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IF K IS A REAL FIELD THEN $cn(PSL(2,K)) = 4$

Let G be a group. The smallest natural number m satisfying $C^m = G$ for each nontrivial conjugacy class C of G is called the covering number of G and is denoted by $cn(G)$.

If there is no such m then it is written $cn(G) = \infty$. It is known that:

(i) If K is a complex field, then $cn(PSL(2,K)) = 2$, (see [4], [5]).

(ii) If K is a finite field and $|K| \geq 4$, then $cn(PSL(2,K)) = 3$, (see [6]).

(iii) If there exists $m \in K^*$such that $m^4 \neq 1$, then $cn(PSL(2,K)) \leq 4$, (see [1], [2]).

(iv) If $K = \mathbf{Q}, \mathbf{R}$. then $cn(PSL(2,K)) \geq 3$, (see[5]).

In this paper we will show that if K is a field in which -1 is not a sum of squares i.e. K is a real field, then $cn(PSL(2,K)) = 4$ and if $K = \mathbf{R}$, then C^3 contain all elements of $PSL(2,R)$ but E.

Furthermore, as we shall see, the theorem that "There is no class C in $PSL(2,\mathbf{Q})$ such CC covers the group", (see [5], Theorem 3.05), which was an additional argument for the statement (iv), is not true. In fact in this paper we will prove that the opposite is true, (see Corollary 3.1). Another proof of (iv) follows by Lemma 3.

We will use the following lemmas:

LEMMA 1. *If* $V = diag(v_1, ..., v_n)$, $W = diag(w_1, ..., w_n)$ *with* $v_i \neq v_j$ *for* $i \neq j$, *and* $V, W \in SL(n,K)$, *then* $SL(n,K) = C_V C_W \cup Z(SL(n,K))$, (see [3], Theorem 1).

LEMMA 2. *Let* $M = \begin{bmatrix} 0 & g \\ -g^{-1} & 0 \end{bmatrix} \in SL(2,K)$. *If* K *is a real field, then there exists no matrices*

$$T_i = \begin{bmatrix} r_i & s_i \\ t_i & u_i \end{bmatrix}, \qquad T_i \in SL(2,K), \quad i = 1,2$$

such that the trace $(M^{T_1} M^{T_2}) = s$ *is equal to zero.*

Proof. It is easy to check that

$$(1)\qquad -s=[g^{-1}(s_1r_2-r_1s_2)]^2+(s_1t_2-r_1u_2)^2+(u_1r_2-t_1s_2)^2 +[g(u_1t_2-u_2t_1)]^2.$$

Suppose, per contra that there are matrices $T_i \in SL(2,K)$ such that $s=0$. Then from (1), assumption that K is a real field, it follows that

$$E_1: s_1r_2-r_1s_2=0,\quad E_2: u_1r_2-t_1s_2=0,$$
$$E_3: s_1t_2-r_1u_2=0,\quad E_4: u_1t_2-u_2t_1=0.$$

Linear combinations

$$-u_1E_1+s_1E_2 \quad\text{and}\quad -u_1E_3+s_1E_4$$

are equivalent to

$$s_2\det(T_1)=0 \quad\text{and}\quad u_2\det(T_1)=0.$$

Since $\det(T_1)\neq 0$, by the assumption, thus $s_2=0$, $u_2=0$ and consequently $\det(T_2)=0$, a contradiction.

LEMMA 3. *If K is a real field, then $cn(PSL(2,K))>3$.*

Proof. The trace (1) is different from 0 in the real field K, by Lemma 2. Thus the set C_M^2 does not contain of $M^{-1}=\begin{bmatrix}0 & -g\\ g^{-1} & 0\end{bmatrix}\in SL(2,K)$ with the trace equal to 0. Therefore $E\notin C_M^2C_M=C_M^3$, so $cn(PSL(2,K))>3$.

THEOREM 1. *If K is a real field, then $cn(PSL(2,K))=4$.*

Proof. From (iii) it follows that $cn(PSL(2,K))\leq 4$, and from Lemma 3 that $cn(PSL(2,K))>3$. Therefore $cn(PSL(2,K))=4$.

THEOREM 2. *If $K=\mathbf{R}$ and C is any non-trivial conjugacy class, then $C^3=PSL(2,\mathbf{R})-E$.*

Proof. Observe that any non-scalar matrix of $SL(2,K)$ is similar to the matrix $B=\begin{bmatrix}0 & b\\ -b^{-1} & a\end{bmatrix}$ in $SL(2,K)$. If we take $X=\begin{bmatrix}0 & m\\ -m^{-1} & 0\end{bmatrix}$, $m^4\neq 1$ and $m\neq b$, then eigenvalues of BB^X are different: $v=-b^2m^{-2}$, $v^{-1}=-m^2b^{-2}$. Eigenvalues of $A=\begin{bmatrix}v^{-1} & 0\\ u & v\end{bmatrix}$ are the same as of BB^X. Since $v\neq v^{-1}$, then A and BB^X are similar in $SL(2,\mathbf{R})$ i. e. $A=B^YB^{XY}$ for some $y\in SL(2,\mathbf{R})$. If we take $S=\begin{bmatrix}0 & -s\\ s^{-1} & -asb^{-1}\end{bmatrix}$, then matrices $B^YB^{XY}B^S=\begin{bmatrix}0 & -m^2s^2b^{-3}\\ m^{-2}s^{-2}b^3 & us^2b^{-1}-ab^2m^{-2}\end{bmatrix}\in C_B^3$, where C_B - conjugacy class of B. In the conjugacy class $C_B\subset PSL(2,\mathbf{R})$ there exists also $-B$. Thus both matrices AB^S and $-AB^S \in C^3_B$. Since $s \in \mathbf{R}^*$ and $u \in \mathbf{R}$ vary over the elements of $\mathbf{R}$, then according to our observation that any non-scalar matrix of SL(2,K) is of the shape B, so $C^3 = PSL(2,\mathbf{R}) - E$.

THEOREM 3. *If* $|K| \geq 4$, *then there exists at least one class* C *in* $PSL(2,K)$ *such that* CC *covers the group.*

Proof. From Lemma 1 it follows that if eigenvalues of $V, W \in PSL(n,K)$ are distinct, then

$$PSL(n,K) = C_V C_W \cup E. \tag{2}$$

By assumption that $|K| \geq 4$ it results that there exists matrices $V = diag(v, v^{-1})$,where $v \neq v^{-1}$, and $W = V^{-1}$, with distinct eigenvalues. Since matrices $V$and V^{-1} are conjugate in $PSL(2,K)$, $E \in C_V C_V$. Therefore, $PSL(n,K) = C_V C_V$, by (2).

COROLLARY 3.1. *If the field* K *has infinite many elements, then there are infinite many conjugacy classes* C *such that* $PSL(2,K) = CC$.

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