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ENVELOPES OF SPECIAL CLASS OF ONE-PARAMETER FAMILIES OF CURVES

Introduction

In our recent paper [6] concernig some relations between the logarithmic and arithmetic means we have obtained a family of solutions ϕ_α of a relevant one-parameter system of functional equations which are of the form

$$\phi_\alpha(x) = g\left(\tfrac{x}{\alpha}\right) + g(\alpha), \qquad x > 0,$$

where $\alpha > 0$ is the parameter, and g is a particular solution of the functional equation corresponding to the parameter $\alpha = 1$. Since g completely describes the one-parameter family of solutions $(\phi_\alpha)_{\alpha>0}$, we call g to be its *generator*. Being interested in the mutual dependence of the position of graphs of the family ϕ_α on the parameter α, it is natural to ask what is the envelope E_g of the family of curves

$$\mathcal{G}(g) := \{\text{graph}(\phi_\alpha) : \alpha > 0\}.$$

This question, in the context of the family $\mathcal{G}(g)$, appears to be interesting. There are some relationships between the envelope E_g of the logarithmic functions.

We also show that there are structural similarities between behaviour of the above mentioned class of curves $\mathcal{G}(g)$ and its envelope E_g, and the classes of three families of curves being the graphs of the functions

$$\begin{aligned}
\phi_\alpha(x) &= g(x-\alpha) + g(\alpha), \qquad x, \alpha \in \mathbb{R},\\
\phi_\alpha(x) &= g(x-\alpha)g(\alpha), \qquad x, \alpha \in \mathbb{R},\\
\phi_\alpha(x) &= g\left(\tfrac{x}{\alpha}\right)g(\alpha), \qquad x > 0,
\end{aligned}$$

and their envelopes, where g is appropriate defined. Some connections, respectively, with linear, exponential, and power functions, will be exhibited.

1. Envelopes for families $\mathcal{G}$(g) of logarithmic type

By $\mathbb{R}$ we denote the set of reals.

For an arbitrary function $g : (0,\infty) \to \mathbb{R}$ define the one-parameter family of functions $\phi_\alpha : (0,\infty) \to \mathbb{R}$ by

$$\phi_\alpha(x) := g\left(\tfrac{x}{\alpha}\right) + g(\alpha), \qquad x, \alpha > 0,$$

of the generator g; by $\mathcal{G}(g)$ denote the family of curves being the graphs of ϕ_α, $\alpha > 0$, and by E_g, the envelope of the family $\mathcal{G}(g)$ (provided it exists).

We often identify a function and its graph. Therefore we write down the envelope E_g in the form $y = E_g(x)$, $x > 0$, when it is possible and convenient.

Remark. 1.1. If $g(x) = c\log x + g(1)$, $x > 0$, where c and $g(1)$ are arbitrary real constant, then $\mathcal{G}(g) = \{g\}$ is a singleton, and E_g, the envelope of g, obviously, coincides with the graph of g.

It turns out that the converse implication holds true:

PROPOSITION 1.1. *Let $g : (0,\infty) \to \mathbb{R}$ be an arbitrary function. Then $\mathcal{G}(g)$ is a singleton if, and only if, the function g satisfies the functional equation*

$$g(xy) + g(1) = g(x) + g(y), \qquad x, y > 0.$$

If moreover g is continuous at least at one point, then there exists a constant $c \in \mathbb{R}$ such that $g(x) = c\log x + g(1)$, $x > 0$.

Proof. The family $\mathcal{G}(g)$ is a singleton if, and only if,

$$g\left(\tfrac{x}{\alpha}\right) + g(\alpha) = g\left(\tfrac{x}{\beta}\right) + g(\beta)$$

for all $x, \alpha, \beta > 0$. Setting $x = \beta$ gives

$$g(\beta/\alpha) + g(\alpha) = g(1) + g(\beta), \qquad \alpha, \beta > 0.$$

Hence, for $\psi : \mathbb{R}_+ \to \mathbb{R}$ defined by the formula

$$\psi(\alpha) := g(\alpha) - g(1), \qquad \alpha > 0,$$

one gets $\psi(1) = 0$ and

$$(1) \qquad \psi\left(\tfrac{\beta}{\alpha}\right) + \psi(\alpha) = \psi(\beta), \qquad \alpha, \beta > 0.$$

Hence, setting $\beta = 1$ in this equation we obtain

$$\psi\left(\tfrac{1}{\alpha}\right) = -\psi(\alpha), \qquad \alpha > 0.$$

Now replacing α by α^{-1} in (1) gives

$$(2) \qquad \psi(\alpha\beta) = \psi(\alpha) + \psi(\beta), \qquad \alpha, \beta > 0,$$

which means that

$$g(\alpha\beta) + g(1) = g(\alpha) + g(\beta), \qquad \alpha, \beta > 0.$$

The converse implication is obvious. Since ψ is a solution of the logarithmic Cauchy functional equation (2), the remainig statement is a well known fact (cf. for instance Aczél [1], p. 41). This completes the proof.

Remark 1.2. Note that the continuity of g at least at one point can replaced by the measurability of g, or by the boundedness above (or below) in a neighbourhood of a point (cf. for instance Kuczma [4], p. 218).

The main result of this section reads as follows:

THEOREM 1.1. *Let $g:(0,\infty)\to\mathbb{R}$ be a differentiable function. Then the graph of the function*

$$(0,\infty)\ni x\to 2g(\sqrt{x}), \tag{3}$$

is contained in the envelope of the family $\mathcal{G}(g)$. If the function

$$(0,\infty)\ni x\to g'(x)x \quad \textit{is one-to-one}, \tag{4}$$

then the envelope E_g has the representation $y = Eg(x) = 2g(\sqrt{x})$, $x > 0$.

Proof. According to a classical method (cf. for instance Favard [2], Chapter III), to find the envelope of the family of curves $\mathcal{G}(g)$ it is enough to eliminate the parameter α from the system of equations

$$y = g(\alpha^{-1}x)+g(\alpha), \quad g'(\alpha^{-1}x)(-\alpha^{-2}x)+g'(\alpha)=0, \qquad x,\alpha>0, y\in\mathbb{R}.$$

The second equation can be written in the following equivalent form

$$g'(\alpha^{-1}x)(\alpha^{-1}x) = g'(\alpha)\alpha, \qquad x,\alpha>0,$$

If the function $(0,\infty)\ni x\to g'(x)x$ is one-to-one, it follows that $\alpha^{-1}x = \alpha$, and consequently, $\alpha=\sqrt{x}$, $x>0$. Setting $\alpha=\sqrt{x}$ into the first of the equations we get the function

$$y = 2g(\sqrt{x}), \qquad x>0. \tag{5}$$

the graph of which is the envelope of the considered family of curves.

If the function $(0,\infty)\ni x\to g'(x)x$ is not one-to-one, then, obviously, every point of the graph of the function (3), is a point of the envelope. This completes the proof.

Remark 1.3. If $g(x) = c\log x + g(1)$ $x>0$, then $g'(x)x = c$ for all $x>0$, i.e. the function (4) is constant. In particular it is not one-to-one. But of course (cf. Remark 1), E_g even coincides with the graph of g.

Remark 1.4. Denote by $\mathcal{F}((0,\infty),\mathbb{R})$ the set of all functions $\psi:(0,\infty)\to\mathbb{R}$. For a given function $F:\mathbb{R}\to\mathbb{R}$ define an operator $T:\mathcal{F}((0,\infty),\mathbb{R})\to\mathcal{F}((0,\infty),\mathbb{R})$ by the formula $T(\psi) := F\circ\psi$. Let $\mathcal{G}(g)$ and $\mathcal{G}(h)$ be the suitable families of curves of continuous generators g and h. Note that $T(\mathcal{G}(g))\subseteq$

$\mathcal{G}(h)$ if, and only if, there exists a function $\beta : (0,\infty) \to (0,\infty)$ such that F, g and h satisfy the functional equation

$$F\,(\,g(x/\alpha) + g(\alpha)\,) = h(x/\beta(\alpha)) + h(\beta(\alpha)), \qquad x, \alpha > 0.$$

Assuming that $g : (0,\infty) \to \mathbb{R}$ is bijective and $\beta(\alpha) = \alpha$ for all $\alpha > 0$, we shall prove that $T\,(\,\mathcal{G}(g)\,) \subseteq \mathcal{G}(h)$ iff T is affine. In fact, as

$$F\,(\,g(x/\alpha) + g(\alpha)\,) = h(x/\alpha) + h(\alpha), \qquad x, \alpha > 0,$$

setting $x := \alpha^2$ gives $F\,(\,2g(\alpha)\,) = 2h(\alpha)$ for all $\alpha > 0$. It follows that

$$F(x) = 2h \circ g^{-1}(x/2), \qquad x \in \mathbb{R}.$$

Substituting this function into the previous relation we get

$$2h \circ g^{-1}\left(\frac{g(x/\alpha) + g(\alpha)}{2}\right) = h(x/\alpha) + h(\alpha), \qquad x, \alpha > 0.$$

Replacing α by y, and x by xy, gives

$$2h \circ g^{-1}\left(\frac{g(x) + g(y)}{2}\right) = h(x) + h(y), \qquad x, y > 0.$$

Replacing here x by $g^{-1}(x)$, and y by $g^{-1}(y), x, y \in \mathbb{R}$, we obtain

$$h \circ g^{-1}\left(\frac{x+y}{2}\right) = \frac{h \circ g^{-1}(x) + h \circ g^{-1}(y)}{2}, \qquad x, y \in \mathbb{R}.$$

It follows that there are $a, b \in \mathbb{R}$ such that $h \circ g^{-1}(x) = ax + b$ for all $x \in \mathbb{R}$ (cf. for instance Aczél [1], p. 43), and consequently

$$F(x) = 2h \circ g^{-1}(x/2) = 2(a(x/2) + b) = ax + 2b, \qquad x \in \mathbb{R},$$

which was to be shown.

Remark 1.5. Note that an element ϕ_α of the family $\mathcal{G}(g)$ coincides with the generator g if, and only if, there is an $\alpha > 0$ such that g satisfies the functional equation

$$g(x) = \phi_\alpha(x) = g(x/\alpha) + g(\alpha), \qquad x > 0.$$

In particular, if the generator g is strictly increasing and $g(1) = 0$, then ϕ_1=g.

Remark 1.6. In general, no member of the family $\mathcal{G}(g)$ will coincide with the envelope E_g. If it is the case, then there exists an $\alpha_0 > 0$ such that g satisfies the functional equation

$$2g(\sqrt{x}) = g(x/\alpha_0) + g(\alpha_0), \qquad x > 0.$$

We shall prove that if g is differentiable at the point $x = \alpha_0$, and satisfies this equation, then there is $ac \in \mathbb{R}$ such that

$$g(x) = c \log x + g(\alpha_0), \qquad x > 0. \tag{6}$$

Replacing x by $\alpha_0^2 x$ we get

$$2g(\alpha_0\sqrt{x}) = g(\alpha_0 x) + g(\alpha_0), \qquad x > 0.$$

Now it is easy to see that the function $\phi : (0,\infty) \to \mathbb{R}$,

$$\phi(x) := g(\alpha_0 x) - g(\alpha_0), \qquad x > 0,$$

satisfies the functional equation

$$2\phi(\sqrt{x}) = \phi(x), \qquad x > 0,$$

and ϕ is differentiable at the point $x = 1$. According to Fubini's result [3] (cf. also [5], p.394), there exists a constant $c \in \mathbb{R}$ such that

$$\phi(x) = c \log x, \qquad x > 0.$$

Hence we get the formula (6).

Thus, under the weak and natural assumption of the differentiability of the function g, the envelope E_g is a member of the family $\mathcal{G}(g)$ if, and only if, $\mathcal{G}(g)$ is a singleton with $g = \log$.

Geometrical comments 1.1. Let $g : (0,\infty) \to \mathbb{R}$ be a differentiable function, and suppose that the function $x \to g'(x)x$ is one-to-one in $\mathbb{R}_+$. Then E_g coincides with the graph of the function $x \to 2g(\sqrt{x})$, so we can write $E_g(x) = 2g(\sqrt{x})$, $x > 0$. Moreover, for every fixed $\alpha > 0$,

$$y = g\left(\tfrac{x}{\alpha}\right) + g(\alpha), \qquad x > 0,$$

the curve $\phi_\alpha \in \mathcal{G}(g)$, touches the envelope E_g at the point $(\alpha^2, 2g(\alpha))$. At this point of contact of the curves ϕ_α and E_g, the common tangent has the slope

$$\phi'(\alpha^2) = E_g'(\alpha^2) = g'(\alpha)/\alpha.$$

EXAMPLE 1. For the generator $g(x) = x$, $x > 0$, we get the family $\mathcal{G}(g)$ of functions ϕ_α,

$$\phi_\alpha(x) = \frac{x}{\alpha} + \alpha, \qquad x > 0.$$

Applying the above commentaries, we get the envelope E_g:

$$E_g(x) = 2\sqrt{x}, \qquad x > 0,$$

points of contact: $(\alpha^2, 2\alpha)$; slope of common tangent: $1/\alpha$.

2. Envelopes for families $\mathcal{G}$(g) of affine type

Analogously as in the previous section, for an arbitrary generator function $g : \mathbb{R} \to \mathbb{R}$ define the one-parameter family of functions $\phi_\alpha : \mathbb{R} \to \mathbb{R}$ by

$$\phi_\alpha(x) := g(x - \alpha) + g(\alpha), \qquad x, \alpha \in \mathbb{R},$$

and introduce the same notations: $\mathcal{G}(g)$ and E_g.

Remark 2.1. If $g(x) = cx + g(0)$, $x \in \mathbb{R}$, where c and $g(0)$ are arbitrary real constants, then $\mathcal{G}(g) = \{g\}$ is a singleton, and E_g, the envelope of g, coincides with the graph of g.

In an analogous way as Proposition 1.1 we can prove:

PROPOSITION 2.1. *Let $g : \mathbb{R} \to \mathbb{R}$ be an arbitrary function. Then $\mathcal{G}(g)$ is a singleton if, and only if, the function g satisfies the functional equation*

$$g(x + y) + g(0) = g(x) + g(y), \qquad x, y \in \mathbb{R}.$$

If moreover g is continuous at least at one point, then there exists a constant $c \in \mathbb{R}$ such that $g(x) = cx + g(0)$, $x \in \mathbb{R}$.

THEOREM 2.1. *Let $g : \mathbb{R} \to \mathbb{R}$ be a differentiable function. Then the graph of the function $\mathbb{R} \ni x \to 2g(\frac{x}{2})$, is contained in the envelope of the family $\mathcal{G}(g)$. If the function $\mathbb{R} \ni x \to g'(x)$ is one-to-one, then $y = E_g(x) = 2g(\frac{x}{2})$, $x \in \mathbb{R}$.*

3. Envelopes for families $\mathcal{G}(g)$ of exponential type

Suppose that $g : \mathbb{R} \to (0, \infty)$ is a generator of the one-parameter family of functions $\phi_\alpha : \mathbb{R} \to (0, \infty)$:

$$\phi_\alpha(x) := g(x - \alpha)g(\alpha), \qquad x, \alpha \in \mathbb{R},$$

and let $\mathcal{G}(g)$ and E_g be defined correspondingly.

Remark 3.1. If $g(x) = g(0)e^{cx}$, $x \in \mathbb{R}$, where $c \in \mathbb{R}$, and $g(0) > 0$, are arbitrary constants, then $\mathcal{G}(g) = \{g\}$ is a singleton, and E_g, the envelope of g, coincides with the graph of g.

PROPOSITION 3.1. *Let $g : \mathbb{R} \to (0, \infty)$ be an arbitrary function. Then $\mathcal{G}(g)$ is a singleton if, and only if, the function g satisfies the functional equation*

$$g(0)g(x + y) = g(x)g(y), \qquad x, y \in \mathbb{R}.$$

If moreover g is continuous at least at one point, then there exists a constant $c \in \mathbb{R}$ such that $g(x) = g(0)e^{cx}$, $x \in \mathbb{R}$.

THEOREM 3.1. *Let $g : \mathbb{R} \to (0,\infty)$ be a differentiable function. Then the graph of the function*

$$\mathbb{R} \ni x \to \left[g\left(\tfrac{x}{2}\right)\right]^2$$

is contained in the envelope of the family $\mathcal{G}(g)$. If the function g'/g is one-to-one, then the envelope E_g has the representation

$$y = E_g(x) = \left[g\left(\tfrac{x}{2}\right)\right]^2, \qquad x > 0.$$

4. Envelopes for families $\mathcal{G}(g)$ of power type

Suppose that $g : (0,\infty) \to (0,\infty)$ is a generator of the one-parameter family of functions $\phi_\alpha : (0,\infty) \to (0,\infty)$:

$$\phi_\alpha(x) := g\left(\tfrac{x}{\alpha}\right) g(\alpha), \qquad x, \alpha > 0,$$

and let $\mathcal{G}(g)$ and E_g be defined correspondingly.

Remark 4.1. If $g(x) = g(1)x^c$, $x > 0$, where $c \in \mathbb{R}$, and $g(1) > 0$, are arbitrary constants, then $\mathcal{G}(g) = \{g\}$ is a singleton, and E_g, the envelope of g, coincides with the graph of g.

PROPOSITION 4.1. *Let $g : (0,\infty) \to (0,\infty)$ be an arbitrary function. Then $\mathcal{G}(g)$ is a singleton if, and only if, the function g satisfies the functional equation*

$$g(1)g(xy) = g(x)g(y), \qquad x, y > 0.$$

If moreover g is continuous at least at one point, then there exists a constant $c \in \mathbb{R}$ such that $g(x) = g(1)x^c$, $x > 0$.

THEOREM 4.1. *Let $g : (0,\infty) \to (0,\infty)$ be a differentiable function. Then the graph of the function*

$$(0,\infty) \ni x \to \left[g(\sqrt{x})\right]^2$$

is contained in the envelope E_g of the family $\mathcal{G}(g)$. If the function

$$(0,\infty) \ni x \to \frac{g'(x)}{g(x)} x$$

is one-to-one, then the envelope curve has the representation

$$y = E_g(x) = \left[g(\sqrt{x})\right]^2, \qquad x > 0.$$

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