DEMONSTRATIO MATHEMATICA
Vol. XXIX No 4 1996

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MORE ON THE EQUATION $\nu * \rho * \mu = \rho$

Let G be an infinite (countable) group and e be its neutral element. By $P(G)$ we denote the convex set of all probability distributions on G. The convolution of two measures ν and μ is denoted by $\nu * \mu$. We recall

$$\nu * \mu(g) = \sum_{h \in G} \nu(gh^{-1})\mu(h) \qquad \Big(= \sum_{h \in G} \nu(h)\mu(h^{-1}g)\Big).$$

It is well known that $P(G)$ with the binary operation $*$ is an associative semigroup. In this paper we shall deal with convolution equations on $(P(G), *)$. Namely, given $\nu, \mu \in P(G)$ we consider when

$$(\diamond) \qquad \nu * \rho * \mu = \rho.$$

The case $\nu = \check{\mu}$ (where $\check{\mu}(g) = \mu(g^{-1})$) has been recently discussed in [1] and [2]. Here we study a nonsymmetric case. We show that $(\diamond)$ has a solution $\rho \in P(G)$ if and only if ν and μ are concentrated on mutually inverse elements of the quotient group by a finite subgroup. Our notation and definitions follow [1] and [2]. For the completeness of the paper we briefly recall the most important ones.

The support $S(\mu)$ of the measure $\mu \in P(G)$ is the set

$$\{g \in G : \mu(g) > 0\}.$$

A measure $\mu \in P(G)$ is said to be adapted if the smallest subgroup $G(\mu)$ containing $S(\mu)$ coincides with G. If μ is adapted then the smallest normal subgroup H of G such that $S(\mu)$ is contained in a coset of H is denoted by $\mathfrak{h}(\mu)$. It has been proved in [3] that the quotient group $G/\mathfrak{h}(\mu)$ is finite or isomorphic to $\mathbb{Z}$. In particular, if $\mathfrak{h}(\mu)$ is finite then

$$G/\mathfrak{h}(\mu) = \{g^j\mathfrak{h}(\mu) : j \in \mathbb{Z}, \quad g \in S(\mu)\}.$$

1991 *Mathematics Subject Classification*: 22D40, 43A05, 47A35, 60B15, 60J15.
Key words and phrases: convolution, concentration function, strict aperiodicity.
I thank the Foundation for Research Development for financial support.

Measures μ, for which $\mathfrak{h}(\mu)$ are finite, are characterized in [1] and [2]. Namely, an adapted μ has finite $\mathfrak{h}(\mu)$ if and only if the equation $\check{\mu} * \rho * \mu = \rho$ has a solution $\rho \in P(G)$, and only if the concentration function $\sup_{g \in G} \mu^{*n}(g)$ does not converge to 0.

It is convenient to look on a mapping $\tau \to \nu * \tau * \mu$ as a linear operator which is defined on $\ell^1(G)$. Therefore we introduce

$$_\nu P_\mu f(g) = \sum_{x,y \in G} f(xgy)\nu(x)\mu(y).$$

We notice that $_\nu P_\mu$ is the composition $_\nu P \circ P_\mu$, where

$$_\nu P f(g) = \sum_{x \in G} f(xg)\nu(x) \qquad \text{and} \qquad P_\mu f(g) = \sum_{y \in G} f(gy)\mu(y).$$

The operators $_\nu P$ and P_μ commute. It is easy to check that

$$_\nu P_\mu^n = {}_{\nu^{*n}} P_{\mu^{*n}} \qquad \text{and} \qquad {}_\nu P_\mu^* = {}_{\check{\nu}} P_{\check{\mu}},$$

where without any confusion $*$ will stand also for the adjoint operation. If ν and μ are probability measures then $_\nu P_\mu$ is a positive linear contraction on $\ell^1(G)$ as well as on $\ell^\infty(G)$. Clearly

$$_\nu P_\mu \mathbf{1} = {}_\nu P_\mu^* \mathbf{1} = \mathbf{1}$$

where $\mathbf{1}$ denotes the identity function (in particular $_\nu P_\mu$ is doubly stochastic). This implies that $_\nu P_\mu$ are positive linear contractions on each $\ell^p(G)$ where $1 \le p \le \infty$. Given $x, y \in G$, $_x\Phi_y$ is the mapping

$$G \ni g \to {}_x\Phi_y(g) = xgy \in G.$$

A subset $F \subseteq G$ is said to be $[\nu, \mu]$–invariant if

$$_x\Phi_y(F) = F \qquad \text{for all} \qquad x \in S(\nu) \qquad \text{and} \qquad y \in S(\mu).$$

We note that F is $[\nu, \mu]$–invariant if and only if it is invariant under actions of the group $\mathbf{\Phi}$ of all bijections generated by the set

$$\{{}_x\Phi_y : x \in S(\nu), \quad y \in S(\mu)\}.$$

Obviously the whole group G may be decomposed onto pairwise disjoint minimal $[\nu, \mu]$–invariant sets. Moreover each minimal set F is of the form

$$F = \{\Phi(g) : \Phi \in \mathbf{\Phi}\}$$

where $g \in F$ is arbitrary.

Given adapted $\nu, \mu \in P(G)$ by $\mathfrak{h}(\nu, \mu)$ we denote the smallest normal subgroup H of G such that $S(\nu)$ and $S(\mu)$ are contained in cosets of H.

Now we are in a position to formulate the main result of the paper.

THEOREM. *Let ν, μ be adapted probability measures on a countable group G. Then the following conditions are equivalent:*

(a) *there exist a finite set* $K \subseteq G$ *and a sequence* $g_n \in G$ *so that for some* $\varepsilon > 0$ *we have*

$$\mu^{*n}(g_n K) \geq \varepsilon \qquad \text{and} \qquad \check{\nu}^{*n}(g_n K) \geq \varepsilon,$$

(b) *there exists a probability measure (with finite support)* ρ *so that*

$$\nu * \rho * \mu = \rho,$$

(c) *the subgroups* $\mathfrak{h}(\nu, \mu)$, $\mathfrak{h}(\nu)$, $\mathfrak{h}(\mu)$ *are finite and equal, and*

$$S(\mu) \cup S(\check{\nu}) \subseteq g\mathfrak{h}(\mu)$$

where $g \in S(\mu)$ *arbitrary.*

Proof. (a) $\Rightarrow$ (b). Let us consider ${}_\nu P_\mu$ to be a positive linear contraction on $\ell^2(G)$. By the von Neumann Mean Ergodic Theorem the Cesaro means $\frac{1}{N} \sum_{n=0}^{N-1} {}_\nu P_\mu^n$ are convergent in the strong operator topology to a projection onto the space of all ${}_\nu P_\mu$–invariant functions. Let f be the characteristic function of the set $K^{-1}K$. Then for each N we have

$$\frac{1}{N} \sum_{n=0}^{N-1} {}_\nu P_\mu^n f(e) = \frac{1}{N} \sum_{n=0}^{N-1} \sum_{x,y} 1_{K^{-1}K}(xy) \nu^{*n}(x) \mu^{*n}(y) \geq \varepsilon^2.$$

This implies that there exists a nonzero ${}_\nu P_\mu$–invariant function $f_* \in \ell^2(G)$ (nonnegative). Obviously it is also invariant for the adjoint operator. Hence

$${}_\nu P_\mu^* f_* = {}_{\check{\nu}} P_{\check{\mu}} f_* = f_*.$$

From the uniform convexity of $\ell^2(G)$ we get

$${}_{\delta_x} P_{\delta_y} f_* = f_*$$

for all pairs x, y satisfying

$$x \in S(\nu^{*n}), \quad y \in S(\mu^{*n}) \qquad \text{or} \qquad x \in S(\check{\nu}^{*n}) \quad \text{and} \quad y \in S(\check{\mu}^{*n}).$$

As a result $f_* \circ \Phi = f_*$ for any $\Phi \in \mathbf{\Phi}$. Since $f_*(e) > 0$, thus the minimal set F containing e (denoted by F_e) must be finite. All the mappings ${}_x\Phi_y$ are 1–1, so we get

$${}_x\Phi_y(F_e) = F_e \qquad \text{for all} \qquad x \in S(\check{\nu}^{*n}) \quad \text{and} \quad y \in S(\check{\mu}^{*n}).$$

As a result the probabilities

$$\nu^{*n} * \mu^{*n} = \nu^{*n} * \delta_e * \mu^{*n}$$

are concentrated on F_e. The Cesaro means $\frac{1}{N} \sum_{n=0}^{n-1} \nu^{*n} * \mu^{*n}$ converge to the probability measure

$$\rho = \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} {}_\nu P_\mu^n \delta_e \qquad (\text{with} \quad S(\rho) \subseteq F_e),$$

where ${}_\nu P_\mu$ is considered as an operator on $\ell^1(G)$. We notice that ρ is ${}_\nu P_\mu$–invariant, what is equivalent to

$$\nu * \rho * \mu = \rho.$$

(b) $\Rightarrow$ (c)

Let $F_{\max}$ denote the set (finite)

$$\{g : \rho(g) = \max_{\tilde{g}\in G} \rho(\tilde{g})\}.$$

Then for each $g \in F_{\max}$ we get

$$\rho(g) = \nu * \rho * \mu(g) = \sum_{x,y} \rho(x^{-1}gy^{-1})\nu(x)\mu(y),$$

which implies that $F_{\max}$ is $[\nu,\mu]$–invariant. In particular

$$S(\nu^{*n})F_{\max}S(\mu^{*n}) = F_{\max}.$$

From this we easily get

$$\lim_{n\to\infty} \#S(\nu^{*n}) \le \#F_{\max} \qquad \text{and} \qquad \lim_{n\to\infty} \#S(\mu^{*n}) \le \#F_{\max}.$$

Now by [1] (or [2]) both groups $\mathfrak{h}(\nu)$ and $\mathfrak{h}(\mu)$ are finite. Since

$$\mathfrak{h}(\nu) = S(\check{\nu}^{*n} * \nu^{*n}) \qquad \text{and} \qquad \mathfrak{h}(\mu) = S(\mu^{*n} * \check{\mu}^{*n})$$

for n large enough (see [2] for all details) we obtain

$$\begin{aligned} F_{\max} &= S(\check{\nu}^{*n})S(\nu^{*n})F_{\max}S(\mu^{*n})S(\check{\mu}^{*n}) \\ &= \mathfrak{h}(\nu)F_{\max}\mathfrak{h}(\mu). \end{aligned}$$

In particular $F_{\max}$ is a finite union of cosets of $\mathfrak{h}(\nu)$ and $\mathfrak{h}(\mu)$, and $\tilde{h}F_{\max} = F_{\max}$ for any $\tilde{h} \in \mathfrak{h}(\nu)$.

Now we consider the quotient group $G/\mathfrak{h}(\mu)$, which is isomorphic to $\mathbb{Z}$ (see [3]). For any $g \in S(\mu)$ we have

$$G = \bigcup_{j=-\infty}^{+\infty} g^j \mathfrak{h}(\mu).$$

Hence, for some integer $j_1 < j_2 < \cdots < j_k$ we get

$$F_{\max} = g^{j_1}\mathfrak{h}(\mu) \cup \cdots \cup g^{j_k}\mathfrak{h}(\mu).$$

Suppose $\mathfrak{h}(\nu) \not\subseteq \mathfrak{h}(\mu)$. Then $g^j h \in \mathfrak{h}(\nu)$ for some $j \neq 0$ and $h \in \mathfrak{h}(\mu)$. We get

$$g^j h \bigcup_{l=1}^{k} g^{j_l}\mathfrak{h}(\mu) = \bigcup_{l=1}^{k} g^{j+j_l}\mathfrak{h}(\mu) \neq \bigcup_{l=1}^{k} g^{j_l}\mathfrak{h}(\mu)$$

what contradicts the invariance of $F_{\max}$.

Similarly we may obtain $\mathfrak{h}(\mu) \subseteq \mathfrak{h}(\nu)$. Hence the subgroups $\mathfrak{h}(\mu)$ and $\mathfrak{h}(\nu)$ are equal. Obviously they coincide with $\mathfrak{h}(\nu, \mu)$. Let $g \in S(\mu)$ and $g_{-1} \in S(\nu)$. We have the representation

$$g_{-1} = g^j h \qquad \text{where} \qquad h \in \mathfrak{h}(\mu) \quad \text{and} \quad j \in \mathbb{Z}.$$

For suitable $h_n, \tilde{h}_n \in \mathfrak{h}(\mu)$ we have

$$F_{\max} = g_{-1}^n F_{\max} g^n = g^{nj} h_n \left(\bigcup_{l=1}^{k} g^{j_l}\mathfrak{h}(\mu) \right) g^n =$$

$$= \tilde{h}_n \bigcup_{l=1}^{k} g^{j_l}\mathfrak{h}(\mu) g^{n(j+1)} = F_{\max} g^{n(j+1)}.$$

Since $F_{\max}$ is a finite union of classes of $\mathfrak{h}(\mu)$ it is possible only if $j = -1$, and we finally get

$$S(\nu) \subseteq g^{-1}\mathfrak{h}(\mu).$$

(c) $\Rightarrow$ (a)

We have $S(\nu^{*n}) \subseteq g^{-n}\mathfrak{h}(\mu)$ where $g \in S(\mu)$ arbitrary. Hence

$$\mu^{*n}(g^n \mathfrak{h}(\mu)) = \check{\nu}^{*n}(g^n \mathfrak{h}(\mu)) \equiv 1 \qquad \text{for all } n,$$

and the proof of theorem is complete.

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Received August 30, 1995.