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DYNAMICS OF LINEAR MAPPING AND INVARIANT MEASURES ON SPHERE

1. Introduction

The description and investigation of measures in relation to a given mapping $\alpha : X \to X$ in a topological space is one of the classical problems in the theory of dynamical systems. In the case of a compact space X the existence and description of such measures has been obtained in the well-known work by N.M. Krilov and N.N. Bogolubov [7] (see also [8]) and the description of measures for other classes of spaces continues to attract attention [6]. As the general description of invariant measures is noneffective, the explicit form of invariant measures for concrete classes of mappings is of interest. In the present paper the measures on the sphere of the space C^m are explicitly described which are invariant and ergodic with respect to mappings of the form $a(x) = \frac{Ax}{\|Ax\|}$, where A is a nonsingular matrix.

The explicit form of measures being invariant with respect to such mappings is needed, e.g. for obtaining explicit conditions for functional differential operators of Fredholm type with partial derivatives and also for the explicit description of spectra of operators of weighted shift [1], [2].

It is known [8] that the structure of invariant ergodic measures is connected with the behaviour of the trajectories of points for iterations of the mapping α, i.e. with dynamics of this mapping.

In the case under consideration, the problem is connected with the dynamics description of a linear operator, i.e. with the behaviour of the sequence of vectors $A^n x$ for $n \to +\infty$. We notice that the information on the behaviour of such sequences is useful in other questions as well, e.g. in subtle analysis of iterative methods [5]. For some concrete forms of matrices A, the dynamics was known, and in the general case the fundamental complexity consisted in obtaining a visual description of all possible cases.

2. Preliminaries

A measure μ on X is said to be invariant with respect to a mapping $\alpha : X \to X$ if for any measurables set ω, the equality $\mu(\alpha^{-1}(\omega)) = \mu(\omega)$ holds.

A measure μ is said to be ergodic with respect to a mapping α if given a measurable subset $\omega \subset X$ invariant with respect to α, it follows that either $\mu(\omega) = 0$ or $\mu(X \setminus \omega) = 0$.

The measure μ is said to be normalized (or a probability measure) if $\mu(X) = 1$. Since every invariant measure on X can be expressed by means of ergodic measures [3], the description of ergodic measures becomes our fundamental problem.

Let

$$T^p = \{z = (z_1, \ldots, z_p) : z_j \in C,\ |z_j| = 1\}$$

be a p-dimensional torus. By a standard shift on the torus T^p generated by an element $w \in T^p$ we mean the mapping of the form $a_w(z) = wz = (w_1 z_1, \ldots, w_p z_p)$.

The invariant, ergodic measures on the torus T^p with respect to a standard shift depend on a closed subgroup W in T^p generated by the element $w \in T^p$. The form of this subgroup depends on the existence of relations, between numbers w_j. In fact, the description of such a subgroup is given in the work by H. Weyl [9]. Let $w_j = e^{i2\pi h_j}, j = 1, \ldots, p$. The numbers $\tau_1, \tau_2, \ldots, \tau_l$ are said to be rationally independend if the equality $q_1\tau_1 + q_2\tau_2 + \ldots + q_l\tau_l = 0$, $q_j \in Q$ is possible only in the case when $q_j = 0$, $j = 1, \ldots, l$. Let m_l be the largest number of rationally independent numbers among the numbers $1, h_1, h_2, \ldots, h_p$. Then the subgroup W is an m_1-dimensional submanifold which is homeomorfic either to torus T^{m_1} or to the product of T^{m_1} by the finite cyclic group Z_N, where the number N depends on the form of rational relations between the numbers $h_1, \ldots, h_p$. On the subgroup W there exists a unique normalized measure invariant with respect to the standard shifts from this subgroup which differs from m_1-dimensional Lebesgue measure only by the factor $\frac{1}{N}$. An invariant measure defined on every coset in the torus with respect to the subgroup W is called a standard invariant measure.

PROPOSITION 1. *A measure μ on the torus T^p is normalized, invariant and ergodic with respect to a standard shift $a_w(z) = wz$ if and only if when its support coincides with one of cosets with respect to the subgroup W and the measure μ coincides with the standard invariant measure on this coset.*

3. Invariant ergodic measures on the sphere

Let $X = \{x \in C^m : \|x\| = 1\}$ be the sphere of the space C^m, let A be a nonsingular matrix and let $\alpha : X \to X$ be a mapping defined by $\alpha(x) = \frac{Ax}{\|Ax\|}$.

We assume that the matrix A has the normal Jordan form. Let q denote the number of different moduli of eigenvalues of A and let these moduli be arranged in a decreasing manner $r_1 > r_2 > \ldots > r_q > 0$. Let $q(k)$ denote the number of Jordan cells whose moduli of their respective eigenvalues are equal to r_k. For each k let such cells be indexed in accordance with their decreasing dimension.

Thus, we obtain a collection of Jordan cells J_{kj}, $1 \le k \le q$, $1 \le j \le q(k)$. We denote the dimension of the cell J_{kj} by $v(k,j)$ and its corresponding eigenvalue by $\lambda(k,j)$. For vectors of the basis in which the matrix A has the Jordan form, we obtain the enumeration with three indices:

$$e(k,j,l),\ 1 \le k \le q,\ 1 \le j \le q(k),\ 1 \le l \le v(k,j).$$

The coordinates of the vector x in this basis we denote by $x(k,j,l)$.

Let L_k be the vector subspace of C^m generated by the eigenvectors of the matrix A corresponding to eigenvalues $\lambda(k,j)$, $1 \le j \le q(k)$. The set $S_k = X \cap L_k$ is the unit sphere of the subspace L_k, its real dimension being equal to $2q(k) - 1$. The sphere S_k undergoes stratification onto manifolds which are invariant with respect to α as follows. Let us consider the mapping from S_k onto $R^{q(k)}$ defined by

$$\pi_k(x) = (|x(k,1,1)|, |x(k,2,1)|, \ldots, |x(k,q(k),1)|).$$

The image of the sphere S_k under π_k is some subset of the $(q(k) - 1)$-dimensional sphere

$$B_k = \{\xi \in R^{q(k)} : \xi_j \ge 0,\ \|\xi\| = 1\}.$$

The inverse image $\pi_k^{-1}(\xi)$ of each point $\xi \in B_k$ posesses the natural structure of the torus whose dimension is equal to the number of non-zero coordinates of the point ξ. On the set S_k we define the mapping α by

$$\alpha(x) = (w(k,1)x(k,1,1), w(k,2)x(k,2,1), \ldots, w(k,q(k))x(k,q(k),1),$$

where

$$w(k,j) = \frac{\lambda(k,j)}{|\lambda(k,j)|}.$$

Therefore on the torus $T_{k,\xi} = \pi_k^{-1}(\xi)$ the action of the mapping α coincides with the standard shift generated by the element w consisting of those numbers $w(k,j)$ for which the corresponding coordinate of the vector ξ is different from zero.

Let $W_{k,\xi}$ denote a subgroup in $T_{k,\xi}$ generated by the element w.

THEOREM 1. *The normalized measure μ on the sphere $X = S^{2m-1}$ is invariant and ergodic with respect to the mapping $\alpha(x) = \frac{Ax}{\|Ax\|}$ if and only if its support belongs to one of invariant toruses $T_{k,\xi}$ and coincides with one of cosets of the torus $T_{k,\xi}$ with respect to the subgroup $W_{k,\xi}$ and the measure μ coincides with the standard invariant measure on this coset.*

4. The dynamics of a linear mapping

The proof of Theorem 1 is based on information about the behaviour of the sequence of vectors $A^n x$ when n increases. For description of this behaviour we introduce some auxiliary objects.

For a vector $x \in C^m, x \neq 0$ let us define the following integral characteristics:

$$k(x) = \min\{k : \exists x(k,j,l) \neq 0\},$$
$$v_j(x) = \max\{l : x(k(x),j,l) \neq 0\},$$
$$v(x) = \max\{v_1(x), \ldots, v^{(x)}_{q(k(x))}\}.$$

For a given vector z let us construct three new vectors, i.e. we define three mappings:

$$\varphi : x \to y, \ \psi : x \to z \text{ and } \eta : x \to u,$$

where

$$y(k,j,l) = \begin{cases} w(k,j)^{v(x)-1} x(k,j,v(x)), & k = k(x), \, l = 1; \\ 0, & k \neq k(x) \text{ or } l \neq 1; \end{cases}$$
$$z(k,j,l) = \begin{cases} x(k,j,l), & k = k(x); \\ 0, & k \neq k(x); \end{cases}$$
$$u = x - z.$$

The point $\widetilde{y} = \frac{\varphi(x)}{\|\varphi(x)\|}$ belongs to the sphere $S^{k(x)}$, the trajectory of this point belongs to the invariant torus $T_{k(x),\xi}$, where $\xi = \pi_k(\widetilde{y})$ and the closure of this trajectory coincides with one of the cosets in $T_{k(x),\xi}$ with respect to the subgroup $W_{k(x),\xi}$. Let us denote this coset by $\theta(x)$.

For two sequences (a_n) and (b_n) of real numbers we shall introduce the following notation

$$a_n \sim b_n \text{ iff } a_n = O(b_n) \text{ and } b_n = O(a_n),$$
$$a_n \approx b_n \text{ iff } \lim_{n\to\infty} \frac{a_n}{b_n} = 1.$$

A point x_0 is said to be a limit point of the trajectory of a point x if for every neighbourhood U of the point x_0 and every integer $n_0 > 1$ there exists an integer $n_1 > n_0$ such that $\alpha^{n_1}(x) \in U$. The set of limit points of the trajectory of the point x will be denoted by $\Omega(x)$.

THEOREM 2. (i) *The dynamics of the linear mapping* $x \to A^n x$, *when* $n \to \infty$ *is characterised by the following relationships*:

$$\|A^n x\| \approx \frac{r_{k(x)}^{-v(x)+1}\|y\|}{(v(x)-1)!} r_{k(x)}^n n^{v(x)-1}, \tag{1}$$

and in particular, $\|A^n x\| \sim r_1^n n^{v_1}$ *holds for vectors of general position.*

(ii) *The trajectory* $\alpha^n(x), n = 1, 2, \dots$ *of the point* x *tends to trajectory of the point* $\widetilde{y}$ *and that the limit set*

$\Omega(x)$ *coincides with the coset* $\theta(x)$ *in the torus* $T_{k(x),\xi}$, *where*, $\xi = \pi_{k(x)}(\widetilde{y})$. *As for the rate of convergence the following estimation holds*:

$$\|\alpha^n(x) - \alpha^n(\widetilde{y})\| \sim \begin{cases} 0, & x = y = z \\ \left[\dfrac{r_{k(u)}}{r_{k(x)}}\right] n^{v(u)-v(x)}, & x \neq z,\ y \neq z \\ \dfrac{1}{n}, & y \neq z. \end{cases} \tag{2}$$

Proof. As it is known [4], the n-th power J^n of the Jordan cell J of dimension v with the eigenvalue λ for $n > v$ is of the form

$$J^n = \begin{pmatrix} \lambda^n & n\lambda^{n-1} & C_n^2\lambda^{n-2} & \dots & C_n^{v-1}\lambda^{n-v+1} \\ 0 & \lambda^n & n\lambda^{n-1} & \dots & C_n^{v-2}\lambda^{n-v} \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & \lambda^n \end{pmatrix},$$

C_n^s $(s = 2, \dots, v-1)$ being binominal coefficients.

Therefore the coordinates of the vector $x^n = A^n x$ are expressed by the formula

$$x^{n(k,j,l)} = \sum_{i=l}^{v(k,j)} \lambda_{(k,j)}^{n-i+l} C_n^{i-l} x(k,j,i). \tag{3}$$

Since $C_n^{i-l} \approx \dfrac{n^{i-l}}{(i-l)!}$ so, for fixed j, k, l the dominating term in sum (3), i.e. a term having the greatest rate of increase or the least rate of decrease must be a nonzero term with the greatest index i. Therefore

$$|x^n(k,j,l)| \approx r_k^{n-p+l} n^{p-l} \frac{|x(k,j,p)|}{(p-l)!},$$

where

$$p = p(k,j) = \max\{i : x(k,j,i) \neq 0\}.$$

Among the coordinates corresponding to the Jordan cell $J(k,j)$, the coordinate $x^n(k,j,l)$, is a dominating one. Among all the coordinates of the vector x^n dominating coordinates are those for which $k = k(x)$, $l = 1$ and j is such that $v_j(x) = v(x)$. The dominating coordinates determine the behaviour of the norm $\|A^n x\|$. Therefore

$$\|A^n x\|^2 \approx \sum_j |x^n(k(x),j,1)|^2 \approx (r_{k(x)}^{n-v(x)} C_n^{v(x)-1})^2 \sum_j |x(k(x),j,v(x))|^2$$
$$= (r_{k(x)}^{n-v(x)} C_n^{v(x)-1})^2 \|y\|^2,$$

from which it follows (1).

Analysing the form of the remaining terms in (3) we assert that

$$\|A^n x\| = r_{k(x)}^{n-v(x)} C_n^{v(x)-1} \|y\|(1+\delta_n), \tag{4}$$

where $\delta_n = O(\frac{1}{n})$. In particular, it follows from (4) that

$$\|A^n x\|^{-1} = (r_{k(x)}^{n-v(x)} C_n^{v(x)-1} \|y\|)^{-1}(1+\gamma_n), \quad \text{where } \gamma_n = O\left(\frac{1}{n}\right). \tag{5}$$

Now we are able to obtain estimation (2).

As the vectors $A^n z$ and $A^n u$ belong to mutually complementing subspaces, we have

$$\left\| \frac{A^n x}{\|A^n x\|} - \frac{A^n y}{\|A^n y\|} \right\| \sim \left\| \frac{A^n z}{\|A^n x\|} - \frac{A^n y}{\|A^n y\|} \right\| + \frac{\|A^n u\|}{\|A^n x\|}.$$

Thus, in view of (1), we obtain

$$\frac{\|A^n u\|}{\|A^n x\|} \sim \left[\frac{r_{k(u)}}{r_{k(x)}}\right]^n \frac{n^{v(u)}}{n^{v(x)}}. \tag{6}$$

Now let us consider the vector

$$\zeta^n = \frac{A^n z}{\|A^n x\|} - \frac{A^n y}{\|A^n y\|}.$$

Let $y = z$. Noticing that $\|A^n y\| = r_{k(x)}^n \|y\|$, we get

$$\|\zeta^n\| = \|A^n y\| \left(\frac{1}{\|A^n x\|} - \frac{1}{\|A^n y\|} \right) = \frac{\|A^n y\| - \|A^n x\|}{\|A^n x\|} \sim \frac{\|A^n u\|}{\|A^n x\|}.$$

Thus in view of (6), estimation (2), for the case $y = z$, is obtained.

Now, let $y \neq z$, i.e. $v(x) > 1$. Since for the vector $y^n = A^n y$ we have $y^n(k(x),j,2) = 0$, so in view of (1) and (3), we assert that $\zeta^n(k(x),j,2) \sim \frac{1}{n}$ for those j which satisfy the equality $v_j(x) = v(x)$. Therefore the order of convergence of the norm $\|\zeta^n\|$ to zero cannot exceed $\frac{1}{n}$.

As for integers k, j and l for which the corresponding coordinate $y(k,j,l) = 0$, the inequality

$$\zeta^n(k,j,1) \leq \frac{c}{n} \quad (c = \text{const} > 0) \tag{7}$$

follows directly from (3).

We now show that estimation (7) also holds for the remaining coordinates, i.e. coordinates with indices (k, j, l), where $k = k(x), l = 1$ and j is such that $v_j(x) = v(x)$.

We have

$$
\begin{aligned}
(8) \quad |\zeta^n(k(x), j, 1)| \leq & \left| \sum_{i=1}^{v(x)-1} \lambda_{(k(x),j)}^{n-i+1} C_n^{i-1} x(k(x), j, i) \right| \\
& + \left| \frac{(\lambda(k(x), j))^{n-v(x)-1} C_n^{v(x)-1} x(k(x), j, v(x))}{\|A^n x\|} \right. \\
& \left. - \frac{\lambda(k(x), j)^n w(k(x), j)^{v(x)-1} x(k(x), j, v(x))}{\|A^n y\|} \right|.
\end{aligned}
$$

In view of (1) the first term in (8) is not greater than $\frac{c}{n}$, where $c > 0$ is some constant. Making use of (5), we estimate the second term in (8) as follows

$$
\begin{aligned}
& \left| \frac{\lambda(k(x), j)^{n-v(x)-1} C_n^{v(x)-1} x(k(x), j, v(x))}{v(x) - 1} \right. \\
& \qquad \left. - (\lambda(k(x), j))^n \frac{w(k(x), j)^{v(x)-1} x(k(x), j, v(x))}{\|A^n y\|} \right| \\
& = \gamma_n \frac{|x(k(x), j, v)(x)|}{\|y\|} = O\left(\frac{1}{n}\right).
\end{aligned}
$$

This completes the proof.

Theorem 2 includes, in particular, the conditions for convergence of the sequence $\left(\frac{A^n x}{\|A^n x\|}\right)$.

Indeed, this sequence converges if and only if, when $\widetilde{y}$ is a fixed point of the mapping α, and the latter holds if and only if, when all eigenvalues $\lambda(k(x), j)$ corresponding to those j for which $v_j(x) = v(x)$, are positive.

Proof of Theorem 1. The assertion of Theorem 1, with the aid of general considerations, follows from Theorem 2 and Proposition 1. Indeed, as it is known, the support of every ergodic probability measure is the limit set of some trajectory [8]. Thus, in the case under consideration according to Theorem 2, the limit set is contained in the torus $T_{k(x),\xi}$. Consequently, we obtain description of measures from Proposition 1. Thus Theorem 1 is proved.

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