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A NOTE ON AUTOMORPHISMS OF PRIME RINGS

We prove algebraic generalizations of some results of M. Awami and A. B. Thaheem concerning the equation $\alpha + \alpha^{-1} = \beta + \beta^{-1}$, where α, β are automorphisms of C^*-algebras. We show that if R is a prime ring of characteristic not equal to 2 and α, β are automorphisms of R satisfying $\alpha + \beta^{-1}\alpha^{-1}\beta = \beta + \beta^{-1}$, then $\alpha = \beta$ or $\alpha = \beta^{-1}$. We also prove a result on the commutativity of α and β.

1. Introduction and Preliminaries

Recently, Brešar [2, Corollary 3] proved the following:

THEOREM A. *Let R be a prime ring of characteristic not equal to 2. Suppose that the automorphisms α, β of R satisfy the equation*

$$(*) \qquad \alpha + \alpha^{-1} = \beta + \beta^{-1}.$$

If α and β commute then either $\alpha = \beta$ or $\alpha = \beta^{-1}$.

In [2], Brešar also proposed a problem as under what conditions the assumption that α and β commute can be removed from Theorem A.

In this note we provide (Proposition 2.1) an algebraic generalization of a result of Awami and Thaheem [1] proved for von Neumann algebras by considering the equation $\alpha + \beta^{-1}\alpha^{-1}\beta = \beta + \beta^{-1}$ on prime rings (of characteristic not equal to 2) and show that either $\alpha = \beta$ or $\alpha = \beta^{-1}$. This also generalizes Theorem A because if α, β commute then we get Theorem A as an immediate corollary.

We also consider here Brešar's problem of commutativity of α and β, in general, and provide (Proposition 2.3) a partial answer to this problem. We essentially show that if R is a unital ring with no nontrivial nilpotent elements and α, β are automorphisms of R satisfying the equation $(*)$ such that either α (or β) is inner, then α and β commute. Remark that similar to Proposition 2.1, this result also gives an algebraic generalization of a well-known result on the commutativity of α and β on C^*-algebras ([4]).

We now recall some preliminaries for the sake of completeness. Throughout R denotes an associative ring with unity 1. R is prime if $axb = 0$, for all $x \in R$, implies $a = 0$ or $b = 0$. For automorphisms α and β of R, an additive mapping α of R into itself is called an (α, β)-derivation if

$$d(xy) = \alpha(x)d(y) + d(x)\beta(y) \quad \text{for all } x, y \in R.$$

d is called an α-derivation if

$$d(xy) = \alpha(x)d(y) + d(x)y \quad \text{for all } x, y \in R.$$

Of course, derivations are (1,1)-derivations where 1 is an identity automorphism of R.

We shall need the following generalization of Posner's result on composition of derivations (see Brešar [2, Corollary 1]).

THEOREM B. *Let R be a prime ring of characteristic not equal to* 2, *d be an (α, β)-derivation of R, and g be a (γ, δ)-derivation of R. Suppose that g commutes with both γ and δ. If the composition dg is an $(\alpha, \gamma, \beta\delta)$-derivation, then either $d = 0$ or $g = 0$.*

The equation $\alpha + \alpha^{-1} = \beta + \beta^{-1}$ has been extensively studied for von Neumann algebras and C^*-algebras during the last decade or so. For more information concerning this equation on von Neumann algebras and C^*-algebras, we refer to [3] which also contains further references.

2. Results

We first prove a generalization of Theorem A and a result of Awami and Thaheem [1].

PROPOSITION 2.1. *Let R be a prime ring of characteristic not equal to 2. Let α, β be automorphisms of R satisfying the equation*

$$\alpha + \beta^{-1}\alpha^{-1}\beta = \beta + \beta^{-1}.$$

Then either $\alpha = \beta$ or $\alpha = \beta^{-1}$.

Proof. Put $d = (\beta\alpha - 1)$ and $g = (\alpha^{-1}\beta - 1)$. Then it is easy to verify that d is a $(\beta\alpha, 1)$-derivation and g is an $(\alpha^{-1}\beta, 1)$-derivation. g commutes with $\alpha^{-1}\beta$ and

$$\begin{aligned} dg &= (\beta\alpha - 1)(\alpha^{-1}\beta - 1) = \beta\alpha\alpha^{-1}\beta - \beta\alpha - \alpha^{-1}\beta + 1 \\ &= \beta[\beta - \alpha - \beta^{-1}\alpha^{-1}\beta + \beta^{-1}] = 0. \end{aligned}$$

Thus dg is obviously a $(\beta\alpha\alpha^{-1}\beta, 1)$-derivation, or in other words, dg is a $(\beta^2, 1)$-derivation. By Theorem B, $d = 0$ or $g = 0$. This shows that either $\alpha = \beta$ or $\alpha = \beta^{-1}$. This completes the proof.

We now come to the problem of the commutativity of automorphisms which is also an algebraic generalization of a result of Thaheem [4] for C^*-algebras. We first prove the following.

LEMMA 2.2. *Let α be an automorphism of a 2-torsion free ring R with no nontrivial nilpotent elements and b be an element of R such that*

$$\text{(i)} \qquad (\alpha+\alpha^{-1})(b) = 2b$$

and

$$\text{(ii)} \qquad (\alpha+\alpha^{-1})(b^2) = 2b^2.$$

Then $\alpha(b) = b$.

Proof. By equation (ii), $(\alpha-1)(b^2)+(\alpha^{-1}-1)(b^2) = 0$. Since $(\alpha-1)$ is an α-derivation and $(\alpha^{-1}-1)$ is an α^{-1}-derivation, therefore we get

$$\text{(iii)} \quad \alpha(b)(\alpha-1)(b)+(\alpha-1)(b)b+\alpha^{-1}(b)(\alpha^{-1}-1)(b)+(\alpha^{-1}-1)(b)b = 0.$$

It follows from (i) and (iii) that

$$\alpha(b)(\alpha-1)(b)+(\alpha-1)(b)b+\alpha^{-1}(b)(1-\alpha)(b)+(1-\alpha)(b)b = 0.$$

That is,

$$\alpha(b)(\alpha-1)(b)-\alpha^{-1}(b)(\alpha-1)(b) = 0$$

or

$$\text{(iv)} \qquad (\alpha(b)-\alpha^{-1}(b))(\alpha-1)(b) = 0.$$

By equation (i), $\alpha^{-1}(b) = 2b-\alpha(b)$ and substituting in (iv), we get

$$(\alpha(b)-2b+\alpha(b))(\alpha-1)(b) = 0$$

or

$$2((\alpha-1)(b))^2 = 0.$$

That R is 2-torsion free and R has no nontrivial nilpotent elements imply that $\alpha(b) = b$. This completes the proof.

PROPOSITION 2.3. *Let α,β be automorphisms of a 2-torsion free ring R with no nontrivial nilpotent elements such that*

$$\alpha(x)+\alpha^{-1}(x) = \beta(x)+\beta^{-1}(x) \quad \textit{for all } x\in R.$$

If β (or α) is inner then α,β commute.

Proof. Assume that β is inner, induced by an element $b\in R$. Then $\beta(x) = bxb^{-1}$ for all $x\in R$. In particular for $x = b$ and b^2, we have $\beta(b) = \beta^{-1}(b) = b$ and $\beta(b^2) = \beta^{-1}(b^2) = b^2$. Thus we have $\alpha(b)+\alpha^{-1}(b) = 2b$ and $\alpha(b^2)+\alpha^{-1}(b^2) = 2b^2$. By Lemma 2.2, $\alpha(b) = b$. Thus for all $x\in R$, we have

$$(\beta\alpha)(x) = \beta(\alpha(x)) = b\alpha(x)b^{-1} = \alpha(b)\alpha(x)\alpha(b^{-1}) = \alpha(bxb^{-1}) = (\alpha\beta)(x).$$

This proves that α, β commute.

We conclude the note with the following corollary which is an immediate consequence of Lemma 2.2.

COROLLARY 2.4. *Let α, β be automorphisms of a 2-torsion free ring R with no non-trivial nilpotent elements such that*

$$\alpha(x) + \alpha^{-1}(x) = \beta(x) + \beta^{-1}(x) \quad \text{for all } x \in R.$$

Then α and β have the same fixed points.

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