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SOME MATHEMATICAL PROBLEMS CONNECTED WITH THE AXIOMATIC OF SIMILARITY GEOMETRY

1. Introduction

This work concerns the axiomatic of the central geometry of similarities. In this paper we use for the description of Euclidean space the notions that are considerably different from the traditional ones. The primitive notions are: the set E, the distinguished point O, the operation of the addition of points and the quaternary relation of similarity P. The operation of the addition of points can be replaced by the ternary relation of parallelogram with a fixed vertex O. In the Euclidean space the formula $P(x, y, x', y')$ denotes the similarity of triplets $\langle 0, x, y \rangle$ and $\langle 0, x', y' \rangle$ with the additional condition: $x \neq 0$. This paper contains an axiom system of dimension-free geometry (for dimensions > 1). An information about how this axiom system has been constructed and the proofs of the representation theorems can be found in [5]. In this work we will briefly discuss these problems.

The paper contains the solution to the problem of independency of the axioms and the answer to the question what axioms are to be added to the axiom system in order to obtain a complete theory. These problems were suggested by L. W. Szczerba.

2. Definitions of notions

We start with the definitions of notions which we will use.

Let us define a linear space with the scalar product (cf. [1]) and a central similarity geometry.

DEFINITION 1. Let $\vartheta = (V, F, 0, +, \circ)$ be a linear space over the field $F = (F, +, \cdot, 0, 1)$ satisfying the additional conditions:

(i) $F \subset V$,

(ii) $+/F^2 = +$,

(iii) $\circ/F^2 = \cdot$.

The structure (ϑ, o) will be called the linear space with the Euclidean scalar product, if the following axioms are satisfied:

1.1 $a \circ a = 0 \Leftrightarrow a = 0,$

1.2 $a \circ (b\lambda) = (a \circ b)\lambda,$

1.3 $a \circ (b + c) = (a \circ b) + (a \circ c),$

1.4 $a \circ b = b \circ a,$

1.5 $\alpha \neq 0 \Rightarrow \exists\lambda\ (a\lambda) \circ (a\lambda) = b \circ b,$

1.6 $a.b \in F \Rightarrow a \circ b = ab.$

We distinguish the spaces satisfying the conditions $dimV \geq 4$, $dimV \geq 2$, $dimV \geq 1$. (Each of these conditions can be presented by an elementary axiom.)

DEFINITION 2. Let ϑ be a linear space over the field $\mathbf{F} = (F, +, \cdot, 0, 1)$ with a scalar product. By a dimension-free central Euclidean geometry of similarities over the field $\mathbf{F}$ we mean the elementary theory of structures $C_F = (\vartheta, R_1, \ldots, R_k, f_1, \ldots, f_s, 0)$ where R_i $(i = 1, \ldots, k)$, f_j $(j = 1, \ldots, s)$ are relations and functions with arguments and values in ϑ, definable by elementary formulae of ϑ, 0 is zero-vector of ϑ and the group of automorfism of C_F structure coincides with the group of the orthogonal transformations of linear space ϑ such that the point 0 is fixed (cf. [2]).

The concept of central geometry of similarities would be defined in a more general way, if the condition of Euclidean property imposed on the scalar product was omitted. The isometries and similarities connected with Minkowskian or Galilean metric form are often discussed in geometry and algebra.

However, in this paper we restrict ourselves to the Euclidean similarities.

Let us introduce the definition of the similarity space.

Let $(\vartheta, \circ)$ be a linear space with a scalar product given by Definition 1. Let P_F be the similarity relation of the pairs of elements of the $(\vartheta, \circ)$ space defined in the following way.

DEFINITION 3. $P_F(x, y, z, u)$

$$\Leftrightarrow [x \neq 0 \wedge \exists\lambda \in F\ (z \circ z = \lambda(x \circ x) \wedge u \circ u = \lambda(y \circ y) \wedge u \circ z = \lambda(y \circ x)].$$

DEFINITION 4. The structure $\Sigma(F) = (\vartheta, +_F, 0_F, P_F)$ is called the similarity space, where $\vartheta = (\vartheta, \circ)$ denotes the linear space according to Definition 1, $+_F$—the operation of addition of vectors, 0_F—the zero-vector and P_F—the relation described by Definition 3.

The dimension of the space ϑ is called the dimension of similarity space $\Sigma(F)$.

3. The axiom system of dimension-free geometry

Let us discuss now the axiom system of dimension-free geometry.

We split this axiom system into two parts. The first is the axiom system **A0** constructed by L. Dubikajtis. He formulated the problem of possibility of the extension of system **A0** to the axiom system of central geometry of similarities. The paper [5] contains the solution of this problem.

The axiom system **A0** can be extended to the axiom system **A2** of plane geometry and to the axiom system **A3** of dimension-free geometry (for dimensions ≥ 3).

Let us now recall the axiom system **A0** (cf. [5]). We consider the structure $(E,+,0,P)$, where $+ : E^2 \to E$, $0 \in E$, $P \subset E^4$, satisfying the following axioms:

A0: $(E,+,0)$ is the abelian group with at least two elements,

A1: $P(x,y,z,u) \Rightarrow x \neq 0$,

A2: $P(x,y,0,y') \Rightarrow y' = 0$,

A3: $P(x,y,x',y') \Rightarrow P(x,y,-x',-y')$,

A4: $\forall x,y,z[x \neq 0 \Rightarrow \exists y',u\ (P(x,y,x,y') \wedge P(x,y',z,u) \wedge P(x,y',x+z,y'+u))]$,

A5: $P(x,y,x',y') \wedge P(x,y,x'',y'') \wedge x' \neq 0 \Rightarrow P(x',y',x'',y'')$,

A6: $P(x,0,y,y') \Rightarrow y' = 0$,

A7: $P(x,x',y,y') \Rightarrow P(x,-x',y,-y')$,

A8: $P(x,x',y,y') \Rightarrow P(x,x+x',y,y+y')$,

A9: $\forall x,x',x'',y,y'[x' \neq 0 \wedge P(x,x',y,y') \Rightarrow \exists y''(P(x,x'',y,y'') \wedge P(x',x'',y',y'')]$,

A10: $\forall x,x',y\ [x \neq 0 \Rightarrow \exists y'(P(x,y,x',y') \wedge P(x,x',y,y')]$,

A11: $P(x,y,x',y') \wedge P(y,y',y',y) \Rightarrow P(x,x',x',x)]$,

We denote by $\mathbf{A0} = \{\mathrm{A0},\ldots,\mathrm{A11}\}$.

We shall describe now three models of the axiom system **A0**.

MODEL I. Let $\mathbf{F} = (F,+,.,0,1)$ be an arbitrary commutative field. We define the structure:

$$M_F^n = (E_F,+_F,0_F,P_F);\ E_F = F^n,\ 0_F = (0,\ldots,0),$$
$$(x_1,\ldots,x_n) +_F (y_1,\ldots,y_n) = (x_1+y_1,\ldots,x_n+y_n),$$

$P_F(x,y,z,u)$

$$\Leftrightarrow [x \neq 0 \wedge \exists\lambda \in F(z \circ z = \lambda(x \circ x) \wedge u \circ u = \lambda(y \circ y) \wedge u \circ z = \lambda(y \circ x))]$$

where $x \circ y = x_1y_1 + \ldots + x_ny_n$ for $x = (x_1,\ldots,x_n), y = (y_1,\ldots,y_n)$. It can be shown that M_F^n is the model for **A0**.

MODEL II. Interpreting E as the set of vectors of the Euclidean real plane which are fixed at the point 0, operation $+$ as the addition of vectors (fig. 1) and the relation P as the relation of similarity of triangles with common vertex (fig. 2) we obtain also the model of **A0** axiomatics.

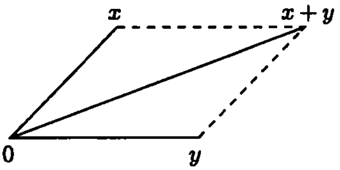

Fig. 1

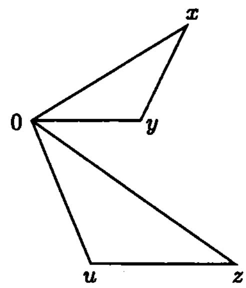

Fig. 2

Using the notation introduced in the model I, we see that the model II is isomorphic with M_R^2.

MODEL III. Let $\mathbf{F} = (F, +, \cdot, 0, 1)$ be an arbitrary commutative field and $P_F(x,y,z,u) \Leftrightarrow [x \neq 0 \wedge u = yx^{-1}z]$. It can be shown that the structure $(F, +, 0, P_F)$ is the model for **A0**.

The other models of axiom system **A0** have been described in [5].

Now, we add the following axioms to the axiom system **A0**.

A12: $\forall x \neq 0\ \exists y \neq x\ P(x,y,y,-x)$
A13: $P(x,y,y,-x) \wedge P(x',y',y',-x') \Rightarrow P(x,y,y',x')$
A14: $\forall x, y \neq 0\ \exists z\ [P(y,z,z,y) \wedge \forall z'(P(x,z,x,z') \Rightarrow z = z')]$
A15: $P(x,z-y,x,y-z) \wedge P(y,z-x,y,x-z) \wedge z \neq 0 \Rightarrow P(z,x-y,z,y-x)$
A16: $P(x,y,-x,y) \wedge P(x,z,-x,z) \Rightarrow P(x,y+z,-x,y+z)$
A17: $P(x,y,y,x) \wedge y \neq -x \Rightarrow P(x+y,x-y,x+y,y-x)$
A18: $\forall x, y\ [x \neq 0 \Rightarrow \exists z\ (P(x,y,x,z) \wedge \forall z'(P(x,y+z,x,z') \Rightarrow y+z = z'))]$
A19: $(P(x,y,x,t) \Rightarrow y = t) \vee (P(x,u,x,v) \wedge P(y,u,y,v) \Rightarrow u = v)$
A20: $\exists x,y,t,u,v(P(x,y,x,t) \wedge y \neq t \wedge P(x,u,x,v) \wedge P(y,u,y,v) \wedge u \neq v]$

Figures 3 and 4 present the interpretation of axioms A14 and A18, respectively, for the second model described above.

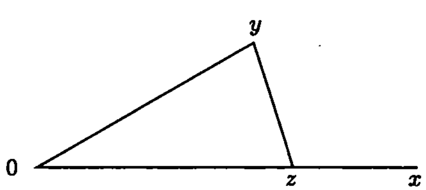

Fig. 3

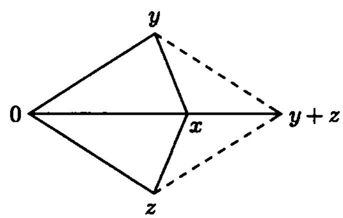

Fig. 4

Let $\mathbf{A1} = \mathbf{A0} \cup \{\text{A12}, \ldots, \text{A18}\}$, $\mathbf{A2} = \mathbf{A1} \cup \{\text{A19}\}$, $\mathbf{A3} = \mathbf{A1} \cup \{\text{A20}\}$.

The axiom system **A2** is the axiomatics of the plane geometry. The axiom system **A3** is the axiomatics of dimension-free geometry (of dimensions ≥ 3).

Now, we shall formulate the representation theorems.

THEOREM 1. *The structure* $(E, +, 0, P)$ *is the model of the system* $\sigma^2 = \mathrm{Cn}(\mathbf{A2})$ *if and only if it is isomorphic with the similarity space* $\Sigma(F) = (\vartheta, +_F, 0_F, P_F)$ *of dimension* 2.

THEOREM 2. *The structure* $(E, +, 0, P)$ *is the model of the system* $\sigma^{3\geq} = \mathrm{Cn}(\mathbf{A3})$ *if and only if it is isomorphic with the similarity space* $\Sigma(F) = (\vartheta, +_F, 0_F, P_F)$ *of dimension* ≥ 3.

The proofs of these theorems can be found in [5].

The axiom A19 can be also written in equivalent form

A19: $\neg\exists x, y, t, u, v(P(x, y, x, t) \wedge y \neq t \wedge P(x, u, x, v) \wedge P(y, u, y, v) \wedge u \neq v)$.

It is the dimension axiom, as it restructs the dimensions of the space to 2. It is also the negation of the condition A20.

Let M be a model for the system $\mathrm{Cn}(\mathbf{A1})$. If the condition A19 is fulfilled in M ($M \models$ A19), then M is the model for $\mathrm{Cn}(\mathbf{A2})$; thus by Theorem 1 it is a similarity space of dimension 2. If the condition A19 is not satisfied in M ($\neg(M \models$ A19) or $M \models \neg$A19), then M is the model for $\mathrm{Cn}(\mathbf{A3})$ hence it is a similarity space of dimension ≥ 3 (Theorem 2).

We have thus

THEOREM 3. A*n arbitrary model of the system* $\mathrm{Cn}(\mathbf{A1})$ *is isomorphic with certain space of similarities of dimension* > 1.

4. Some properties of the relation P and the relation collinearity L

We shall now consider the relation of similarity P and the additional relation of collinearity L. Let us define a relation $L(0, x, y)$.

DEFINITION 5. $L(0, x, y) \Leftrightarrow (P(x, y, x, z) \Rightarrow y = z))$.

Now we prove some properties of relations P and L which we shall use in the sequel.

LEMMA 1. $x \neq 0 \Rightarrow P(x, x, y, y)$.

Proof. Let $x \neq 0$. By A10

$$\exists t : P(x, x, y, t).$$

By A7, A8, A6:

$$P(x, x, y, t) \Rightarrow P(x, -x, y, -t) \Rightarrow P(x, 0, y, y - t) \Rightarrow t = y. \blacksquare$$

On the basis of these axioms it is quite simple to prove:

LEMMA 2. $x \neq 0 \Rightarrow P(x, y, x, y)$.

LEMMA 3. $P(x, y, x', y') \wedge x' \neq 0 \Rightarrow P(x', y', x, y)$.

Proof. By A1 and Lemma 2

$$P(x, y, x', y') \Rightarrow x \neq 0 \Rightarrow P(x, y, x, y).$$

By A5

$$P(x, y, x', y') \wedge P(x, y, x, y) \wedge x' \neq 0 \Rightarrow P(x', y', x, y). \blacksquare$$

LEMMA 4. $P(x, y, x', y') \wedge x' \neq 0 \Rightarrow P(y, x, y', x')$.

Proof. The assumptions and the axiom A9 imply:

$$\exists u : (P(x, x, x', u) \wedge P(y, x, y', u)).$$

We have to prove that: $x' = u$. By A6, A7, A8:

$$P(x, x, x', u) \Rightarrow P(x, 0, x', x' - u) \Rightarrow x' = u. \blacksquare$$

By A2, Lemmas 3 and 4 we prove

LEMMA 5. $P(x, y, x', y') \wedge \neq 0 \Rightarrow P(y', x', y, x)$.

LEMMA 6. $P(x, y, x', y') \Leftrightarrow P(x, -y, -x', y')$.

The proof of $\Rightarrow$. By A3 and A7

$$P(x, y, x', y') \Rightarrow P(x, -y, x', -y') \Rightarrow P(x, -y, -x', y').$$

The proof of converse implication is analogous. ■

LEMMA 7. $P(x, y, x', y') \Rightarrow P(-x, y, -x', y')$.

Proof. So, if $P(x, y, x', y')$, then by A1, Lemma 2 and A3 we get $P(x, y, -x, -y)$, hence, by A5 it follows that

$$(P(x, y, -x, -y) \wedge P(x, y, x', y') \Rightarrow P(-x, -y, x', y')).$$

By Lemma 6 we get the thesis. ■

LEMMA 8. $\forall x\ [L(0, x, 0) \wedge L(0, x, x) \wedge L(0, x, -x)]$.

Proof. The case $x = 0$ is obvious so, if $x \neq 0$ then by A6 we obtain $L(0, x, 0)$ and by A6, A7, A8 we prove the other properties. ■

In simple way by A7 we prove

LEMMA 9. $L(0, x, y) \Rightarrow L(0, x, -y)$.

By Lemma 7 and A7 we deduce

LEMMA 10. $L(0, x, y) \Rightarrow L(0, -x, -y)$.

By A7 and A8 we get

LEMMA 11. $L(0, x, y) \Rightarrow L(0, x, x + y)$.

An interpretation of the next property is the following: if the triangles $\langle 0, x, y\rangle$ and $\langle 0, x', y'\rangle$ are similar and the points $0, x, y$ are collinear then the points $0, x', y'$ are collinear too.

LEMMA 12. $L(0,x,y) \wedge P(x,y,x',y') \Rightarrow L(0,x',y')$.

Proof. The cases $x' = 0$ and $y' = 0$ are obvious.

Thus we assume that: $y' \neq 0$. We have to prove that $P(x',y',x',z) \Rightarrow y' = z$. From the assumptions and $P(x',y',x',z)$ by Lemma 3 and Lemma 5 we have

$$(1)\ P(y',x',y,x), \quad (2)\ P(x',y',x',y), \quad (3)\ P(x',z,x',y').$$

From (2) by A9 we get

$$(4)\ \exists u : (P(y',z,y,u) \wedge P(x',z,x,u).$$

Applying A5 and from (2), (3) and (4) we obtain $P(x,y,x,u)$.

The assumption $L(0,x,y)$ and the condition $P(x,y,x,u)$ imply $y = u$ hence from (4) we have $P(y',z,y,y)$.

Now we can apply Lemma 11 because $y' \neq 0$ hence $y \neq 0$ and we get $P(y,y,y',z)$.

By A6, A7, A8 we prove that $y' = z$. ∎

LEMMA 13. $P(x,y,z,u) \wedge L(0,x,y) \Rightarrow P(x,y,x+z,y+u)$.

Proof. From the assumption $P(x,y,z,u)$ by A1 and A4 we get

$$\exists v,w(P(x,y,x,v) \wedge P(x,v,z,w) \wedge P(x,v,x+z,v+w)$$

since $L(0,x,y)$ then from Definition 1 it follows that: $y = v$. The case $z = 0$ is trivial. When $z \neq 0$ by A5 and Lemma 12 we prove that $w = u$ hence $P(x,y,x+z,y+u)$. ∎

LEMMA 14. $L(0,x,x') \Rightarrow L(0,x',x)$.

Proof. The cases $x' = 0$ and $x = 0$ are obvious, thus we assume

$$(1)\ x \neq 0 \wedge x' \neq 0 \wedge P(x',x,x',z).$$

We have to prove that $x = z$. By Lemma 4 and A3, (1) implies $P(x,x',-z,-x')$.

Applying the assumption $L(0,x,x')$ and Lemma 13 we obtain $P(x,x',x-z,0)$. Then by Lemma 4 and A2, (1) implies $x = z$. ∎

LEMMA 15. $x \neq 0 \wedge L(0,x,x') \wedge L(0,x,x'') \Rightarrow L(0,x',x'')$.

Proof. Putting particular cases aside we assume that $x' \neq 0$ and $x'' \neq 0$. We have to prove that $P(x',x'',x',u) \Rightarrow x'' = u$. By A9 we have

$$\exists z\ (P(x',x,x',z) \wedge P(x'',x',u,z))$$

then from the assumptions, Lemma 14, Lemma 4 and Definition 1 we obtain that $x = z$ and $P(x, x'', x, u)$. On the basis of Definition 1 we get $x'' = u$. ∎

As a direct consequence of Lemma 14 and Lemma 15 we obtain:

LEMMA 16. $y \neq 0 \wedge L(0, x, y) \wedge L(0, y, z) \Rightarrow L(0, x, z)$.

LEMMA 17. $L(0, x, y) \wedge L(0, x, y') \Rightarrow L(0, x, y + y')$.

Proof. The case $x = 0$ is obvious. So we assume $x \neq 0$. From the assumptions by Lemma 9, Lemma 11, Lemma 15 we get

$$L(0, x, x + y) \wedge L(0, x, x - y') \Rightarrow L(0, x + y, x - y') \Rightarrow L(0, x + y, y + y').$$

When $y \neq -x$ by Lemma 16 we get the thesis. If $y = -x$ then from assumption $L(0, x, y')$ by Lemma 9, Lemma 10 and Lemma 11 we obtain

$$L(0, x, y + y'). \ ∎$$

LEMMA 18. $P(x, y, x', y') \wedge P(y, z, y', z') \wedge L(0, y, z) \Rightarrow P(x, z, x', z')$.

Proof. A1 and $P(y, z, y'z')$ imply $y \neq 0$ thus by A9 we get:

$$(1) \ \exists u \ (P(x, z, x', u) \wedge P(y, z, y', u).$$

If $y' = 0$ from (1) and the assumption $P(y, z, y', z')$ by A2 it follows that $u = z'$. If $y' \neq 0$ from (1) and $P(y, z, y', z')$ by A5 it follows that $P(y', z', y', u)$, by Lemma 12 we get $L(0, y', z')$. We get $z' = u$ by virtue of Definition 1 and this completes the proof. ∎

The fig. 5 presents the interpretation of Lemma 18 for the model II.

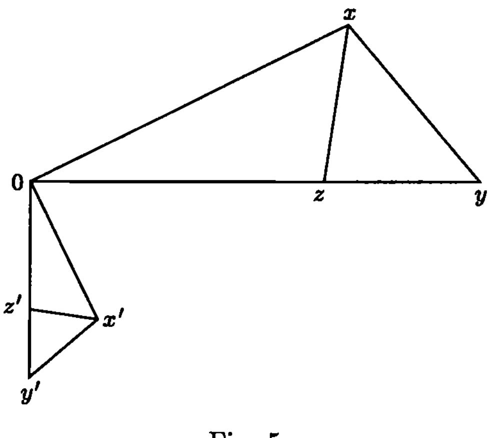

Fig. 5

LEMMA 19. $x + x = 0 \Rightarrow x = 0$.

Proof. Suppose $x \neq 0$. By Lemma 1 we get $P(x,x,x,x)$ since $x=-x$ hence $P(x,x,x,-x)$. So $x \neq 0$ then by A12 we have

(1) $\exists y \neq x : P(x,y,y,-x)$

thus by A13 $P(x,x,y,x)$.

By A6, A7, A8 we prove that $P(x,x,y,x) \Rightarrow P(x,0,y,y-x) \Rightarrow y=x$ what contradicts (1). ■

5. The axiom system of the complete theory

In this section we show the solution of a problem of an extension of the introduced in section 3 systems of axioms to complete theories.

We shall construct the theory for which models will be the similarity spaces over the real-closed fields.

We add the axiom AE to the axiom system **A1**:

AE $\forall x,y\ [x \neq 0 \Rightarrow \exists z\ (P(x,z,z,y) \vee P(x,-z,z,y))$.

Now, we verify the condition AE in an algebraic model.

Let $\Sigma(F)$ be a similarity space over Euclidean field F. We prove that the condition AE is satisfied in this space. To this aim the following lemma will be used:

LEMMA 20. *Let $x,y \in V$. The vectors x,y are linearly independent if and only if*

$$w(x,y) = (x \circ x)(y \circ y) - (x \circ y)^2 \neq 0.$$

Proof. If $y = \alpha x$ and $\alpha \in F$, then $w(x,y)=0$. We assume

(1) $\alpha x + \beta y = 0 \Rightarrow \alpha = \beta = 0$,

and suppose

(2) $w(x,y)=0$,

then the system of equations

$$\alpha(x \circ x) + \beta(x \circ y) = 0, \qquad \alpha(y \circ x) + \beta(y \circ y) = 0$$

has a solution different from zero.

There exists $\beta \neq 0$ such that the pair $(-\beta(y \circ x)/(x \circ x), \beta)$ is the solution of this system of equations.

If $t = -\beta(y \circ x)/(x \circ x)x + \beta y$ then $t \circ t = 0$, hence by Definition 1 (1.1) $t=0$. There exists a pair $(\alpha,\beta) \neq (0,0)$ such that $\alpha x + \beta y = 0$ what contradicts the condition (1). ■

THEOREM 4. $x \neq 0 \Rightarrow \exists z\ (P_F(x,z,z,y) \vee P_F(x,-z,z,y))$, *where P_F is the algebraic equivalent of relation P (see Definition 3).*

Proof. We have to prove that there exist λ, μ, z such that

$$(*)\qquad \begin{aligned} y \circ y = \lambda(z \circ z) \wedge z \circ z = \lambda(x \circ x) \wedge y \circ z = \lambda(x \circ z) \vee \\ y \circ y = \mu(z \circ z) \wedge z \circ z = \mu(x \circ x) \wedge y \circ z = -\mu(x \circ z). \end{aligned}$$

Let us notice that λ and μ satisfy the condition

$$\lambda^2 = \mu^2 = y \circ y / x \circ x.$$

We consider two possibilities:

(i) x, y are linearly independent.

Let us notice that $[\lambda(x \circ x) + (x \circ y)][\lambda(x \circ x) - (x \circ y)] = w(x, y)$ thus by Lemma 20 we have $\lambda(x \circ x) + (x \circ y) \neq 0$.

Let $s = \frac{y \circ y}{2(\lambda(x \circ x) + (x \circ y))}$. $\exists \alpha : \alpha^2 = s$ or $\alpha^2 = -s$ because F is the Euclidean field. Let $z = \alpha(x + 1/\lambda y)$. If we take $\mu = \lambda$ when $\alpha^2 = s$ and $\mu = -\lambda$ when $\alpha^2 = -s$ it is easy to check that the conditions (*) are satisfied.

(ii) $y = \beta x$.

There exist γ such that $\gamma^2 = \beta$ or $\gamma^2 = -\beta$ because F is the Euclidean field. If we take $z = \gamma x$ and $\mu = \lambda = \beta$ when $\gamma^2 = \beta$ or $z = \gamma x$ and $\mu = \lambda = -\beta$ when $\gamma^2 = -\beta$ we can easily check that the conditions $(*)$ are satisfied. ∎

We have proved that AE is satisfied in the similarity space over Euclidean field.

Let $\Sigma(F)$ be a similarity space for $\mathrm{Cn}(\mathbf{A1} \cup \{\mathrm{AE}\})$.

Now we prove that F is an Euclidean field.

THEOREM 5. $\forall \alpha \in F\ \exists \beta \in F\ (\beta^2 = \alpha \vee \beta^2 = -\alpha)$.

Proof. Let $x = 1$ and $y = \alpha$. By AE and Def. 1 (1.6) $\exists \lambda, \mu, z$ satisfying the conditions $(z^2 = \lambda \wedge \alpha^2 = \lambda z^2 \wedge \alpha z = \lambda z) \vee (z^2 = \mu \wedge \alpha^2 = \mu z^2 \wedge \alpha z = -\mu z)$. If $\alpha = 0$ then it suffices to take $\beta = 0$, if $\alpha \neq 0$, then $z \neq 0$. So we get $\alpha = z^2$ or $\alpha = -z^2$. ∎

Let us define an ordering in F as:

$$x \geq 0 \Leftrightarrow \exists r \in F : x = r^2.$$

From the condition AE is results that

$$\forall x\ (x \geq 0 \vee x \leq 0).$$

The monotony of multiplication is obvious. The monotony of addition results from the fact that F is a Pitagorean field because

$$x \geq 0 \wedge y \geq 0 \Rightarrow x = r^2 \wedge y = s^2 \Rightarrow x + y = r^2 + s^2 = t^2 \Rightarrow x + y \geq 0.$$

Thus an addition of the condition AE ensures that $\Sigma(F)$ is a similarity space over the orderly field.

Let us define Tarski's relation B in $\mathrm{Cn}(\mathbf{A2} \cup \{\mathrm{AE}\})$ system (cf. [4]).

DEFINITION 6.

$$B(0,x,y) \Leftrightarrow [\exists u \neq 0\ \forall t\ (P(u,t,t,u) \Rightarrow P(x,t,t,y)) \vee x = 0),$$
$$B(x,y,z) \Leftrightarrow B(0,y-x,z-x).$$

Now we verify the correctness of this definition.

Let $\Sigma(F)$ be a similarity space over a Euclidean field.

LEMMA 21. $[P_F(x,y,x,z) \Rightarrow y = z] \Leftrightarrow \exists \alpha \in F\ y = \alpha x$.

Proof $\Leftarrow$. In agreement with Def. 3 we assume that $z \circ z = y \circ y$ and $x \circ z = x \circ y$. We prove that $(z - \alpha x) \circ (z - \alpha x) = 0$ using the conditions (1.1), (1.2), (1.3) from Definition 1. Hence $z = \alpha x$ thus $z = y$.

Proof $\Rightarrow$. We define the function $\Pi : V \times V \to F$:

$$\Pi(x,y) = \begin{cases} \dfrac{x \circ y}{x \circ x}, & \text{when } x \neq 0 \\ 0, & \text{when } x = 0 \end{cases}$$

(cf. [1]). We assume that: $P_F(x,y,x,z) \Rightarrow y = z$ then by Definition 3 the conditions $z \circ z = y \circ y$ and $x \circ z = y \circ z$ are satisfied. Let us notice that $u = 2\Pi(x,y)x - y$ satisfies these conditions too hence $u = y$ i.e. $y = \Pi(x,y)x$. ■

Let us define $B_F(0,x,y) \Leftrightarrow \exists k \neq 0\ y = k^2 x$.

THEOREM 6.

$$x \neq 0 \wedge B_F(0,x,y) \Leftrightarrow [\exists u \neq 0\ \forall t\ (P_F(u,t,t,u) \Rightarrow P_F(x,t,t,y))].$$

Proof $\Leftarrow$. We assume that $\exists u \neq 0\ \forall t\ (P_F(u,t,t,u) \Rightarrow P_F(x,t,t,y))$, in particular we have $P_F(u,u,u,u) \Rightarrow P_F(x,u,u,y)$ hence $x \neq 0$. By A14 we have $\exists x'(P_F(u,x',x',u) \wedge L_F(0,x,x'))$, but by the assumption

$$P_F(u,x',x',u) \Rightarrow P_F(x,x',x',y).$$

By Lemma 12 we get

$$P_F(x,x',x',y) \wedge L_F(0,x,x') \Rightarrow L_F(0,x',y).$$

By Definition 3, Definition 5 and Lemma 21 we get

$$y \circ y = \lambda(x' \circ x') \wedge x' \circ x' = \lambda(x \circ x) \wedge x' \circ y = \lambda(x' \circ x) \wedge y = \beta x' \wedge x' = \gamma x$$

so $\beta(x' \circ x') = \lambda\gamma(x \circ x)$, what implies $\beta(x' \circ x') = \gamma(x' \circ x')$, since $x' \neq 0$ then $\beta = \gamma$.

Finally we get $y = \gamma^2 x$ and $\gamma \neq 0$.

Proof $\Rightarrow$. If $x \neq 0$ and $B_F(0,x,y)$, that is $y = k^2 x$, then it is easy to check that $u = 1/ky$ satisfies the required conditions.

Let us add the primary schema to axioms of the system $\mathrm{Cn}(\mathbf{A2} \cup \{\mathrm{AE}\})$ in the form:

AC: All sentences in the form:

$$\exists a \forall x, y(\varphi \wedge \psi \Rightarrow [\exists u \neq 0\, \forall t\, (P(u,t,t,u) \Rightarrow P(x-a,t,t,y-a)) \vee x = a])$$
$$\Rightarrow \exists c \forall x, y(\varphi \wedge \psi \Rightarrow [\exists w \neq 0\, \forall v\, (P(w,v,v,w) \Rightarrow P(c-x,v,v,y-x)) \vee x = c]),$$

where φ, ψ are expressions in which variables a, c, y and a, c, x, respectively, cannot be free.

THEOREM 7. *A theory* $\sigma^2 = \mathrm{Cn}(\mathbf{A2} \cup \{\mathrm{AE}\})$ *is the complete theory and each of its models is a two-dimensional similarity space* $\Sigma(F)$ *over a real-closed field* F.

COROLLARY. *Any model of the theory* σ^2 *is primary elementary equivalent to a two-dimensional similarity space* $\Sigma(R)$ *over the real field.*

Replacing an elementary schema by a nonprimary elementary axiom of continuity in the form $\exists a\ B(a, X, Y) \Rightarrow \exists c\ B(X, c, Y)$, we shall get a categorical theory σ^2.

($B(a, X, Y)$ is a short notation of:

$$B(a, X, Y) \Leftrightarrow$$
$$\forall x, y(x \in X \wedge y \in Y \Rightarrow [\exists u \neq 0\, \forall t\, (P(u,t,t,u) \Rightarrow P(x-a,t,t,y-a)) \vee x = a])$$

and by analogy for $B(X, c, Y)$.)

THEOREM 8. A*ny model of the* σ^2 *theory is isomorphic with two-dimensional similarity space* $\Sigma(R)$ *over the real field.*

The dimension free geometries cannot be complete, because when we add the axiom of actual dimension we get the consistent theory as before.

THEOREM 9. *The unique models of* $\mathrm{Cn}(\mathbf{A1} \cup \{\mathrm{AE}, \mathrm{AC}\})$ *system are the similarity space* $\Sigma(F)$ *over the real-closed fields of dimension* > 1.

If we add the axiom of dimension, we obtain the completeness of this theory.

Thus we have given rather simple procedure of extension of n-dimensional central geometry of similarities to the complete theory.

6. The problem of independency of axioms

Now, we discuss the problem of independency of axioms.

We introduce independence models for some axioms of axiom system **A1**.

The independence model for A12:

We assume $F = R$ in the model III. The relation P_R does not satisfy A12 because for $x \neq 0$ and $y \neq x$ $\neg(-x = yx^{-1}y)$.

The remaining axioms from **A1** are satisfied.

The independence model for A13:

We assume $F = C$ in the model III and $x = 1, y = i, x' = -i, y' = 1$. We get $-1 = i \cdot 1 \cdot i$ hence $P_c(x, y, y, -x)$ and $i = 1 \cdot \frac{-1}{i} \cdot 1$ hence $P_c(x', y', y', -x')$ but $-i \neq i \cdot 1 \cdot 1$ then $\neg P(x, y, y', x')$.

The remaining axioms from **A1** are satisfied.

The independence model for A14:

Let $F = Q(i), +-$ the addition in this field and

$$P_Q(x, y, z, u) \Leftrightarrow [x \neq 0 \wedge (xu = yz \vee x\overline{u} = y\overline{z})]$$

where if $z = a + bi$ then $\overline{z} = a - bi$.

We assume $x = 1,\ y = 1 + i$.

In order to satisfy the axiom A14 there must exist

$$z = \sqrt{2} \text{ or } z = \sqrt{-2} \text{ but } \sqrt{\pm 2} \notin Q(i).$$

The remaining axioms from **A1** are satisfied.

As the axioms A19 and A20 are the dimension axioms, they are independent.

Now we shall prove that the axiom A17 depends on the others.

THEOREM 10. $P(x, y, y, x) \wedge y \neq -x \Rightarrow P(x + y, x - y, x + y, y - x)$.

Proof. If $y = x$ then by Lemma 2 and Lemma 19 we get the thesis. So we assume

(i) $y \neq x$.

By A7, A8, Lemma 4 and (i) we get

$$P(x, y, y, x) \Rightarrow P(x, -y, y, -x) \Rightarrow P(x, x - y, y, y - x) \Rightarrow P(x - y, x, y - x, y).$$

By A1, Lemma 1 and A8 we prove that $P(x, x + x, y, y + y)$. By Lemma 8, Lemma 11 and Lemma 18 we get $P(x - y, x + x, y - x, y + y)$ hence by A7 and A8 it follows that $P(x - y, -y - x, y - x, -x - y)$.

By A7 and Lemma 4 obtain the thesis. ∎

The axiom A15 follows from the condition of simple form:

WA 15: $P(x, y, y, x) \wedge P(x, z, z, x) \Rightarrow P(y, z, z, y)$.

Therefore A15 can be replaced by WA15. The fig. 6 presents the interpretation of WA15 in the model II.

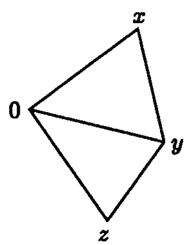

Fig. 6

THEOREM 11.

$$P(x, y-z, x, y-z) \wedge P(y, z-x, y, x-z) \wedge z \neq 0 \Rightarrow P(z, x-y, z, y-x)$$

Proof. We denote the conditions from the antecedent of the implication by (1), (2), (3) respectively.

Let us suppose that $z = x + y$. From (1) by A7, A8, A6 we get $x + x = 0$ which by Lemma 19 and A1 leads to contradiction, hence

(4) $z \neq x + y$.

From (1) and (2) by A3 and A8 we get

$$P(x, z-y+x, -x, z-y-x) \text{ and } P(y, z-x+y, -y, z-x-y).$$

Applying to these formulae and (4) by Lemma 5 and Lemma 6 we obtain

$$P(z-y-x, x, y-x-z, x) \text{ and } P(z-x-y, y, x-y-z, y).$$

By Lemma 1 and A11 we have

$$P(z-y-x, y-x-z, y-x-z, z-x-y) \text{ and}$$
$$P(z-y-x, x-y-z, x-y-z, z-x-y)$$

then by WA15 we have $P(y-x-z, x-y-z, x-y-z, y-x-z)$ hence by Theorem 10 it follows that $P(-z, -z, -x-x+y+y, z+z, y-y-x-x)$. Then by Lemma 18, Lemma 7 and Lemma 6 we obtain the thesis. ∎

It is possible to prove that if we add the axiom A19 to the axiom system **A1** then the axioms A16 and A18 become dependent on the others.

At first we prove the additional property of relations P and L, which we get using the axiom A19.

As a direct consequence of A19 we obtain

THEOREM 12. $P(x, x', x, x'') \wedge P(y, x', y, x'') \wedge x' \neq x'' \Rightarrow L(0, x, y)$.

Fig. 7 presents the interpretation of this theorem in the model II.

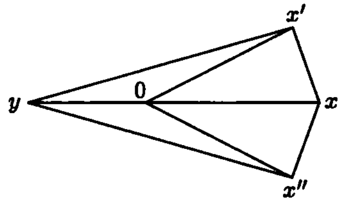

Fig. 7

THEOREM 13. $P(x, x', x, x'') \wedge x' \neq x'' \Rightarrow L(0, x, x' + x'')$.

Proof. By Lemma 8 the case $x'' = -x'$ is obvious so we assume:

(1) $x'' \neq -x'$. Then by A6, Lemma 1, Lemma 4 and A11 we prove that $P(x', x'', x'', x')$ hence by A8 we get $P(x', x' + x'', x'', x' + x'')$ which by (1) and Lemma 4 gives $P(x' + x'', x', x' + x'', x'')$.

Then the thesis follows from Theorem 12. ∎

Now, we present the proof of the dependence for A16.

THEOREM 14. $P(x, y, -x, y) \wedge P(x, z, -x, z) \Rightarrow P(x, y + z, -x, y + z)$.

Proof. The cases $y = 0$ and $z = 0$ are obvious. Let us assume $y \neq 0$ and $z \neq 0$. Applying the properties of Lemma 4, Lemma 19 and Theorem 12 we obtain $L(0, y, z)$, thus the thesis follows from the assumption $P(x, y, -x, y)$ and the properties of Lemma 17 and Lemma 18. ∎

Now we present the proof of the dependence for A18, which by Def. 8 is equivalent to:

THEOREM 15. $x \neq 0 \Rightarrow \exists z\ P(x, y, x, z) \wedge L(0, x, y + z)$.

Proof. By A10 and A12 we prove that: $\exists z P(x, y, x, z) \wedge y \neq z$ then the Theorem 13 yields $L(0, x, y + z)$. ∎

Taking into consideration these results we notice that it is possible to obtain the axiom system of plane by adding to **A0** the four independent axioms A12, A13, A14, A19 and the condition WA15.

7. Final remarks

The paper contains the axiom system **A1** of dimension-free geometry and the representation Theorems 1, 2 and 3).

In the paper we have presented the construction of the axiom system $\mathbf{A1} \cup \{\mathrm{AE}, \mathrm{AC}\}$ of the complete theory and the representation theorem (Theorem 9).

The problem of independence of axioms has been also considered. The paper contains the models of independence for the axioms A12, A13, A14 and the proof od dependence of the axiom A17.

In the paper we have discussed the problem of dependence of the axioms A16 and A18.

It is also interesting that the system of primitive notions considered here can lead to the axiomatics of one dimensional geometry and universal axiomatics that does not neglect any case of axiomatics of dimension-free geometry. Some problems will be discussed in a separate paper.

Let us notice that from 4-ary relation P we can pass to some 5-ary relation of similarity concerning the pair of similar triangles with common vertex (not necessarily fixed). Thus the obtained results can be used for axiomatization of the non-central Euclidean geometry.

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