathbb{C}(f,g)}$, and $l(\varphi^{\mathbb{C}(f,g)}([b]_{\theta_{fg}})) = \varphi^{\mathbb{D}}(l([b]_{\theta_{fg}}))$. This shows that l is indeed a conformism. ∎

Let us now consider already at this place the general case of the existence of limits and colimits, before we discuss the existence and structure of multiple pullbacks and multiple pushouts. The discussion of the existence of limits and colimits in general in any of the categories under consideration already at this place is possible because of well known facts from category theory (see e.g. HERRLICH and STRECKER [HS73]).

Corresponding to the state of our investigations, where we have considered so far terminal and initial objects, products and coproducts for non-empty index sets, and equalizers and coequalizers, we can already decide about completeness and cocompleteness of the categories under consideration. The following results are best read from Table 1 in Part I (or from Table 2 at the end of this note), in particular Theorem 4 from lines 1, 2 and 4, and Theorem 4d from lines 1d, 2d and 4d, where in each case one has to take the conjunction of all the conditions for the similarity type.

The entries in Table 1 in Part I only refer to the existence of limits and colimits with non-empty index sets. Yet, the other entries in this table show that there is no difference on the conditions for the arities, whether or not the empty index set is allowed. Since permission of the empty index set allows us to speak about completeness and cocompleteness, respectively, of the categories under consideration, we choose this case in what follows. For an explicit description of the constructions of limits and colimits in the case of their (general) existence see e.g. MAC LANE, [Ml71], Chapter V (limits) and the dualization.

THEOREM 4 (Completeness, limits).
(1) $\mathfrak{Hom}(\tau)$ *is complete for all similarity types* τ.
(2) $\mathfrak{C}$-$\mathfrak{Hom}(\tau)$ *is complete, iff* $\tau = (1)_{\varphi\in\Omega}$.
(3) $\mathfrak{Quom}(\tau)$ *is complete, iff* $\Omega = \emptyset$.
(4) $\mathfrak{C}$-$\mathfrak{Quom}(\tau)$ *is complete, iff* $\tau = (0)_{\varphi\in\Omega}$ *or* $\tau = (1)_{\varphi\in\Omega}$.
(5) $\mathfrak{Conf}(\tau)$ *is complete, iff* $\tau = (1)_{\varphi\in\Omega}$. ∎

THEOREM 4d (Cocompleteness, colimits).
(1*d*) $\mathfrak{Hom}(\tau)$ *is cocomplete for all similarity types* τ.
(2*d*) $\mathfrak{C}$-$\mathfrak{Hom}(\tau)$ *is cocomplete, iff* $\tau = (1)_{\varphi\in\Omega}$.
(3*d*) $\mathfrak{Quom}(\tau)$ *is cocomplete, iff* $\Omega = \emptyset$.
(4*d*) $\mathfrak{C}$-$\mathfrak{Quom}(\tau)$ *is cocomplete, iff* $\tau = (0)_{\varphi\in\Omega}$ *or* $\tau = (1)_{\varphi\in\Omega}$.
(5*d*) $\mathfrak{Conf}(\tau)$ *is cocomplete, iff* $\tau = (1)_{\varphi\in\Omega}$. ∎

Since some interesting constructions may exist in general for some similarity types, even when the corresponding category is not complete or not cocomplete, we add the consideration of some further constructions.

First we investigate the existence of multiple pullbacks and multiple pushouts for non-empty index sets.

THEOREM 5 (Multiple pullbacks for non-empty index sets). *Let the index sets be non-empty*:

- *In the categories* $\mathfrak{Hom}(\tau)$ *and* $\mathfrak{C}$-$\mathfrak{Hom}(\tau)$ *multiple pullbacks always exist for all similarity types* τ.
- *In* $\mathfrak{Quom}(\tau)$ *multiple pullbacks always exist, iff* $\Omega = \emptyset$.
- *In* $\mathfrak{C}$-$\mathfrak{Quom}(\tau)$ *multiple pullbacks always exist, iff* $\tau = (0)_{\varphi\in\Omega}$ *or* $\tau = (1)_{\varphi\in\Omega}$.
- *In* $\mathfrak{Conf}(\tau)$ *multiple pullbacks always exist, iff* $\tau = (1)_{\varphi\in\Omega}$.

More precisely: *Let* $(f_i : \mathbb{A}_i \rightharpoonup \mathbb{B})_{i\in I}$ $(I \neq \emptyset)$ *be any non-empty family of morphisms in any of the categories under consideration.*

In the categories $\mathfrak{Hom}(\tau)$ *and* $\mathfrak{C}$-$\mathfrak{Hom}(\tau)$ *a typical representative of the multiple pullback is* $(\mathbb{P}, (p_i : \mathbb{P} \rightarrow \mathbb{A}_i)_{i\in I})$, *where* $\mathbb{P}$ *is the subalgebra of the direct product* $\prod_{i\in I} \mathbb{A}_i$ *with carrier set*

$$P := \{(a_i \mid i \in I) \mid a_i \in A_i \text{ and } f_i(a_i) = f_j(a_j) \text{ for all } i, j \in I\};$$

and, for each $i \in I$, $p_i = \mathrm{pr}_i\,|_P$ *is the restriction to* P *of the* i*-th projection.*

In $\mathfrak{Quom}(\emptyset)$, *i.e. if we are in the category* $\mathfrak{Par}$, *then, for a typical representative* $(P, (p_i : P \rightharpoonup A_i)_{i\in I})$, P *is given as the subset*

$$P := \{(a_i \mid i \in I) \mid a_i \in \mathrm{dom}\, f_i \text{ and } f_i(a_i) = f_j(a_j) \text{ for all } i, j \in I\} \cup \\ \cup \bigcup_{\emptyset \neq J \subseteq I} \mathop{\times}_{j\in J} (A_i \setminus \mathrm{dom}\, f_i),$$

of the product object in $\mathfrak{Par}$, *and* $p_i = \mathrm{pr}^*{}_i|_P$ *with respect to the product* $(A^*, (\mathrm{pr}^*{}_i : A^* \rightharpoonup A_i)_{i\in I})$ *as described in Theorem 2 of Part I.*

This construction from $\mathfrak{Par}$ *also applies in the case of* $\mathfrak{C}$-$\mathfrak{Quom}((0)_{\varphi\in\Omega})$, *where one only has to observe that, for* $\varphi \in \Omega$, $\varphi^{\mathbb{P}}$ *exists, iff* $\varphi^{\mathbb{A}_i} \in \mathrm{dom}\, f_i$, *for all* $\varphi \in \Omega$, *or* $\varphi^{\mathbb{A}_i} \notin \mathrm{dom}\, f_i$, *for all* $\varphi \in \Omega$.

Since the pullbacks in the categories $\mathfrak{C}$-$\mathfrak{Quom}((1)_{\varphi\in\Omega})$ *and in* $\mathfrak{Conf}((1)_{\varphi\in\Omega})$ *can always be constructed as multiple equalizers w.r.t. the family* $(f_j \circ \mathrm{pr}_j : \prod_{i\in I} \mathbb{A}_i \rightarrow \mathbb{B})_{j\in I})$ *of morphisms starting from the direct product of the family* $(\mathbb{A}_i)_{i\in I}$, *and since the equalizers and products in these categories have quite difficult descriptions (cf. Theorems 2 and 3), we do not give more details in these cases.*

Proof. Except for the closed homomorphisms the existence statements directly follow from the completeness statements in Theorem 4, and the descriptions follow – in connection with well the known general category construction mentioned above – from Theorems 2 and 3. What remains to show for the case of closed homomorphisms is the fact that all homomorphisms p_i are closed. However this fact follows from Theorem 10.1.2.(viii) and Proposition 10.2.8.(i) in [B86], since the class $\mathfrak{CH}(\tau)$ of all closed homomorphisms in the category $\mathfrak{Hom}(\tau)$ is just $\Lambda(\mathfrak{Epi}(\tau))$, i.e. the "right hand partner" of the class of all epimorphisms in $\mathfrak{Hom}(\tau)$ in a factorization system, and therefore closed with respect to multiple pullbacks. Yet it can also easily be proved directly: Use the notation from the theorem, and fix some $i \in I$, $\varphi \in \Omega$ and $\underline{a}_1, \ldots, \underline{a}_{\tau(\varphi)} \in \mathbb{E}_{\mathcal{F}}$, such that $p_i(\underline{a}_1), \ldots, p_i(\underline{a}_{\tau(\varphi)})) \in \operatorname{dom} \varphi^{\mathbb{A}_i}$. Then $((f_i \circ p_i)(\underline{a}_1), \ldots, (f_i \circ p_i)(\underline{a}_{\tau(\varphi)})) \in \operatorname{dom} \varphi^{\mathbb{B}}$ and $((f_i \circ p_i)(\underline{a}_1), \ldots, (f_i \circ p_i)(\underline{a}_{\tau(\varphi)})) = ((f_j \circ p_j)(\underline{a}_1), \ldots, (f_j \circ p_j)(\underline{a}_{\tau(\varphi)}))$, for each $j \in I$. Since each f_j is closed, one has $(p_j(\underline{a}_1), \ldots, p_j(\underline{a}_{\tau(\varphi)})) \in \operatorname{dom} \varphi^{\mathbb{A}_j}$, for each $j \in I$. Since $\mathbb{E}_{\mathcal{F}}$ is a subalgebra of the product $\prod_{i \in I} \mathbb{A}_i$, this implies $(\underline{a}_1, \ldots, \underline{a}_{\tau(\varphi)})) \in \operatorname{dom} \varphi^{\mathbb{E}_{\mathcal{F}}}$, showing that p_i is closed for each $i \in I$.

Let us now discuss the non-existence statements included in the theorem:

Since $\mathfrak{Quom}(\tau)$ has a terminal object for each similarity type, the non-existence of products, if $\Omega \neq \emptyset$ (cf. Theorem 2), and Theorem 4 imply that $\mathfrak{Quom}(\tau)$ cannot have multiple pullbacks, if $\Omega \neq \emptyset$.

The same argumentation applies to the categories $\mathfrak{C}$-$\mathfrak{Quom}(\tau)$ and $\mathfrak{Conf}(\tau)$ in the cases of the similarity type τ, where we claim that multiple pullbacks do not exist. ∎

THEOREM 5d (Multiple pushouts for non-empty index sets). *Let the index sets be non-empty*:

- *In the category $\mathfrak{Hom}(\tau)$ multiple pushouts always exist for all similarity types τ.*
- *In $\mathfrak{Quom}(\tau)$ multiple pushouts always exist, iff $\Omega = \emptyset$.*
- *In the categories $\mathfrak{C}$-$\mathfrak{Hom}(\tau)$, $\mathfrak{C}$-$\mathfrak{Quom}(\tau)$ and $\mathfrak{Conf}(\tau)$ multiple pushouts always exist, iff one has for all arities: $n_\varphi \leq 1$ $(\varphi \in \Omega)$.*

More precisely: Let $(f_i : \mathbb{A} \rightharpoonup \mathbb{B}_i)_{i \in I}$ $(I \neq \emptyset)$ be any non-empty family of morphisms in any of the categories under consideration. And let $((q_i : \mathbb{B}_i \to \mathbb{Q})_{i \in I}, \mathbb{Q})$, be the candidate for the multiple pushout.

In the category $\mathfrak{Hom}(\tau)$ $\mathbb{Q}$ is the quotient algebra of $\coprod_{i \in I} \mathbb{B}_i$ – in the coproduct $((\iota_j : \mathbb{B}_j \to \coprod_{i \in I} \mathbb{B}_i)_{j \in I}, \coprod_{i \in I} \mathbb{B}_i)$ in $\mathfrak{Hom}(\tau)$ (see Theorem 2d in Part I) – with respect to the congruence relation θ on $\coprod_{i \in I} \mathbb{B}_i$ generated by the relation $\{((\iota_i \circ f_i)(a), (\iota_j \circ f_j)(a)) \mid i, j \in I,\ a \in A\}$. As morphisms take $(q_i := \operatorname{nat}_\theta \circ \iota_i : \mathbb{B}_i \to \mathbb{Q})_{i \in I}$.

In $\mathfrak{Quom}(\emptyset)$, i.e. if we are in the category $\mathfrak{Par}$ of sets with partial mappings, the pushout object Q can be constructed as follows: Let $B := \bigcup_{i \in I} B_i \times \{i\}$ and $\iota_i : B_i \to B$ with $\iota_i(b) := (b, i)$ for each $i \in I$ describe the coproduct of the family $(B_i)_{i \in I}$. Then the desired pushout object in $\mathfrak{Conf}(\emptyset) = \mathfrak{Par}$ is given by the multiple coequalizer object $Q := B / \operatorname{nat}_{\theta_{\mathcal{F}}}$ of the family $\mathcal{F} := (\iota_i \circ f_i : A \rightharpoonup B)_{i \in I}$ (cf. Theorem 3d and the remark following it with its notation). And the family of partial mappings of the pushout is given as $(q_i := \operatorname{nat}_{\theta_{\mathcal{F}}} \circ \iota_i : B_i \rightharpoonup Q)_{i \in I}$.

In connection with the categories $\mathfrak{C}$-$\mathfrak{Hom}(\tau)$, $\mathfrak{C}$-$\mathfrak{Quom}(\tau)$ and $\mathfrak{Conf}(\tau)$

with $n_\varphi \leq 1$, *for all* $\varphi \in \Omega$, *introduce the* $\Omega^{(1)}$*-reduction* $\tau_1 := \tau|_{\Omega^{(1)}}$ *of the given similarity type, and for a partial algebra, say* $\mathbb{D}$, *in any of these categories let* $\mathbb{D}^{(1)} := (D, (\varphi^{\mathbb{D}})_{\varphi \in \Omega^{(1)}})$ *designate its* τ_1*-reduct.*[1] *For the pushouts in any of these categories first construct the coproduct* $((\iota_i^{(1)} : \mathbb{B}_i^{(1)} \to \mathbb{B}^{(1)})_{i \in I}, \mathbb{B}^{(1)})$ – *i.e.* $B := \bigcup_{i \in I} B \times \{i\}$ (*cf. Theorem 2d*) – *in the reduct categories* $\mathfrak{C}$-$\mathfrak{Hom}(\tau_1)$, $\mathfrak{C}$-$\mathfrak{Quom}(\tau_1)$ *or* $\mathfrak{Conf}(\tau_1)$, *respectively. And then consider the multiple coequalizer* (*cf. Theorem 3d*) *with respect to these categories for the family* $\mathcal{F} := (\iota_i^{(1)} \circ f_i : \mathbb{A} \rightharpoonup \mathbb{B})_{i \in I}$. *And in each of the categories* $\mathfrak{C}$-$\mathfrak{Hom}(\tau)$, $\mathfrak{C}$-$\mathfrak{Quom}(\tau)$ *and* $\mathfrak{Conf}(\tau)$ *set* $(q_i := \mathrm{nat}_{\mathcal{F}}^{(1)} \circ \iota_i^{(1)} : \mathbb{B}_i \rightharpoonup \mathbb{Q})_{i \in I}$ *for the pushout morphisms. As far as the constants are concerned, define for* $\varphi \in \Omega^{(0)}$

– *in* $\mathfrak{C}$-$\mathfrak{Hom}(\tau)$:

$$\varphi^{\mathbb{Q}} := \begin{cases} \mathrm{nat}_{\theta\mathcal{F}}^{(1)} \circ \iota_i^{(1)} \circ f_i(\varphi^{\mathbb{A}}), & \textit{for any } i \in I, \textit{ if } \varphi^{\mathbb{A}} \textit{ exists},\\ \textit{undefined}, & \textit{else.} \end{cases}$$

– *in* $\mathfrak{C}$-$\mathfrak{Quom}(\tau)$ *and* $\mathfrak{Conf}(\tau)$:

$$\varphi^{\mathbb{Q}} := \begin{cases} \mathrm{nat}_{\theta\mathcal{F}}^{(1)} \circ \iota_i^{(1)} \circ f_i(\varphi^{\mathbb{A}}), & \textit{for any } i \in I, \textit{ if } \varphi^{\mathbb{B}_j} \textit{ exists for all } j \in I, \textit{ and}\\ & \textit{if } \emptyset \neq \bigcup_{i \in I} f_i^{-1}([(\varphi^{\mathbb{B}_i}, i)]_{\theta'_{\mathcal{F}}}) \subseteq \bigcap_{i \in I} \mathrm{dom}\, f_i,\\ \textit{undefined}, & \textit{else.} \end{cases}$$

P r o o f. With respect to homomorphisms and quomorphisms, and for the τ_1-reducts in $\mathfrak{C}$-$\mathfrak{Hom}(\tau_1)$, $\mathfrak{C}$-$\mathfrak{Quom}(\tau_1)$ or $\mathfrak{Conf}(\tau_1)$ the existence statements directly follow from the completeness statements in Theorem 4d, and in all these cases the descriptions – as coeqalizers of families of morphisms starting from a coproducts – follow from Theorems 2d and 3d.

We still have to show that $((q_i : \mathbb{B}_i \to \mathbb{Q})_{i \in I}, \mathbb{Q})$ is also a multiple pushout in the categories $\mathfrak{C}$-$\mathfrak{Hom}(\tau)$, $\mathfrak{C}$-$\mathfrak{Quom}(\tau)$ and $\mathfrak{Conf}(\tau)$, respectively. Because of the definitions in the theorem the q_i are also morphisms of the corresponding categories with respect to the constants, and $q_i \circ f_i = q_j \circ f_j$ is also true, for all $i, j \in I$. Now, assume that $(g_i : \mathbb{B}_i \to \mathbb{D})_{i \in I}$ is a sink, for which $g_i \circ f_i = g_j \circ f_j$ holds, for all $i, j \in I$. Let $g : \mathbb{Q}^{(1)} \rightharpoonup \mathbb{D}^{(1)}$ be the induced morphism for the unary reducts. And let $\varphi \in \Omega^{(0)}$.

First assume, that $\varphi^{\mathbb{D}}$ exists. Then $\varphi^{\mathbb{D}} \in (g_i \circ f_i)(A)$ for each $i \in I$. Hence $\varphi^{\mathbb{A}}$ as well as $\varphi^{\mathbb{B}_i}$ exist, for each $i \in I$, and we have $\varphi^{\mathbb{A}} \in \bigcup_{i \in I} f_i^{-1}([(\varphi^{\mathbb{B}_i}, i)]_{\theta'_{\mathcal{F}}}) \subseteq \bigcap_{i \in I} \mathrm{dom}\, f_i$. Therefore, $\varphi^{\mathbb{Q}}$ exists, and it is mapped by g onto $\varphi^{\mathbb{D}}$. This argument is already sufficient for $\mathfrak{Conf}(\tau)$.

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[1] We do not use the superscript for the morphisms except for those cases, where we want to indicate that they refer particularly to the reduct category.

For the cases of the categories $\mathfrak{C}$-$\mathfrak{Hom}(\tau)$ and $\mathfrak{C}$-$\mathfrak{Quom}(\tau)$ assume that $\varphi^{\mathbb{B}_i} \in \operatorname{dom} g_i$, for some $i \in I$. Since g_i is a homomorphism or quomorphism, respectively, we have $g_i(\varphi^{\mathbb{B}_i}) = \varphi^{\mathbb{D}}$. Then, in particular, $\varphi^{\mathbb{D}}$ has to exist, and therefore we can repeat the argumentation from above, showing that g is a closed homomorphism or closed quomorphism, respectively.

Finally, let us consider the non-existence statements:
In the categories $\mathfrak{C}$-$\mathfrak{Hom}(\tau)$, $\mathfrak{C}$-$\mathfrak{Quom}(\tau)$ and $\mathfrak{Conf}(\tau)$ multiple pushouts always exist for at most unary operations. If they would exist, when in addition at least one at least binary operation were around, we would have completeness of the full subcategory, where no nullary constants were allowed or existing, since then we would also have initial objects (in this subcategory). Moreover, in each of these cases the "local" pushout would also be a "global" one; namely the non-existence of nullary constants in all the partial algebras $\mathbb{B}_i$ $(i \in I)$ would imply, that in each partial algebra, say $\mathbb{C}$, allowing morphisms from the $\mathbb{B}_i$ into it, no nullary constants could be defined. Since therefore these subcategories would be cocomplete, this would contradict the non-existence (in general) of coproducts or coequalizers in such categories (which then also would be "global" ones).

Since in $\mathfrak{Quom}(\tau)$ initial objects always exist, while coequalizers only exist in general, when only nullary constants are specified, and while coproducts exist in general only, when no nullary constants are specified, multiple pushouts (for non-empty index sets) cannot exist in general, when Ω is non-empty (cf. Theorem 4d and Table 2). ∎

In particular, in the case of *closed homomorphisms*, the above argumentation can be used to prove the following extension of Theorem 2d of Part II (while this does not work in the case of partial morphisms, as can be concluded e.g. from the examples in figures 5 and 6 in Part II):

COROLLARY. *Let the arities of the similarity type τ be at most unary, i.e. $n_\varphi \leq 1$, for all $\varphi \in \Omega$. Then the coproduct of a non-empty family $(\mathbb{B}_i)_{i\in I}$ $(I \neq \emptyset)$ exists in the category $\mathfrak{C}$-$\mathfrak{Hom}(\tau)$, if, for all $\varphi \in \Omega^{(0)}$,*

$$\varphi^{\mathbb{B}_i} \textit{ exists, iff } \varphi^{\mathbb{B}_j} \textit{ exists, for all } i, j \in I.$$

And this coproduct is defined as in Theorem 2d of Part II for the case of homomorphisms. ∎

A particular case of limits and colimits is given by inverse limits and directed colimits. These exist in all categories under consideration for all possible types.

THEOREM 6 (Inverse limits). *In all five kinds of categories under consideration there exist for all similarity types inverse limits of non-empty inverse systems.*

More precisely:
Let $\mathbb{I} := (I, \leq)$ *be any non-empty upward directed set, and let* $\mathfrak{A} := ((\mathbb{A}_i)_{i\in I}, (f_{ij} : \mathbb{A}_i \rightharpoonup \mathbb{A}_j \mid i \geq j \text{ in } \mathbb{I}))$ *be any inverse system of partial algebras of type* τ *in any of the categories under consideration to which the following constructions always refer. Moreover, in what follows let* $(B := \times_{i\in I} A_i, (\mathrm{pr}_i)_{i\in I})$ *be the cartesian product of the carrier sets of the partial algebras involved together with the family of all natural projections. And let* $(\mathbb{A}, (f_i : \mathbb{A} \rightharpoonup \mathbb{A}_i)_{i\in I})$ *be a candidate of the inverse limit of* $\mathfrak{A}$. *Then the constructions are as follows*:

In $\mathfrak{Hom}(\tau)$ *and* $\mathfrak{C}$-$\mathfrak{Hom}(\tau)$ *let be*

$$A := \{\underline{a} = (a_i \mid i \in I) \mid \text{for } i \geq j \text{ in } \mathbb{I} \text{ one has } f_{ij}(a_i) = a_j\} \subseteq B,$$

let A *be provided with the relative substructure of the direct product* $\mathbb{B} = \prod_{i\in I} \mathbb{A}_i$ *– as a matter of fact* A *is a closed subset of* $\mathbb{B}$ *–, thus getting the inverse limit object* $\mathbb{A}$. *Moreover, for each* $i \in I$, *let* $f_i : \mathbb{A} \to \mathbb{A}_i$ *be the restriction of the natural projection*: $f_i := \mathrm{pr}_i|_A$.

For the categories with partial mappings underlying the morphisms call a sequence $\underline{a} := (a_j \mid j \in J) \in \times_{j\in J} A_j$ $(J \subseteq I)$ *a* filament[2] *of* $\mathfrak{A}$, *if*

(*F1*) $J =: J(\underline{a})$ *is a non-empty 'order filter' (i.e.* $J \neq \emptyset$, *and* $i \in J$ *and* $i \leq j$ *imply* $j \in J$*) of* $\mathbb{I}$,
(*F2*) *for* i *and* j *in* J *with* $i \geq j$ *one has* $a_i \in \operatorname{dom} f_{ij}$ *and* $f_{ij}(a_i) = a_j$, *and*
(*F3*) $i \in J$, $i \geq j$ *and* $a_i \in \operatorname{dom} f_{ij}$ *imply* $j \in J$.

Let A *be the set of all filaments of* $\mathfrak{A}$.
For each $i \in I$ *let the filament* $(a_j \mid j \in J)$ *belong to* $\operatorname{dom} f_i$ *if and only if* $i \in J$, *and if* $(a_j \mid j \in J)$ *belongs to* $\operatorname{dom} f_i$, *then* $f_i(a_j \mid j \in J) := a_i$ *– this yields the limiting cone* $(f_i : A \rightharpoonup A_i)_{i\in I}$ *within the category of all sets and partial mappings as morphisms.*
For $\varphi \in \Omega$, *and for a sequence* $\mathfrak{a} := (\underline{a}_1, \ldots, \underline{a}_{n_\varphi}) \in A^{n_\varphi}$ *of filaments define* $J(\mathfrak{a}) := \bigcap\{J(\underline{a}_k) \mid 1 \leq k \leq n_\varphi\}$ *(then this is always a non-empty set).*
As far as the structure is concerned, one has to distinguish between the different categories:

In the categories $\mathfrak{Quom}(\tau)$ *and* $\mathfrak{C}$-$\mathfrak{Quom}(\tau)$ *let a sequence* $\mathfrak{a} := (\underline{a}_1, \ldots \ldots, \underline{a}_{n_\varphi}) \in A^{n_\varphi}$ *of filaments belong to* $\operatorname{dom} \varphi^{\mathbb{A}}$ *if and only if there is some* k *in* $J(\mathfrak{a})$ *such that* $(*)_k$ *holds, where*

$$(*)_k \quad \text{for all } l \in J(\mathfrak{a}) \text{ with } l \geq k \text{ one has } f_l \circ \mathfrak{a} \in \operatorname{dom} \varphi^{\mathbb{A}_l}, \; \varphi^{\mathbb{A}_l}(f_l \circ \mathfrak{a}) \in \operatorname{dom} f_{lk}, \text{ and } f_{lk}(\varphi^{\mathbb{A}_l}(f_l \circ \mathfrak{a})) = \varphi^{\mathbb{A}_k}(f_k \circ \mathfrak{a}).$$

[2] We have slightly changed the definition in comparison to [P73], but in principle we get an isomorphic construction.

And if $\mathfrak{a} = (\underline{a}_1, \ldots, \underline{a}_{n_\varphi}) \in \operatorname{dom} \varphi^{\mathbb{A}}$ *then* $\varphi^{\mathbb{A}}(\mathfrak{a}) =: \underline{a}$, *where*

$$J(\underline{a}) := \{j \in I \mid \textit{there is } k \geq j \textit{ in } J(\mathfrak{a}) \textit{ satisfying } (^*)_k \textit{ and such that } \varphi^{\mathbb{A}_k}(f_k \circ \mathfrak{a}) \in \operatorname{dom} f_{kj}\}$$

and

$$\underline{a} := (a_j \mid j \in J(\underline{a}) \textit{ and there is } k \in J(\underline{a}) \textit{ satisfying } k \geq j, (*)_k \textit{ and } a_j = f_{kj}(\varphi^{\mathbb{A}_k}(f_k \circ \mathfrak{a}))).$$

In the category $\mathfrak{Conf}(\tau)$ *a sequence* $\mathfrak{a} := (\underline{a}_1, \ldots, \underline{a}_{n_\varphi}) \in A^{n_\varphi}$ *of filaments belongs to* $\operatorname{dom} \varphi^{\mathbb{A}}$, *if and only if there is* $j \in J(\mathfrak{a})$ *such that* $f_j \circ \mathfrak{a} \in \operatorname{dom} \varphi^{\mathbb{A}_j}$, *and if* $\mathfrak{a} \in \operatorname{dom} \varphi^{\mathbb{A}}$, *then define*

$$J' := \{i \in I \mid \textit{there is } l \geq i \textit{ such that } l \in J(\mathfrak{a}), f_l \circ \mathfrak{a} \in \operatorname{dom} \varphi^{\mathbb{A}_l} \textit{ and } \varphi^{\mathbb{A}_l}(f_l \circ \mathfrak{a}) \in \operatorname{dom} f_l\},$$

choose for each $i \in J'$ *some* $l(i) \geq i$ *– as it has to exist according to the definition of* J' *– and set* $\varphi^{\mathbb{A}}(\mathfrak{a}) := (f_{l(i)i}(\varphi^{\mathbb{A}_{l(i)}}(f_{l(i)} \circ \mathfrak{a}) \mid i \in J')$ *– observe that this definition is independent from the choice function* $l : J' \to I$.

Proof. We use the notation introduced in the theorem. In the case of total mappings and total algebras it is well known (cf. e.g. [G79]) that the construction given for $\mathfrak{Hom}(\tau)$ and $\mathfrak{C}$-$\mathfrak{Hom}(\tau)$ really yields the inverse limit. Since the case of partial algebras and (*closed*) *homomorphisms* is quite similar, we leave the details to the reader – it will also be easy to transform the proof for (closed) quomorphisms below to the one of (closed) homomorphisms.

The idea of the construction of inverse limits in connection with morphisms based on partial mappings is taken from V.S.Poythress, [P73], where he describes inverse limits for *conformisms* (which he calls p-morphisms). Therefore we do not repeat the proof here but concentrate our considerations on (*closed*) *quomorphisms*:

In what follows let always $\varphi \in \Omega$ be a fixed operation symbol, and let $n := n_\varphi$ be its arity. Moreover, let $\mathfrak{a} := (\underline{a}_1, \ldots, \underline{a}_n) \in A^n$ be an arbitrary sequence. Then $J(\mathfrak{a})$ is non-empty, since it is the finite intersection of non-empty order ideals of an upward directed set (if φ is nullary, then $J(\mathfrak{a}) = I$ by the usual definition of empty intersections within some given set). Assume $\mathfrak{a} \in \operatorname{dom} \varphi^{\mathbb{A}}$, let $\underline{a} := \varphi^{\mathbb{A}}(\mathfrak{a}) =: (a_j \mid j \in J(\underline{a}))$ be its value as constructed in the theorem, and let us first show that $J(\underline{a})$ is really a filament: It is non-empty, since $J(\mathfrak{a})$ is non-empty, and since the (non-empty) order filter of $J(\mathfrak{a})$ induced by some $k \in J(\mathfrak{a})$ satisfying $(^*)_k$ obviously forms a subset of $J(\underline{a})$ (since, for $l \geq k$ $(^*)_l$ is then satisfied, too, and $f_{ll} = \mathrm{id}_{A_l}$ is always an isomorphism). Moreover one has:

(F1): Assume $j \in J(\underline{a})$ and $j' \in I$ with $j' \geq j$. Then, by the definition of $\underline{a}$, there are $k \in J(\mathfrak{a})$, $k \geq j$ such that $(*)_k$ is satisfied, and such that $a_j = f_{kj}(\varphi^{\mathbb{A}_k}(f_k \circ \mathfrak{a}))$. Since $\mathbb{I}$ is directed, there is $k' \geq k, j'$. Obviously, k' still belongs to $J(\mathfrak{a})$, and satisfies $(*)_{k'}$ and in particular $\varphi^{\mathbb{A}_{k'}}(f_{k'} \circ \mathfrak{a}) \in \operatorname{dom} f_{k'j'}$, since $f_{k'j} = f_{kj} \circ f_{k'k} = f_{j'j} \circ f_{k'j'}$.

(F2): This follows immediately from the definition of $J(\underline{a})$ and $\underline{a}$ (compare also the argumentation above in connection with (F1)).

(F3): Let $j \in J(\underline{a})$ and $j \geq j'$ such that $a_j \in \operatorname{dom} f_{jj'}$. Then there is $k \in J(\mathfrak{a})$ such that $k \geq j$ satisfying $(*)_k$ and $\varphi^{\mathbb{A}_k}(f_k \circ \mathfrak{a}) \in \operatorname{dom} f_{kj}$. But $j \geq j'$ and $a_j \in \operatorname{dom} f_{jj'}$ together with $f_{kj'} = f_{jj'} \circ f_{kj}$ imply $j' \in J(\underline{a})$.

Next let us prove that each f_i is a quomorphism: Therefore, with the notation from above, let $i \in I$ be an arbitrary but fixed index, and assume that $i \in J(\mathfrak{a}) \cap J(\underline{a})$, i.e. f_i is defined on each $\underline{a}_k$, $k = 1 \ldots, n$, and on $\underline{a}$. Then, there is $l \geq i$ such that $(*)_l$ holds, and therefore e.g. one has, for each $l' \in J(\mathfrak{a})$ with $l' \geq l$, $f_{l'} \circ \mathfrak{a} \in \operatorname{dom} \varphi^{\mathbb{A}_{l'}}$ and $\varphi^{\mathbb{A}_{l'}}(f_{l'} \circ \mathfrak{a}) \in \operatorname{dom} f_{l'k}$. Since $f_i = f_{li} \circ f_l$, since f_{li} is a quomorphism, and since $f_i(\underline{a}) = f_i(\varphi^{\mathbb{A}}(\mathfrak{a})) = (f_{li} \circ f_l)(\varphi^{\mathbb{A}}(\mathfrak{a})) = f_{li}(f_l(\varphi^{\mathbb{A}}(\mathfrak{a}))) = f_{li}(\varphi^{\mathbb{A}_l}(f_l \circ \mathfrak{a})), \varphi^{\mathbb{A}_i}(f_{li} \circ f_l \circ \mathfrak{a})$ has to exist, and one has to have $f_i(\underline{a}) = f_i(\varphi^{\mathbb{A}}(\mathfrak{a})) = f_{li}(\varphi^{\mathbb{A}_l}(f_l \circ \mathfrak{a})) = \varphi^{\mathbb{A}_i}(f_{li} \circ f_l \circ \mathfrak{a}) = \varphi^{\mathbb{A}_i}(f_i \circ \mathfrak{a}))$, showing that f_i is indeed a quomorphism.

With respect to $\mathfrak{C}$-$\mathfrak{Quom}(\tau)$ we still have to show that f_i is closed, if each f_{lk} is closed: For this purpose assume that $f_i \circ \mathfrak{a} \in \operatorname{dom} \varphi^{\mathbb{A}_i}$, for some $i \in I$. Then, for every $k \geq i$, we have $f_i \circ \mathfrak{a} = f_{ki} \circ f_k \circ \mathfrak{a}$, and since f_{ki} is closed, we have $f_k \circ \mathfrak{a} \in \operatorname{dom} \varphi^{\mathbb{A}_k}$, and $f_{ki}(\varphi^{\mathbb{A}_k}(f_k \circ \mathfrak{a})) = \varphi^{\mathbb{A}_i}(f_{ki} \circ f_k \circ \mathfrak{a})) = \varphi^{\mathbb{A}_i}(f_i \circ \mathfrak{a}))$. Let $j \in J(\mathfrak{a})$. Then there is some $k \in J(\mathfrak{a})$ such that $k \geq i, j$. Then, from what we just have proved, there follows that $(*)_k$ is satisfied. Hence $\mathfrak{a} \in \operatorname{dom} \varphi^{\mathbb{A}}$, what was to be shown in order to prove that f_i is closed.

Next, let $\mathfrak{D} := (\mathbb{D}, (g_i : \mathbb{D} \rightharpoonup \mathbb{A}_i)_{i \in I})$ be a system of (closed) quomorphisms compatible with the given inverse system $\mathfrak{A}$ (i.e. $f_{ki} \circ g_k = g_i$, for all $i, k \in I$ with $i \leq k$). And let $g : D \rightharpoonup A$ be the partial mapping induced by the family $\mathfrak{D}$ in $\mathfrak{Par}$. Then we have to show that g is a (closed) quomorphism $g : \mathbb{D} \rightharpoonup \mathbb{A}$: Let φ and n be given as before, and let $\underline{d} := (d_1, \ldots, d_n) \in D^n$. Assume first that $\underline{d} \in \operatorname{dom} \varphi^{\mathbb{D}}$, and that, for some $i \in I$, $d_m, \varphi^{\mathbb{D}}(\underline{d}) \in \operatorname{dom} g_i$, for each $m \in \{1, \ldots, n\}$. Let $\underline{a}_k := g(d_k)$, $1 \leq k \leq n$, be the filament generated by the sequence $(g_j(d_k) \mid j \in I,\ d_k \in \operatorname{dom} g_j)$, and let $\mathfrak{a} := (\underline{a}_1, \ldots, \underline{a}_n)$. Then – since g_i is a quomorphism –, $g_i \circ \underline{d} \in \operatorname{dom} \varphi^{\mathbb{A}_i}$ and $g_i(\varphi^{\mathbb{D}}(\underline{d})) = \varphi^{\mathbb{A}_i}(g_i \circ \underline{d})$. Moreover, for every $k \in I$ with $k \geq i$, we have by $g_i = f_{ki} \circ g_k$ that $d_m, \varphi^{\mathbb{D}}(\underline{d}) \in \operatorname{dom} g_k$, for each $m \in \{1, \ldots, n\}$. And therefore we also have $g_k \circ \underline{d} \in \operatorname{dom} \varphi^{\mathbb{A}_k}$, $g_k(\varphi^{\mathbb{D}}(\underline{d})) = \varphi^{\mathbb{A}_k}(g_k \circ \underline{d})$, and $f_{ki}(\varphi^{\mathbb{A}_k}(g_k \circ \underline{d})) = \varphi^{\mathbb{A}_i}(g_i \circ \underline{d})$. This shows that $J_{\underline{d}} := \{i \in I \mid d_m, \varphi^{\mathbb{D}}(\underline{d}) \in$

$\operatorname{dom} g_k$ for $1 \leq m \leq n\} \subseteq \{i \in I \mid \varphi^{\mathbb{A}_i}(g_i \circ \underline{d})$ exists$\} \subseteq J(\mathfrak{a})$ is a non-empty order filter of $\mathbb{I}$, and that, for each $j \in J(\mathfrak{a})$, there is k – as an upper bound of i and j – such that $(*)_l$ is satisfied for all $l \geq k$. This shows that $\mathfrak{a} = g \circ \underline{d} \in \operatorname{dom} \varphi^{\mathbb{A}}$, and that – obviously – $\varphi^{\mathbb{A}}(g \circ \underline{d}) = g(\varphi^{\mathbb{D}}(\underline{d}))$.

Finally, assume that all f_{ki} and g_i are closed ($k \geq i$ in $\mathbb{I}$), that $\underline{d} \in (\operatorname{dom} g)^n$, and that $g \circ \underline{d} \in \operatorname{dom} \varphi^{\mathbb{A}}$. Then there is some $i \in I$ such that $g \circ \underline{d} \in (\operatorname{dom} f_i)^n$ and $f_i \circ g \circ \underline{d} = g_i \circ \underline{d} \in \operatorname{dom} \varphi^{\mathbb{A}_i}$ (since each f_i is a quomorphism). Since g_i is closed, we get $\underline{d} \in \operatorname{dom} \varphi^{\mathbb{D}}$. This shows that g is then a closed quomorphism, and this ends the proof. ■

THEOREM 6d (Direct limits, i.e. directed colimits). *In all five kinds of categories under consideration there exist for all similarity types directed colimits of non-empty directed systems.*

The constructions run as follows:
Let $\mathbb{I} := (I, \leq)$ be any non-empty upward directed set, and let $\mathfrak{A} := ((\mathbb{A}_i)_{i \in I}, (f_{ij} : \mathbb{A}_i \rightharpoonup \mathbb{A}_j \mid i \leq j \text{ in } \mathbb{I}))$ be any directed system of partial algebras of type τ in any of the categories under consideration to which the following constructions always refer. Let $((f_i : \mathbb{A}_i \rightharpoonup \mathbb{A})_{i \in I}, \mathbb{A})$ be a candidate for the directed colimit of $\mathfrak{A}$. Define, for each $i \in I$,

$$D_i := \bigcap\{\operatorname{dom} f_{ij} \mid j \geq i \text{ in } \mathbb{I}\},$$

and let $\mathbb{D}_i$ be the corresponding relative subalgebra of $\mathbb{A}_i$ with carrier set D_i. Let $g_{ij} := f_{ij}\,|_{D_i} : \mathbb{D}_i \to \mathbb{D}_j$ be the restriction of f_{ij} to D_i, for $i \leq j$ in $\mathbb{I}$. Then

- *$\mathfrak{D}_0 := ((D_i)_{i \in I}, (g_{ij} : D_i \to D_j \mid i \leq j \text{ in } \mathbb{I}))$ is a directed system of sets with mappings;*
- *in the categories $\mathfrak{Hom}(\tau)$ and $\mathfrak{C}$-$\mathfrak{Hom}(\tau)$ one just has $\mathbb{D}_i = \mathbb{A}_i$ and $g_{ij} = f_{ij}$, for all $i, j \in I$ with $i \leq j$.*
- *and in the categories $\mathfrak{Quom}(\tau)$, $\mathfrak{C}$-$\mathfrak{Quom}(\tau)$ and $\mathfrak{Conf}(\tau)$ $(g_{ij} : \mathbb{D}_i \to \mathbb{D}_j \mid i \leq j \text{ in } \mathbb{I})$ is a family of homomorphisms, closed homomorphisms or totally defined conformisms, respectively.*

In what follows let $((\iota_i : D_i \to B)_{i \in I}, B := \bigcup_{i \in I} D_i \times \{i\})$ be the set theoretical coproduct (disjoint union) of the carrier sets of the partial algebras involved together with the family of all natural injections. Define on B the following equivalence relation θ:

$$\theta := \{((a,i),(b,j)) \mid i,j \in I,\ a \in D_i,\ b \in D_j \text{ and there is } m \in I \text{ such that } i,j \leq m \text{ and } g_{im}(a) = g_{jm}(b)\},$$

and set $A := B/\theta$. Finally define, for each $i \in I$, $g_i := \operatorname{nat}_\theta \circ \iota_i$, and let f_i be the partial mapping from A_i into A with the same graph as g_i.

The construction of the structure in the different categories is as follows: Let $\varphi \in \Omega$ *and* $\underline{\mathfrak{a}} := (\mathfrak{a}_1, \ldots, \mathfrak{a}_{n_\varphi}) \in A^{n_\varphi}$. *Then we set* $\underline{\mathfrak{a}} \in \operatorname{dom} \varphi^{\mathbb{A}}$, *iff*

- *in* $\mathfrak{Hom}(\tau)$, $\mathfrak{C}$-$\mathfrak{Hom}(\tau)$, $\mathfrak{Quom}(\tau)$ *and* $\mathfrak{C}$-$\mathfrak{Quom}(\tau)$: *there are an index* $i \in I$ *and some sequence* $\underline{a} := (a_1, \ldots, a_{n_\varphi}) \in \operatorname{dom} \varphi^{D_i}$ *such that* $\underline{\mathfrak{a}} = g_i \circ \underline{a}$; *and if such an index* i *and such a sequence* $\underline{a}$ *exist, then we define* $\varphi^{\mathbb{A}}(\underline{\mathfrak{a}}) := f_i(\varphi^{\mathbb{D}_i}(\underline{a}))$;
- *in* $\mathfrak{Conf}(\tau)$: *for all indices* $i \in I$, *for which there exists a sequence* $\underline{a} := (a_1, \ldots, a_{n_\varphi}) \in D_i^{n_\varphi}$ *such that* $\underline{\mathfrak{a}} = g_i \circ \underline{a}$, *one has* $\underline{a} \in \operatorname{dom} \varphi^{D_i}$; *and if* $\underline{\mathfrak{a}} \in \operatorname{dom} \varphi^{\mathbb{A}}$, *and if* i *is any index and* $\underline{a} \in D_i^{n_\varphi}$ *any sequence such that* $\underline{\mathfrak{a}} = g_i \circ \underline{a}$, *then we define* $\varphi^{\mathbb{A}}(\underline{\mathfrak{a}}) := f_i(\varphi^{\mathbb{D}_i}(\underline{a}))$.

Proof. The construction of directed colimits of *homomorphisms* has been treated in [B86], Proposition 4.4.4 (and its proof is also obtained as a special case of the construction given above, when quomorphisms are treated). From Corollaries 1 and 2 of Proposition 11.3.1 in [B86] one can conclude that this also works for *closed homomorphisms*, yet it will follow directly, too, from our argumentation below. Therefore we concentrate considerations in this proof to the case of morphisms based on partial mappings.

First we show – using the notation from the theorem – that $\mathfrak{D}_0 := ((D_i)_{i\in I}, (g_{ij} : D_i \to D_j \mid i \leq j \text{ in } \mathbb{I}))$ is indeed a directed system of sets with mappings, and that $((f_i : A_i \rightharpoonup A)_{i\in I}, A)$, as constructed above, is its direct limit in the category $\mathfrak{Par}$: Assume that $d \in D_i$, for some $i \in I$, and let $j \in I$ with $i \leq j$. Set $d' := f_{ij}(d)$. We have to show that $d' \in D_j$: Let $k \in I$ with $j \leq k$. Then $f_{ik}(d) = (f_{jk} \circ f_{ij})(d) = f_{jk}(d')$. This shows that $d' \in \operatorname{dom} f_{jk}$, for all $k \geq j$ in $\mathbb{I}$. Therefore indeed $d' \in D_j$, and each g_{ij} is a total mapping $g_{ij} : D_i \to D_j$. That $\mathfrak{D}_0$ is a directed system of sets with mappings then follows from the fact that $\mathfrak{A}$ is a directed system. It is well-know, too, – since all g_i are mappings – that θ is indeed an equivalence relation on $B = \bigcup_{i\in I} D_i \times \{i\}$.

Next, let $(h_i : A_i \rightharpoonup C)_{i\in I}$ be a family of partial mappings with the same target set C compatible with $(f_{ij} : A_i \rightharpoonup A_j \mid i \leq j \text{ in } I)$. This means that

$$h_j \circ f_{ij} = h_i, \text{ for all } i \leq j \text{ in } \mathbb{I}. \tag{1}$$

We define an induced partial mapping $h : A \rightharpoonup C$ by

- $\operatorname{dom} h := \{\mathfrak{d} \in A \mid \text{there are } i \in I \text{ and } d \in \operatorname{dom} h_i \text{ such that } f_i(d) = \mathfrak{d}\}$,
- and if $\mathfrak{d} \in \operatorname{dom} h$ with $\mathfrak{d} = f_i(d)$, then define $g(\mathfrak{d}) := h_i(d)$.

We have to show first that $(h_i|_{D_i} : D_i \rightharpoonup C)_{i\in I}$ is compatible with $\mathfrak{D}_0$: Consider $a \in \operatorname{dom} h_i$, for some $i \in I$. Then, for every $k \in I$ with $i \leq k$, we have, by (1), that $h_k \circ f_{ik}(a) = h_i(a)$. This shows that $a \in \bigcap_{k\in I,\, k\geq i} \operatorname{dom} f_{ik} = D_i$.

Therefore, for $k \geq i$, we get $h_i(a) = (h_k \circ f_{ik})(a) = (h_k \circ g_{ik})(a)$, what was to be shown.

By the definition of h there follows immediately that $h \circ f_i = h_i$ is satisfied for all $i \in I$. The uniqueness of the induced morphism is also obvious: If $h' : A \rightharpoonup C$ satisfies $h' \circ f_i = h_i$, for all $i \in I$, then $\operatorname{dom} h \subseteq \operatorname{dom} h'$ and $h'(\mathfrak{d}) = h(\mathfrak{d})$ for all $\mathfrak{d} \in \operatorname{dom} h$. If there were $\mathfrak{d}' \in \operatorname{dom} h' \setminus \operatorname{dom} h$, then $\mathfrak{d}'$ would not belong to any $f_i(\operatorname{dom} h_i)$, and therefore $h'(\mathfrak{d}')$ could be an arbitrary element of C, i.e. the uniqueness requirement would be violated.

Next let us show that all the structure of $\mathbb{A}$ is correctly defined, – then the f_i ($i \in I$) will be morphisms of the corresponding category just by the corresponding definition –:
Let $\varphi \in \Omega$, $\underline{\mathfrak{a}} := (\mathfrak{a}_1, \ldots, \mathfrak{a}_{n_\varphi}) \in \operatorname{dom} \varphi^{\mathbb{A}}$, and $\underline{a} := (a_1, \ldots, a_{n_\varphi}) \in \operatorname{dom} \varphi^{\mathbb{D}_i}$ and $\underline{a}' := (a'_1, \ldots, a'_{n_\varphi}) \in \operatorname{dom} \varphi^{\mathbb{D}_j}$ such that $\underline{\mathfrak{a}} = g_i \circ \underline{a} = g_j \circ \underline{a}'$. Then the directedness of $\mathbb{I}$ and the definition of θ imply the existence of some $m \geq i, j$ in I such that $g_{im}(a_k) = g_{jm}(a'_k)$ for $1 \leq k \leq n_\varphi$ and $g_{im}(\varphi^{\mathbb{D}_i}(\underline{a}) = g_{jm}(\varphi^{\mathbb{D}_j}(\underline{a}')$ (possibly by applying the directedness in several steps) – the existence of; say, $\varphi^{\mathbb{D}_m}(g_{im} \circ \underline{a})$ follows in the first four kinds of categories from the fact that the g_i are homomorphisms; in $\mathfrak{Conf}(\tau)$ it follows from the assumption that $\underline{\mathfrak{a}} = f_i \circ \underline{a} = f_j \circ \underline{a}' \in \operatorname{dom} \varphi^{\mathbb{A}}$, which, by definition, implies this existence.

Now, let us show that h is a morphism in each of the categories under consideration: Let $\varphi \in \Omega$ and $n := n_\varphi$.

- In connection with $\mathfrak{Hom}(\tau)$, $\mathfrak{C}$-$\mathfrak{Hom}(\tau)$, $\mathfrak{Quom}(\tau)$ and $\mathfrak{C}$-$\mathfrak{Quom}(\tau)$ consider first $\underline{\mathfrak{a}} := (\mathfrak{a}_1, \ldots, \mathfrak{a}_n) \in \operatorname{dom} \varphi^{\mathbb{A}} \cap (\operatorname{dom} h)^n$, and $\underline{a} := (a_1, \ldots, a_{n_\varphi}) \in \operatorname{dom} \varphi^{\mathbb{D}_i}$ such that $\underline{\mathfrak{a}} = g_i \circ \underline{a}$, and also $\varphi^{\mathbb{A}} \in \operatorname{dom} h$. Then there is $k \geq i$ in I such that $g_{ik} \circ \underline{a} \in \operatorname{dom} \varphi^{\mathbb{D}_k} \cap (\operatorname{dom} h_k)^n$. Since h_k is a quomorphism, and since $h \circ g_k = h_k$, we get that $h \circ \underline{\mathfrak{a}} = h \circ g_i \circ \underline{a} = h_k \circ g_{ik} \circ \underline{a} \in \operatorname{dom} \varphi^{\mathbb{C}}$, and $h(\varphi^{\mathbb{A}}(\underline{\mathfrak{a}})) = h_k(\varphi^{\mathbb{D}_k}(g_{ik} \circ \underline{a})) = \varphi^{\mathbb{C}}(h_k \circ g_{ik} \circ \underline{a}) = \varphi^{\mathbb{C}}(h \circ \underline{\mathfrak{a}})$. This shows that h is a quomorphism.
- In connection with $\mathfrak{C}$-$\mathfrak{Hom}(\tau)$ and $\mathfrak{C}$-$\mathfrak{Quom}(\tau)$ assume (with the notation from above) that $h \circ \underline{\mathfrak{a}} \in \operatorname{dom} \varphi^{\mathbb{C}}$. Then, since each $h_j : \mathbb{D}_j \to \mathbb{C}$ is a closed homomorphism, and since, for suitable $i, k \in I$ with $i \leq k$, $h \circ \underline{\mathfrak{a}} = h \circ g_i \circ \underline{a} = h_k \circ g_{ik} \circ \underline{a}$, we get $g_{ik} \circ \underline{a} \in \operatorname{dom} \varphi^{\mathbb{D}_k}$. Since g_k is a quomorphism containing $g_{ik} \circ \underline{a}$ and $\varphi^{\mathbb{D}_k}(g_{ik} \circ \underline{a})$ in its domain, this implies that $\underline{\mathfrak{a}} \in \operatorname{dom} \varphi^{\mathbb{A}}$, showing that h is closed.
- Finally consider the category $\mathfrak{Conf}(\tau)$: Choose $\underline{\mathfrak{a}} := (\mathfrak{a}_1, \ldots, \mathfrak{a}_n) \in (\operatorname{dom} h)^n$ such that $h \circ \underline{\mathfrak{a}} \in \operatorname{dom} \varphi^{\mathbb{C}}$, and let $i \in I$ and $\underline{a} := (a_1, \ldots, a_n) \in (\operatorname{dom} f_i)^n$ such that $\underline{\mathfrak{a}} = f_i \circ \underline{a}$. Then $h \circ \underline{\mathfrak{a}} = h \circ f_i \circ \underline{a} = h_i \circ \underline{a} \in \operatorname{dom} \varphi^{\mathbb{C}}$. Since h_i is a conformism, this implies $\underline{a} \in \operatorname{dom} \varphi^{\mathbb{D}_i}$. Therefore, by the

definition of the structure of $\mathbb{A}$, we may conclude that $\underline{a} \in \operatorname{dom} \varphi^{\mathbb{A}}$, and this shows that h is indeed a conformism. ∎

Finally we collect all results on different kinds of morphisms investigated and known so far (cf. e.g. [AMRS95], where quomorphisms on closed domains and closed quomorphisms on closed domains[3] have been studied, and [M93], where quomorphisms with initial segments as domains and totally defined conformisms have been investigated). In order that the table does not become too wide we use the following abbreviations for the categories with the class of all partial algebras of type τ as object class:

$\mathfrak{H}$ for $\mathfrak{Hom}(\tau)$; $\mathfrak{cH}$ for $\mathfrak{C}$-$\mathfrak{Hom}(\tau)$; $\mathfrak{Q}$ for $\mathfrak{Quom}(\tau)$; $\mathfrak{cQ}$ for $\mathfrak{C}$-$\mathfrak{Quom}(\tau)$; $\mathfrak{cdQ}$ for $\mathfrak{CD}$-$\mathfrak{Quom}(\tau)$, where the class of all quomorphisms, of which the domain is a closed subalgebra of the start object, is the class of all morphisms; $\mathfrak{cdcQ}$ for $\mathfrak{CDC}$-$\mathfrak{Quom}(\tau)$, where the class of all '*closed-domain closed quomorphisms*', i.e. of all closed quomorphisms, of which the domain is a closed subalgebra of the start object, is the class of all morphisms; $\mathfrak{iQ}$ for $\mathfrak{I}$-$\mathfrak{Quom}(\tau)$, where the class of all quomorphisms, of which the domain is an initial segment of the start object, is the class of all morphisms; $\mathfrak{C}$ for $\mathfrak{Conf}(\tau)$; $\mathfrak{tC}$ for $\mathfrak{T}$-$\mathfrak{Conf}(\tau)$, where the class of all totally defined conformisms, is the class of all morphisms.

The entries in Table 2 mean:

- $+$: no restrictions on the similarity type;
- $-$: the construction does not exist, not even in the case of sets, i.e. of $\Omega = \emptyset$;
- $\emptyset$: the construction exists in general iff $\Omega = \emptyset$;
- $= 0$: the construction exists in general iff $\Omega = \Omega^{(0)}$, i.e. all fundamental operations have to be nullary;
- $= 1$: the construction exists in general iff $\Omega = \Omega^{(1)}$, i.e. all fundamental operations have to be unary;
- ≤ 1 : the construction exists in general iff all fundamental operations are at most unary;
- ≥ 1 : the construction exists in general iff all fundamental operations are at least unary (i.e. no nullary constants are allowed);

while combinations of such restrictions mean that the construction exists in general, iff at least one of the conditions on the similarity type is satisfied.

[3] See $\mathfrak{cdcQ}$ below; we use the notation *closed-domain closed quomorphisms* for them, as is also done in [AMRS95]. In [RV95] these morphisms are called *partial closed homomorphisms*, and in a related paper [ABRVW95] *closed-domain quomorphic conformisms.*

	$\mathfrak{H}$	$\mathfrak{cH}$	$\mathfrak{Q}$	$\mathfrak{cQ}$	$\mathfrak{cdQ}$	$\mathfrak{cdcQ}$	$\mathfrak{iQ}$	$\mathfrak{C}$	$\mathfrak{tC}$
zero-object	–	–	+	≥ 1	≥ 1	≥ 1	+	≥ 1	–
terminal object	+	$= 1$	+	+	≥ 1	≥ 1	+	+	+
product*	+	$= 1$	$\emptyset$	$= 0, = 1$	$= 0$	$= 1$	$\emptyset$	$= 1$	$= 1$
(m.) equalizer*	+	+	+	$= 0, = 1$	+	≤ 1	+	$= 1$	$= 1$
limits	+	$= 1$	$\emptyset$	$= 0, = 1$	$\emptyset$	$= 1$	$\emptyset$	$= 1$	$= 1$
(m.) pullback*	+	+	$\emptyset$	$= 0, = 1$	$= 0$	≤ 1	$\emptyset$	$= 1$	$= 1$
inverse limit*	+	+	+	+	+	+	+	+	+
initial object	+	≥ 1	+	$= 0, \geq 1$	+	≥ 1	+	≥ 1	≥ 1
coproduct*	+	$= 1$	≥ 1	$= 0, = 1$	$\leq 1, \geq 1$	$= 1$	≥ 1	$= 1$	$= 1$
(m.) coeqalizer*	+	≤ 1	$= 0$	≤ 1	$\emptyset$	$= 1$	$= 0$	≤ 1	≤ 1
colimits	+	$= 1$	$\emptyset$	$= 0, = 1$	$\emptyset$	$= 1$	$\emptyset$	$= 1$	$= 1$
(m.) pushout*	+	≤ 1	$\emptyset$	≤ 1	$\emptyset$	$= 1$	$\emptyset$	≤ 1	≤ 1
direct limit*	+	+	+	+	+	+	+	+	+

Table 2. Enlarged table on the existence of category theoretical constructions in nine "types of categories" of partial algebras

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Received October 23, 1995.