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THE GENERAL SOLUTION OF THE EQUATION $\varphi(\varphi(\varphi(\alpha,x),y),z)=\varphi(\alpha,x\cdot\mu(y)\cdot z)$

Let $n\geq 2$ be a natural number, Γ an arbitrary set and $(G,[\ldots])$ denote an arbitrary n-adic group (see [5]). By a result of Hosszu [1], the n-group operation in G can be expressed by

$$[x_1,x_2,\ldots,x_n]=x_1\cdot\mu(x_2)\cdot\mu^2(x_3)\cdot\ldots\cdot\mu^{n-1}(x_n)\cdot a,$$

where „$\cdot$" is a binary group operation on G and μ is an automorphism of $(G,\cdot)$; $a\in G,\mu(a)=a$ and $\mu^{n-1}(x)=a\cdot x\cdot a^{-1}$ (μ^v denotes the v-th iteration). Therefore the translation equation on the n-group G

$$(1)\qquad \varphi(\varphi(\varphi(\ldots\varphi(\varphi(\alpha,x_1),x_2),\ldots),x_{n-1}),x_n)=\varphi(\alpha,[x_1,x_2,\ldots,x_n]),$$

has the equivalent form

$$(1')\qquad \varphi(\varphi(\varphi(\ldots\varphi(\varphi(\alpha,x_1),x_2),\ldots),x_{n-1}),x_n)$$
$$=\varphi(\alpha,x_1\cdot\mu(x_2)\cdot\mu^2(x_3)\cdot\ldots\cdot\mu^{n-2}(x_{n-1})\cdot a\cdot x_n).$$

The characterization of all solutions of the equation ($1'$) by the solutions of the translation equation

$$(2)\qquad F(F(\alpha,x),y)=F(\alpha,x\cdot y)$$

is given in [2].

The paper [3] contains the construction of all solutions of the equation ($1'$) under the additional suppositions: $a=e$, where e is the unit element of $(G,\cdot)$, and μ is the identity on G.

In this paper we solve the problem of solutions of the equation

$$(3)\qquad \varphi(\varphi(\varphi(\alpha,x),y),z)=\varphi(\alpha,x\cdot\mu(y)\cdot z),$$

which is the special case of equation (1') for $n=3$ and $a=e$. In this case $\mu^2(x)=x$ for every $x\in G$.

First we will present two theorems giving a characterization of solutions of the equation (3) by the solutions of the translation equation (2). The above mentioned theorems are corollaries of results from the paper [2], but we will present them with proofs for a comfort of the readers.

Remark 1. If $\varphi : \Gamma \times G \to \Gamma$ is a solution of equation (3), then the function $f(\alpha) := \varphi(\alpha, e)$ satisfies equality

$$f^3 = f. \tag{4}$$

Indeed, by equation (3), we have $f^3(\alpha) = f(f(f(\alpha))) = \varphi(\varphi(\varphi(\alpha, e), e), e) = \varphi(\alpha, e) = f(\alpha)$.

Let us define $F : \Gamma \times G \to \Gamma$ as follows

$$F(\alpha, x) := \varphi[f(\alpha), \mu(x)]. \tag{5}$$

THEOREM 1. *If $\varphi : \Gamma \times G \to \Gamma$ is a solution of the equation (3) and $f(\alpha) := \varphi(\alpha, e)$, then the function $F : \Gamma \times G \to \Gamma$ defined by (5) is a solution of the translation equation (2) satisfying equalities*

$$F(\alpha, e) = f^2(\alpha), \tag{6}$$

$$f(F(\alpha, x)) = F(f(\alpha), \mu(x)), \tag{7}$$

$$\varphi(\alpha, x) = f(F(\alpha, x)). \tag{8}$$

Proof. Since $F(\alpha, e) = \varphi(f(\alpha), e) = f^2(\alpha)$, then (6) holds. Moreover, by (3) and (5), we have

$$\begin{aligned} F(\alpha, x \cdot y) &= \varphi[f(\alpha), \mu(x \cdot y)] \quad \text{and} \\ F(F(\alpha, x), y) &= F(\varphi[f(\alpha), \mu(x)], y) = \varphi[f(\varphi[f(\alpha), \mu(x)]), \mu(y)] \\ &= \varphi[f(\alpha), \mu(x) \cdot \mu(y)] \end{aligned}$$

which means that F satisfies (2). Further, we have

$$\begin{aligned} F(f(\alpha), \mu(x)) &= \varphi[f^2(\alpha), \mu^2(x)] = \varphi[f^2(\alpha), x] \\ &= \varphi(\varphi(\varphi(\alpha, e), e), x) = \varphi(\alpha, x), \\ f(F(\alpha, x)) &= f(\varphi[f(\alpha), \mu(x)]) = \varphi(\varphi(\varphi(\alpha, e), \mu(x)), e) \\ &= \varphi(\alpha, \mu^2(x)) = \varphi(\alpha, x), \end{aligned}$$

i.e., (8) and (7) hold.

THEOREM 2. *If $f \in \Gamma^\Gamma, f^3 = f$ and $F : \Gamma \times G \to \Gamma$ is a solution of the translation equation (2) fulfilling conditions (6), (7), then the function $\varphi : \Gamma \times G \to \Gamma$ defined by $\varphi(\alpha, x) := f(F(\alpha, x))$ is a solution of equation (3) such that $\varphi(\alpha, e) = f(\alpha)$.*

Proof. By (2), (7), (4), (6), we have

$$\begin{aligned} \varphi(\alpha, x \cdot \mu(y) \cdot z) &= f[F(\alpha, x \cdot \mu(y) \cdot z)] = F(f(\alpha), \mu(x) \cdot y \cdot \mu(z)], \\ \varphi(\varphi(\varphi(\alpha, x), y), z) &= f[F(f[F(f[F(\alpha, x)], y)], z)] \end{aligned}$$

$$= F(f^2[F(f[F(\alpha,x)],y)],\mu(z)) = F(F(f[F(\alpha,x)],y),\mu(z))$$
$$= F(F(F(f(\alpha),\mu(x)),y),\mu(z)) = F(f(\alpha),\mu(x)\cdot y\cdot\mu(z)),$$

then φ satisfies (3). Moreover, on account of condition (6), we get $\varphi(\alpha,e) = f[F(\alpha,e)] = f^3(\alpha) = f(\alpha)$. The proof of Theorem 2 is finished.

COROLLARY 1. *In order to characterize all solutions of equation (3), it is necessary and sufficient to describe all solutions of the translation equation (2) satisfying conditions (6) and (7).*

First, let us remember that the general solution of the translation equation (2) has been given in [6] by the following construction.

CONSTRUCTION C_1. 1° The function $q : \Gamma \to \Gamma$ is such that $q\circ q = q$.

2° $q(\Gamma) = \bigcup_{k\in K}\Gamma_k$ is a disjoint union of non-empty sets (transitive fibres) such that for every $k \in K$ there exists a subgroup $G_k \le G$ and a bijection $g_k : G/G_k \to \Gamma_k$, where G/G_k is the set of right cosets of the group G with respect to subgroup G_k.

3° Then $F(\alpha,x) = g_k(g_k^{-1}(q(\alpha))\cdot x)$, when $q(\alpha)\in\Gamma_k$.

The function $f : \Gamma \to \Gamma$ satisfying equality (4) is fixed in the sequel. Let $(G,\cdot)$ be a binary group and μ be an automorphism of $(G,\cdot)$ such that $\mu^2(x) = x$ for $x \in G$, and $F : \Gamma\times G \to \Gamma$ be a solution of the translation equation (2) fulfilling conditions (6) and (7).

Remark 2. It is clear, by virtue of (4), that the equality $f^2(\Gamma) = f(\Gamma)$ holds true.

According to the point 2° of Construction C_1, we denote by $\{\Gamma_k\}_{k\in K}$ the family of fibres of the solution F.

LEMMA 1. *For every fibre Γ_k only one of the following equalities is fulfilled:*

$$f(\Gamma_k) = \Gamma_k \tag{9}$$

or

$$\exists_{l\in K} : l \ne k \text{ and } f(\Gamma_k) = \Gamma_l. \tag{10}$$

Proof. Let us choose $\alpha_0 \in \Gamma_k$ and let us consider two undermentioned possibilities:

(a) $f(\alpha_0)\in\Gamma_k$, (b) $f(\alpha_0)\in\Gamma_l$ and $l\ne k$.

Ad (a). We will prove (9). Let $\alpha\in\Gamma_k$. Hence $F(\alpha_0,x) = \alpha$ for certain $x \in G$. By (7), we have $F(f(\alpha_0),\mu(x)) = f(F(\alpha_0,x)) = f(\alpha)$ and, by supposition (a), we get $f(\alpha)\in\Gamma_k$. Therefore, $f(\Gamma_k)\subseteq\Gamma_k$. Since $\forall_{\alpha\in\Gamma_k} f^2(\alpha) = F(\alpha,e) = \alpha$, then $\Gamma_k = f^2(\Gamma_k) = f(f(\Gamma_k)) \subset f(\Gamma_k) \subset \Gamma_k$, whence $f(\Gamma_k) = \Gamma_k$.

Ad (b). We will prove (10). For every $\alpha\in\Gamma_k$ there exists $x \in G$ such that $F(\alpha_0,x) = \alpha$, hence $F(f(\alpha_0),\mu(x)) = f(F(\alpha_0,x)) = f(\alpha)$, whence, by (b), $f(\alpha) \in \Gamma_l$. Then $f(\Gamma_k) \subseteq \Gamma_l$. Using the equality $f(f(\alpha_0)) = \alpha_0$, we get easily the converse inclusion.

Let us define $A_f : \{\alpha \in \Gamma : f(\alpha) = f^2(\alpha)\}, B_f := \Gamma \setminus A_f = \{\alpha \in \Gamma : f(\alpha) \neq f^2(\alpha)\}$.

Remark 3. It is evident that, if a fibre Γ_k satisfies (10), then $\Gamma_k \subseteq B_f$.

Let us denote

$$E_\alpha(\beta) := \{x \in G : F(\alpha, x) = \beta\}, \ \alpha, \beta \in \Gamma, \text{ and } G_\alpha := E_\alpha(\alpha).$$

LEMMA 2. *For every $\alpha \in f^2(\Gamma)$ the set G_α is a subgroup of the group G and the equalities*

$$\mu(G_\alpha) = G_{f(\alpha)}, \tag{11}$$

$$E_{f(\alpha)}(\alpha) = \mu[E_\alpha(f(\alpha))] = \mu(G_\alpha) \cdot \mu(x_0), \ \text{if } x_0 \in E_\alpha(f(\alpha)), \tag{12}$$

hold.

Proof. Since $\alpha \in f^2(\Gamma)$, then $\alpha = f^2(\beta)$ for certain $\beta \in \Gamma$. Therefore,

$$F(\alpha, e) = F(f^2(\beta), e) = F(F(\beta, e), e) = F(\beta, e) = f^2(\beta) = \alpha.$$

By the general construction of solutions of the translation equation in [6], we have that G_α is a subgroup of group G. The equality (11) can be verified as follows:

$$x \in G_\alpha \Rightarrow F(\alpha, x) = \alpha \Rightarrow F(f(\alpha), \mu(x)) = f(F(\alpha, x)) = f(\alpha) \Rightarrow \mu(x) \in G_{f(\alpha)},$$

$$x \in G_{f(\alpha)} \Rightarrow F(f(\alpha), x) = f(\alpha) \Rightarrow f[F(\alpha, \mu(x))] = f(\alpha) \Rightarrow f^2[F(\alpha, \mu(x))] = f^2(\alpha) \Rightarrow F(\alpha, \mu(x)) = \alpha \Rightarrow \mu(x) \in G_\alpha.$$

We will prove (12). Let $x \in E_{f(\alpha)}(\alpha)$. We have

$$F(f(\alpha), x) = \alpha \Rightarrow f[F(\alpha, \mu(x))] = \alpha \Rightarrow f^2[F(\alpha, \mu(x))] = f(\alpha) \Rightarrow F(\alpha, \mu(x)) = f(\alpha),$$

then $\mu(x) \in E_\alpha(f(\alpha))$ and, since $x = \mu(\mu(x))$, then from the above we get $x \in \mu[E_\alpha(f(\alpha))]$. Hence the inclusion $E_{f(\alpha)}(\alpha) \subseteq \mu[E_\alpha(f(\alpha))]$. Similarly $\mu[E_\alpha(f(\alpha))] \subseteq E_{f(\alpha)}(\alpha)$.

By construction of solutions of the translation equation in [6], there results that $E_\alpha(f(\alpha)) = G_\alpha \cdot x_0$, where $x_0 \in E_\alpha(f(\alpha))$. Therefore $E_{f(\alpha)}(\alpha) = \mu[E_\alpha(f(\alpha))] = \mu(G_\alpha) \cdot \mu(x_0)$ and the proof of Lemma 2 is finished.

COROLLARY 2. *In virtue of Lemma 2 and by general construction of solutions of the translation equation, we get*

$$x_0 \cdot \mu(G_\alpha) = G_\alpha \cdot \mu(x_0^{-1}) \tag{13}$$

for $\alpha \in f^2(\Gamma)$ and $x_0 \in E_\alpha(f(\alpha))$.

Let us remark yet that for every $W = G_\alpha \cdot a \in G/G_\alpha$ we have

$$x_0 \cdot \mu(W) = x_0 \cdot \mu(G_\alpha) \cdot \mu(a) = G_\alpha \cdot \mu(x_0^{-1} \cdot a) \in G/G_\alpha.$$

Remark 4. If the fibre Γ_k satisfies (9) and $\alpha \in \Gamma_k$, then the function $h_\alpha : G/G_\alpha \to \Gamma_k$ defined by $h_\alpha(W) := F(\alpha, W)$ satisfies $h_\alpha(x_0 \cdot \mu(W)) = f(h_\alpha(W))$, for every $W \in G/G_\alpha$, where x_0 is a fixed element of the set $E_\alpha(f(\alpha))$.

Proof. For arbitrary $W \in G/G_\alpha$ we have

$$\begin{aligned} h_\alpha(x_0 \cdot \mu(W)) &= F(\alpha, x_0 \cdot \mu(W)) = F(F(\alpha, x_0), \mu(W)) = F(f(\alpha), \mu(W)) \\ &= f(F(\alpha, W)) = f(h_\alpha(W)). \end{aligned}$$

COROLLARY 3. *If Γ_k satisfies (9), $\alpha \in \Gamma_k, x_0 \in E_\alpha(f(\alpha))$ and $h_\alpha(W) = \beta \in \Gamma_k \cap A_f$ (respectively $h_\alpha(W) = \beta \in \Gamma_k \cap B_f$), then $x_0 \cdot \mu(W) = W$ (respectively $x_0 \cdot \mu(W) \neq W$).*

Remark 5. If the fibre Γ_k satisfies (10), $\alpha \in \Gamma_k, f(\alpha) \in \Gamma_l$, the functions $h_\alpha : G/G_\alpha \to \Gamma_k$, $h_{f(\alpha)} : G/\mu(G_\alpha) \to \Gamma_l$ are defined by $h_\alpha(W) := F(\alpha, W)$ and $h_{f(\alpha)}(W) := F(f(\alpha), W)$, respectively, then $h_{f(\alpha)}(\mu(W)) = f(h_\alpha(W))$ for every $W \in G/G_\alpha$. Indeed, $h_{f(\alpha)}(\mu(W)) = F(f(\alpha), \mu(W)) = f(F(\alpha, W)) = f(h_\alpha(W))$.

DEFINITION. The solution $F : \Gamma \times G \to \Gamma$ of the translation equation (2) satisfying (6), (7) is called f-compatible (respectively not f-compatible), if all its fibres satisfy condition (9) (respectively (10)).

First we will describe solutions which are not f-compatible.

THEOREM 3. *Let $(G, \cdot, e)$ be a group, μ an automorphism of G such that $\mu^2 = id_G$ (identity), Γ an arbitrary non-empty set and a function $f \in \Gamma^\Gamma$ satisfy equalities: $f^3 = f$, $A_f = \emptyset$. All not f-compatible solutions $F : \Gamma \times G \to \Gamma$ of the translation equation (2) satisfying (6), (7) and only such we obtain by construction presented below.*

CONSTRUCTION C_2. 1° Let us denote $B_f^\circ := \{\{f(\alpha), f^2(\alpha)\} : \alpha \in \Gamma\}$ and let us decompose the set B_f° in a disjoint union $\bigcup_{t \in T} \Gamma_t^\circ$ of non-empty sets such that $\forall_{t \in T} \exists_{G_t \leq G} card \Gamma_t^\circ = card G/G_t$.

2° For $t \in T$ let us take a selection Γ_t of the set Γ_t° and define $\overline{\Gamma}_t := f(\Gamma_t)$. (By (4), $\overline{\Gamma}_t$ is a selection of Γ_t° and $\Gamma_t \cap \overline{\Gamma}_t = \emptyset$, by $A_f = \emptyset$).

3° For every $t \in T$ let us take a bijection $g_t : G/G_t \to \Gamma_t$, put $\overline{G}_t := \mu(G_t)$ and define the bijection $\overline{g}_t : G/\overline{G}_t \to \overline{\Gamma}_t$ as follows

$$\overline{g}_t(\mu(W)) := f(g_t(W)). \tag{14}$$

4° The family of sets $\{\Gamma_t\}_{t\in T} \cup \{\overline{\Gamma}_t\}_{t\in T}$ we denote by $\{\Gamma_k\}_{k\in K}$. The family of subgroups $\{G_t\}_{t\in T} \cup \{\overline{G}_t\}_{t\in T}$ we denote respectively by $\{G_k\}_{k\in K}$. The family of bijection $\{g_t\} \cup \{\overline{g}_t\}_{t\in T}$ we denote by $\{g_k\}_{k\in K}$ respectively.

5° We put $F(\alpha, x) := g_k[g_k^{-1}(f^2(\alpha)\cdot x]$, when $f^2(\alpha) \in \Gamma_k$.

Proof. It is evident that the function $F : \Gamma \times G \to \Gamma$ obtained by construction C_2 is a solution of the translation equation not f-compatible satisfying condition (6). We will verify the condition (7) only. Let $\alpha \in \Gamma, x \in G, f^2(\alpha) \in \Gamma_k, f(\alpha) \in \Gamma_l$. There exists $t \in T$ such that $\Gamma_k = \Gamma_t$ and $\Gamma_l = \overline{\Gamma}_t$. We have

$$f(F(\alpha,x)) = f[g_k(g_k^{-1}(f^2(\alpha))\cdot x)] = f[g_t(g_t^{-1}(f^2(\alpha))\cdot x)],$$
$$F(f(\alpha),\mu(x)) = g_l[g_l^{-1}(f(\alpha))\cdot \mu(x)] = \overline{g}_t[\overline{g}_t^{-1}(f(\alpha))\cdot \mu(x)].$$

If $W \in G/G_t$ and $g_t(W) = f^2(\alpha)$, then, by (14), $\overline{g}_t(\mu(W)) = f(\alpha)$. From the above $f(F(\alpha, x)) = f(g_t(W\cdot x))$ and $F(f(\alpha),\mu(x)) = \overline{g}_t(\mu(W\cdot x))$, hence, by (14), we get $f(F(\alpha,x)) = F(f(\alpha),\mu(x))$. This ends the first part of the proof.

Let us suppose now that $f^3 = f, A_f = \emptyset$ and $F : \Gamma \times G \to \Gamma$ is not f-compatible solution of the translation equation (2) fulfilling conditions (6), (7), where μ is an automorphism of G such that $\mu^2 = id_G$. By construction C_1, we have: the family $\{\Gamma_k\}_{k\in K}$ of fibres such that $f^2(\Gamma_k) = \bigcup_{k\in K}\Gamma_k$, the family $\{G_k\}_{k\in K}$ of subgroups of G such that $card\Gamma_k = cardG/G_k$ and the family $\{g_k\}_{k\in K}$ of bijections $g_k : G/G_k \to \Gamma_k$. Let $\{\Gamma_t\}_{t\in T}$ denote the selection of the set $\{\{\Gamma_k, f(\Gamma_k)\} : k \in K\}$ and $\{g_t\}_{t\in T}$ the family of bijections adequate to $\{\Gamma_t\}_{t\in T}$. For every $t \in T$ we put $\overline{\Gamma}_t := f(\Gamma_t)$, $\overline{G}_t := \mu(G_t)$, $\overline{g}_t(\mu(W)) := f(g_t(W))$ for $W \in G/\overline{G}_t$, $\Gamma_t^\circ := \{\{f(\alpha),\alpha\} : \alpha \in \Gamma_t\}$, $B_f^\circ := \{\{f(\alpha), f^2(\alpha)\} : \alpha \in \Gamma\}$.

For arbitrary $t \in T$ and arbitrary $W \in G/G_t$ we have $g_t(W) = F(\alpha_t, W) = h_{\alpha_t}(W)$, where $\alpha_t := g_t(G_t)$, and, by Remark 5, $\overline{g}_t(W) = f(g_t(W)) = f(h_{\alpha_t}(W)) = h_{f(\alpha_t)}(\mu(W)) = F(f(\alpha_t),\mu(W))$.

By virtue of the proof of Theorem 1 in [6], the proof of Theorem 3 is finished.

We will describe now solutions which are f-compatible. First, let us define $G/G_k^{\mu,x_k} := \{W \in G/G_k : x_k\cdot\mu(W) = W\}$, if G_k is the subgroup of group $G, x_k \in G$ and $x_k\cdot\mu(G_k) = G_k\cdot\mu(x_k^{-1})$.

THEOREM 4. *Let* $(G,\cdot,e)$ *be a group,* μ *an automorphism of* G *such that* $\mu^2 = id_G, \Gamma$ *an arbitrary non-empty set and let a function* f *satisfy equality:* $f^3 = f$. *All* f*-compatible solutions of the translation equation* (2) *satisfying* (6), (7) *and only such we obtain by construction presented below.*

CONSTRUCTION C_3. 1° Let us denote $B_f^\circ := \{\{f(\alpha), f^2(\alpha)\} : \alpha \in \Gamma\}$. We decompose $B_f^\circ = \bigcup_{k\in K}\Gamma_k^0$ into a disjoint union of non-empty sets such that

$$\forall_{k\in K}\exists_{G_k\leq G}\exists_{x_k\in G} x_k\cdot\mu(G_k)=G_k\cdot\mu(x_k^{-1}),$$
$$card(\Gamma_k^\circ\cap\{\{f(\alpha)\}:\alpha\in A_f\})=card(G/G_k^{\mu,x_k}),$$
$$2card(\Gamma_k^\circ\cap\{\{f(\alpha),f^2(\alpha)\}:\alpha\in B_f\})=card(G/G_k\setminus G/G_k^{\mu,x_k}).$$

2° For every k from K we denote by Γ_k^1 a selection of the set Γ_k° and by S_k a selection of the set $\{\{W,x_k\cdot\mu(W)\}:W\in G/G_k\}$.

3° Let us put $\Gamma_k:=\Gamma_k^1\cup f(\Gamma_k^1)$ and define the bijection $g_k:G/G_k\to\Gamma_k$ by the formula

$$g_k(W):=\begin{cases} g_k^*(W): W\in S_k,\\ f(g_k^*(x_k\cdot\mu(W))): W\notin S_k,\end{cases}\tag{15}$$

where $g_k^*:S_k\to\Gamma_k^1$ is an arbitrary bijection satisfying conditions

$$W\in G/G_k^{\mu,x_k}\Rightarrow g_k^*(W)\in\Gamma_k^1\cap A_f,\tag{16}$$
$$W\in G/G_k\setminus G/G_k^{\mu,x_k}\Rightarrow g_k^*(W)\in\Gamma_k^1\cap B_f.\tag{17}$$

4° The function $F:\Gamma\times G\to\Gamma$ is defined by equality

$$F(\alpha,x):=g_k(g_k^{-1}(f^2(\alpha))\cdot x),\ \text{when } f^2(\alpha)\in\Gamma_k.\tag{18}$$

Proof. By (4) and by Construction C_1, it results that the function $F:\Gamma\times G\to\Gamma$ obtained by Construction C_3 is a solution of the translation equation (2) satisfying condition (6). Since $f(\Gamma_k)=f(\Gamma_k^1)\cup f^2(\Gamma_k^1)=f(\Gamma_k^1)\cup\Gamma_k^1=\Gamma_k$, $k\in K$, the function F is f-compatible. We will prove the condition (7) only. Let $\alpha\in\Gamma$, $x\in G$, $f(\alpha)$, $f^2(\alpha)\in\Gamma_k$. We have $F(f(\alpha),\mu(x))=g_k(g_k^{-1}(f^3(\alpha))\cdot\mu(x))=g_k(g_k^{-1}(f(\alpha))\cdot\mu(x))$ and $f(F(\alpha,x))=f[g_k(g_k^{-1}(f^2(\alpha))\cdot x)]$.

Let us consider two cases: (a) $\alpha\in A_f$, (b) $\alpha\in B_f$.

Ad (a). In this case $f(\alpha)=f^2(\alpha)$ and $g_k(W)=f(\alpha)$ for $W\in G/G_k^{\mu,x_k}$. Therefore we have $F(f(\alpha),\mu(x))=g_k(W\cdot\mu(x))$ and $f(F(\alpha,x))=f[g_k(W\cdot x)]$. From the above we state:

(i) if $W\cdot\mu(x)\in S_k$ and $x_k\cdot\mu(W\cdot\mu(x))=x_k\cdot\mu(W)\cdot x=W\cdot x\in S_k$, then, by definition of $S_k, W\cdot x=W\cdot\mu(x)$ and $g_k^*(W\cdot x)\in A_f$, hence, by (15), $F(f(\alpha),\mu(x))=g_k(W\cdot\mu(x))=g_k^*(W\cdot\mu(x))=g_k^*(W\cdot x)$ and $f(F(\alpha,x))=f[g_k(W\cdot x)]=f[g_k^*(W\cdot x)]=f^2[g_k^*(W\cdot x)]=g_k^*(W\cdot x)$.

(ii) if $W\cdot\mu(x)\in S_k$ and $x_k\cdot\mu(W\cdot\mu(x))=W\cdot x\notin S_k$, then, by (15), $F(f(\alpha),\mu(x))=g_k^*(W\cdot\mu(x))\in B_f$ and $f(F(\alpha,x))=f[g_k(W\cdot x)]=f^2[g_k^*(W\cdot\mu(x))]=g_k^*[W\cdot\mu(x)]$.

In the case $W\cdot\mu(x)\notin S_k$ the reasoning is similar.

Ad (b). We have $f(\alpha)\neq f^2(\alpha), g_k(W)=f(\alpha)$ and $g_k(x_k\cdot\mu(W))=f^2(\alpha)$ for $W\notin G/G_k^{\mu,x_k}$. By (18), we get $F(f(\alpha),\mu(x))=g_k(W\cdot\mu(x))$ and $f(F(\alpha,x))=f[g_k(x_k\cdot\mu(W)\cdot x)]$.

If (i) holds, i.e., $W \cdot \mu(x) \in S_k$ and $x_k \cdot \mu(W) \cdot x \in S_k$, then, by definition of S_k and by (15), we get $W \cdot \mu(x) = x_k \cdot \mu(W) \cdot x$ and $g_k(W \cdot \mu(x)) = g_k^*(W \cdot \mu(x)) \in A_f$, therefore $F(f(\alpha), \mu(x)) = g_k^*(W \cdot \mu(x))$ and $f(F(\alpha, x)) = f[g_k^*(W \cdot \mu(x))] = f^2[g_k^*(W \cdot \mu(x))] = g_k^*(W \cdot \mu(x))$.

If (ii) holds, i.e., $W \cdot \mu(x) \in S_k$ and $x_k \cdot \mu(W) \cdot x \notin S_k$, then, by (15) and by definition of x_k, we get $F(f(\alpha), \mu(x)) = g_k^*(W \cdot \mu(x)) \in B_f$ and $f(F(\alpha, x)) = f[g_k(x_k \cdot \mu(W) \cdot x)] = f^2[g_k^*(x_k \cdot \mu(x_k) \cdot W \cdot \mu(x))] = f^2[g_k^*(W \cdot \mu(x))] = g_k^*(W \cdot \mu(x))$.

In the case $W \cdot \mu(x) \notin S_k$ the reasoning is similar. This ends first part of the proof of Theorem 4.

Let us assume now that $F : \Gamma \times G \to \Gamma$ is a solution of the translation equation (2) fulfilling conditions (6), (7) and f-compatible. By Construction C_1, we have the following parameters: the family $\{\Gamma_k\}_{k \in K}$ of fibres such that $f^2(\Gamma) = \bigcup_{k \in K} \Gamma_k$, the family of subgroups $\{G_k\}_{k \in K}$ and the family of bijection $g_k : G/G_k \to \Gamma_k$ for which the equality (18) is satisfied. Let $\alpha_k := g_k(G_k)$ for $k \in K$. Evidently $\forall_{k \in K} G_{\alpha_k} = G_k$. Let us choose an arbitrary element x_k of the set $E_{\alpha_k}(f(\alpha_k))$. By Corollary 2, we have $x_k \cdot \mu(G_k) = G_k \cdot \mu(x_k^{-1})$.

Define $B_f^0 := \bigcup_{k \in K} \Gamma_k^0$, where $\Gamma_k^0 := \{\{f(\alpha), f^2(\alpha)\} : \alpha \in \Gamma_k\}$. By Corollary 3, we have $card(\Gamma_k^0 \cap \{\{f(\alpha), f^2(\alpha)\} : \alpha \in A_f\} = card(G/G_k^{\mu, x_k})$ and $2card(\Gamma_k^0 \cap \{\{f(\alpha), f^2(\alpha)\} : \alpha \in B_f\} = card(G/G_k \setminus G/G_k^{\mu, x_k})$, Therefore the point 1° of construction C_3 is fulfilled. Let S_k be a selection of the set $\{\{W, x_k \cdot \mu(W)\} : W \in G/G_k\}$ such that $G_k \in S_k$. We define g_k^* as follows $g_k^*(W) := F(\alpha_k, W)$ for $W \in S_k$. By Corollary 3, the conditions (16), (17) are true. Let $\Gamma_k^1 := \{g_k^*(W) : W \in S_k\}$. By Remark 4, the condition (15) is fulfilled, because $g_k(W) = F(\alpha_k, W) = h_{\alpha_k}(W), W \in G/G_k$. The proof of Theorem 4 is finished.

THEOREM 5. *Let $(G, \cdot, e)$ be a group, Γ an arbitrary non-empty set and let the function $f \in \Gamma^\Gamma$ satisfy equality $f^3 = f$. All solutions $F : \Gamma \times G \to \Gamma$ of the translation equation (2) satisfying (6), (7) and only such can be obtained by the following construction.*

CONSTRUCTION C_4. 1° Let $A_f := \{\alpha \in \Gamma : f(\alpha) = f^2(\alpha)\}$ and $B_f := \Gamma \setminus A_f$. We decompose the set $\{\{f(\alpha), f^2(\alpha)\} : \alpha \in \Gamma\} = E_1 \cup E_2$ into a disjoint union of sets E_1, E_2 such that $E_1 \subseteq \{\{f(\alpha), f^2(\alpha)\} : \alpha \in B_f\}$. Let $R_i := \{\alpha \in \Gamma : \{f(\alpha), f^2(\alpha)\} \in E_i\}$, $i = 1, 2$.

2° Let $F_1 : R_1 \times G \to R_1$ be a not f-compatible solution of the translation equation (2) satisfying (6), (7). This means that F_1 is obtained by construction C_2.

3° Let $F_2 : R_2 \times G \to R_2$ be a f-compatible solution of the translation equation (2) satisfying conditions (6), (7). This means that F_2 is obtained by construction C_3.

4° We put $F := F_1 \cup F_2$.

Proof. It is evident that F obtained by construction C_3 is a solution of (2) fulfilling (6), (7). Assume that $F : \Gamma \times G \to \Gamma$ is a solution of (2) satisfying (6), (7). By Lemma 1, all fibres of F satisfy (9) or (10). Let us define

$$E_1 := \{\{f(\alpha), f^2(\alpha)\} : \alpha \in B_f \text{ and } \exists_{k,l \in K, k \neq l} f(\alpha) \in \Gamma_k \text{ and } f^2(\alpha) \in \Gamma_l\},$$

$$E_2 := \{\{f(\alpha), f^2(\alpha)\} : \alpha \in \Gamma\} \setminus E_1,$$

and

$$R_i := \{\alpha \in \Gamma : \{f(\alpha), f^2(\alpha)\} \in E_i\}, \quad F_i := F|_{R_i \times G}, \quad i = 1, 2.$$

It is visible that $F = F_1 \cup F_2$, where A_f, R_1, R_2, F_1, F_2 are such as in Construction C_4.

EXAMPLE 1. Let $\Gamma := R, G := \langle R, +, 0\rangle$ and $f : \Gamma \to \Gamma$ be defined by $f(\alpha) := -\alpha$. Let $\mu \in AutG$ defined by equality $\mu(x) := -x$. Evidently $f^3 = f$ and $\mu^2 = id_G$. Let $F : \Gamma \times G \to \Gamma$ be defined by the formula $F(\alpha, x) := \alpha + x$. Then F is a solution of the translation equation (2) satisfying equalities $F(\alpha, 0) = \alpha = f^2(\alpha), F(f(\alpha), \mu(x)) = F(-\alpha, -x) = -\alpha - x = -(\alpha + x) = f(F(\alpha, x))$.

The function $\varphi(\alpha, x) := f(F(\alpha, x)) = -\alpha - x$ is a solution of the equation

$$\varphi(\alpha, x - y + z) = \varphi(\varphi(\varphi(\alpha, x), y), z).$$

Indeed, we have $\varphi(\alpha, x - y + z) = -\alpha - x + y - z$ and $\varphi(\varphi(\varphi(\alpha, x), y), z) = \varphi(\varphi(-\alpha - x, y), z) = \varphi(\alpha + x - y, z) = -\alpha - x + y - z$.

EXAMPLE 2. Let $\Gamma := \{\alpha_0, \alpha_1, \beta_0, \beta_1, \gamma_0, \gamma_1, \gamma_2, \gamma_3\}$, and let $G := \langle\{0, 1, 2, 3\}, +_4, 0\rangle$ be the group of residues modulo 4. Let $f : \Gamma \to \Gamma$ be defined as follows:

$$f(\alpha_0) = \alpha_0, f(\alpha_1) = \alpha_1, f(\beta_0) = \beta_1, f(\beta_1) = \beta_0,$$
$$f(\gamma_0) = \gamma_1, f(\gamma_1) = \gamma_0, f(\gamma_2) = \gamma_3, f(\gamma_3) = \gamma_2.$$

Evidently $f^3 = f$ and $A_f = \{\alpha_0, \alpha_1\}$. We define also $\mu \in AutG$ as follows: $\mu(x) := x$ for $x \in \{0, 2\}, \mu(x) := x +_4 2$ for $x \in \{1, 3\}$. Evidently $\mu^2 = id_G$. According to the Construction C_4, we decompose the set $\{\{f(\alpha), f^2(\alpha)\} : \alpha \in \Gamma\} = E_1 \cup E_2$ into a disjoint union of sets: $E_1 := \{\{\gamma_0, \gamma_1\}, \{\gamma_2, \gamma_3\}\}$, $E_2 := \{\{\alpha_0\}, \{\alpha_1\}, \{\beta_0, \beta_1\}\}$. Therefore $R_1 = \{\gamma_0, \gamma_1, \gamma_2, \gamma_3\}, R_2 = \{\alpha_0, \alpha_1, \beta_0, \beta_1\}$.

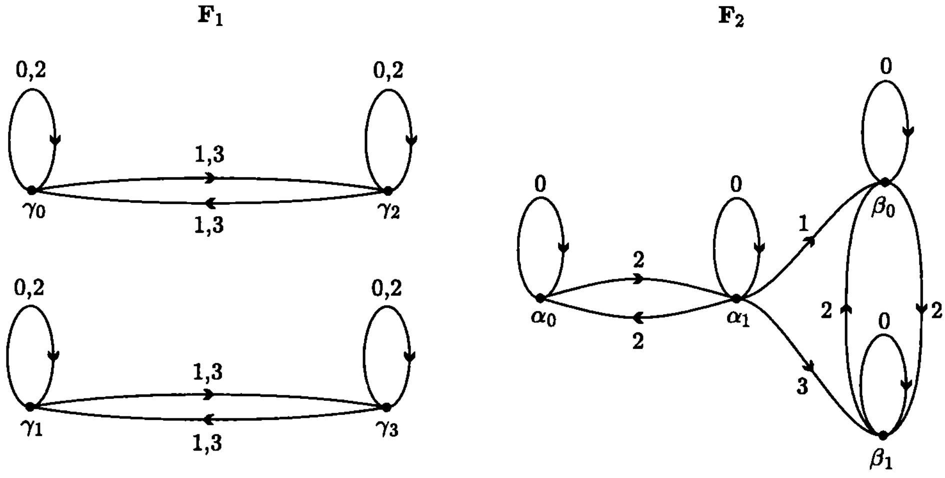
We define first, according to Construction C_2, a solution not f-compatible, namely $F_1 : R_1 \times G \to R_1$. Let $\Gamma_1^0 := B_f^0 = \{\{\gamma_0, \gamma_1\}, \{\gamma_2, \gamma_3\}\}, G_1 :=$

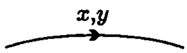
$\{0,2\}$. Then $card\Gamma_1^0 = cardG/G_1 = 2$. Therefore, the point 1° is satisfied. In accordance with the point 2° of the Construction C_2, we take the selection $\Gamma_1 := \{\gamma_0, \gamma_2\}$ of Γ_1^0 and put $\overline{\Gamma}_1 := f(\Gamma_1) = \{\gamma_1, \gamma_3\}$. As in the point 3° we take the bijection $g_1 : G/G_1 \to \Gamma_1$ defined by equalities $g_1(\{0,2\}) := \gamma_0, g_1(\{1,3\}) := \gamma_2$. Let $\overline{G}_1 := \mu(G_1) = G_1$ and put $\overline{g}_1 : G/G_1 \to \overline{\Gamma}_1, \overline{g}_1(\{0,2\}) := f(g_1(\{0,2\})) = \gamma_1, \overline{g}_1(\{1,3\}) := f(g_1(\{1,3\}) = \gamma_3$. We denote $\{\Gamma_1, \overline{\Gamma}_1\} =: \{\Gamma_1, \Gamma_2\}, \{G_1, \overline{G}_1\} =: \{G_1, G_2\}, \{g_1, \overline{g}_1\} =: \{g_1, g_2\}$ and we put $F(\alpha, x) = g_k(g_k^{-1}(f^2(\alpha)) +_4 x)$, $f^2(\alpha) \in \Gamma_k$, $k \in \{1,2\}$, $\alpha \in R_1$.

Now we will define a solution f-compatible, namely $F_2 : R_2 \times G \to R_2$. We will use the Construction C_3. We have $B_f^0 = \{\{\alpha_0\}, \{\alpha_1\}, \{\beta_0, \beta_1\}\}$ and we define $B_f^0 =: \Gamma_1^0, G_1 := \{0\}, x_1 := 2$. We have $x_1 + \{0\} = \{0\} + x_1, card(\Gamma_1^0 \cap \{\{\alpha_0\}, \{\alpha_1\}\}) = 2 = card(G/G_1^{\mu,x_1}) = card(\{\{1\}, \{3\}\})$ and $2card(\Gamma_1^0 \cap \{\{\beta_0, \beta_1\}\}) = 2 = card(G/G_1 \setminus G/G_1^{\mu,x_1}) = card(\{\{0\}, \{2\}\})$.

According to the point 2° of Construction C_3, we denote $\Gamma_1^1 := \{\alpha_0, \alpha_1, \beta_0\}$ and $S_1 := \{\{0\}, \{3\}, \{1\}\}$. As in the point 3° we put $\Gamma_1 := \Gamma_1^1 \cup f(\Gamma_1^1) = \{\alpha_0, \alpha_1, \beta_0, \beta_1\}$ and $g_1^*(\{0\}) := \beta_0, g_1^*(\{1\}) := \alpha_0, g_1^*(\{3\}) := \alpha_1$. By (15), $g_1(\{2\}) = \beta_1$ and the conditions (16), (17) are satisfied. In accordance with the point 4°, we define $F(\alpha, x) := g_1(g_1^{-1}(f^2(\alpha)) +_4 x)$, when $\alpha \in \{\alpha_0, \alpha_1, \beta_0, \beta_1\}$.

One can present this solution on the figure as below.



By α 

β we understand that $F(\alpha, x) = \beta, F(\alpha, y) = \beta$.

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