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ON THE STABILITY OF THE GENERAL EULER–LAGRANGE FUNCTIONAL EQUATION

In 1940 S.M. Ulam [3] imposed at the University of Wisconsin the problem: "*Give conditions in order for a linear mapping near an approximately linear mapping to exist*". In 1978 P.M. Gruber [1] imposed the general problem: "*Suppose that a mathematical object satisfies a certain property approximately. Is it then possible to approximate this object by objects satisfying the property exactly?*" In 1989 J.M. Rassias [2] solved the above Ulam problem, or equivalently the Gruber problem for linear mappings. In this paper the author solves an analogous stability problem for the general 2-dimensional Euler–Lagrange functional inequality

$$(1)\quad \|f(a_1x_1+a_2x_2)+f(a_2x_1-a_1x_2)-(a_1^2+a_2^2)[f(x_1)+f(x_2)]\| \le c,$$

for all 2-dimensional vectors $(x_1, x_2) \in X^2$, with a normed linear space X, a constant c (independent of x_1, x_2) ≥ 0, mapping $f : X \to Y$ (where Y is a complete normed linear space), and any fixed reals a_1, a_2 such that $0 < m = a_1^2 + a_2^2 \neq 0$. Besides he introduces *the 2-dimensional quadratic weighted means*. According to P.M. Gruber [1] the afore-mentioned stability problems are of particular interest in probability theory and in the case of functional equations of different types.

DEFINITION 1. For X, Y as above **a non-linear mapping** $Q_2^a : X \to Y$, such that the functional equation

$$(1)'\quad Q_2^a(a_1x_1+a_2x_2)+Q_2^a(a_2x_1-a_1x_2)=(a_1^2+a_2^2)[Q_2^a(x_1)+Q_2^a(x_2)]$$

holds for all vectors $(x_1, x_2) \in X^2$ and for any fixed reals a_1, a_2 with $m = a_1^2 + a_2^2 > 1$, is called **2-dimensional quadratic.**

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Note that mapping Q_2^a may be called **quadratic**, as well, because the following **Euler–Lagrange identity**

$$(a_1x_1 + a_2x_2)^2 + (a_2x_1 - a_1x_2)^2 = (a_1^2 + a_2^2)[x_1^2 + x_2^2]$$

holds with any fixed reals a_1, a_2, and because the functional equation

$$Q_2^a(m^nx) = m^{2n}Q_2^a(x), \tag{2}$$

holds for all $x \in X$, all $n \in N$ and any fixed reals a_1, a_2 such that $m = a_1^2 + a_2^2 > 1$. In fact, substitution $x_1 = x_2 = 0$ in equation $(1')$ yields $2(1-m)Q_2^a(0) = 0$, or

$$Q_2^a(0) = 0, \quad m > 1. \tag{1a$'$}$$

Substituting $x_1 = x, x_2 = 0$ in $(1)'$ and employing $(1a)'$ one gets $Q_2^a(a_1x) + Q_2^a(a_2x) = m[Q_2^a(x) + Q_2^a(0)]$, or

$$Q_2^a(a_1x) + Q_2^a(a_2x) = mQ_2^a(x), \quad m > 1, \quad \text{for all } x \in X. \tag{2a}$$

Moreover, substituting $x_1 = a_1x, x_2 = a_2x$ in $(1')$ and using $(1a)'$, one finds that $Q_2^a(mx) + Q_2^a(0) = m[Q_2^a(a_1x) + Q_2^a(a_2x)]$, or

$$Q_2^a(a_1x) + Q_2^a(a_2x) = m^{-1}Q_2^a(mx), \ m > 1. \quad \text{for all } x \in X. \tag{2b}$$

The functional equations (2a), (2b) yield

$$Q_2^a(mx) = m^2Q_2^a(x), \quad m > 1, \quad \text{for all } x \in X. \tag{2c}$$

Then induction on $n \in N$ with $x \to m^{n-1}x$ yields equation (2).

DEFINITION 2. For X, Y as above a non-linear mapping $(\overline{Q}_2^a)_{w^2} : X \to Y$, such that

$$(\overline{Q}_2^a)_{w^2}(x) = \frac{Q_2^a(a_1x) + Q_2^a(a_2x)}{a_1^2 + a_2^2}, \quad m = a_1^2 + a_2^2 > 1, \tag{3}$$

holds for all $x \in X$ is called *a 2-dimensional quadratic weighted mean* for $m > 1$.

Note that in the case of equation $(1')$, formula (3), by (2a), is of the form

$$(\overline{Q}_2^a)_{w^2}(x) = Q_2^a(x), \quad m > 1, \quad \text{for all } x \in X. \tag{3a}$$

THEOREM 1. *Assume that $f : X \to Y$ with X, Y as above is a mapping for which there exists a constant $c \geq 0$, independent of x_1, x_2, such that the Euler–Lagrange functional inequality (1) holds for all vectors $(x_1, x_2) \in X^2$ and for any fixed reals a_1, a_2 such that $m = a_1^2 + a_2^2 > 1$. Then the limit*

$$Q_2^a(x) = \lim_{n\to\infty} m^{-2n}f(m^nx), \quad m > 1, \tag{4}$$

exists for all $x \in X$ and $Q_2^a : X \to Y$ is the unique 2-dimensional quadratic mapping satisfying functional equation (1′) *such that*

$$\|f(x) - Q_2^a(x)\| \le \frac{1}{2}\frac{3m^2-1}{(m-1)^2(m+1)}c, \ m > 1, \tag{5}$$

and

$$Q_2^a(x) = m^{-2n}Q_2^a(m^n x), \tag{5a}$$

hold for all $x \in X$, all $n \in N$, and any fixed reals a_1, a_2 such that $m = a_1^2 + a_2^2 > 1$.

Proof of existence. Substitution $x_1 = x_2 = 0$ in inequality (1) yields that

$$\|f(0)\| \le \frac{c}{2(m-1)}, \quad m > 1. \tag{6}$$

Moreover substituting $x_1 = x, x_2 = 0$ in inequality (1) and employing (6) and triangle inequality one concludes functional inequality

$$\|f(a_1x) + f(a_2x) - m[f(x) + f(0)]\| \le c, \tag{7}$$

or

$$\|\overline{f}_{w^2}(x) - f(x)\| \le \frac{c}{m} + \|f(0)\| \le \frac{c}{m} + \frac{c}{2(m-1)},$$

or

$$\|\overline{f}_{w^2}(x) - f(x)\| \le \frac{3m-2}{2m(m-1)}c, \quad m > 1, \tag{8}$$

where

$$\overline{f}_{w^2}(x) = \frac{f(a_1x) + f(a_2x)}{a_1^2 + a_2^2}, \quad m > 1, \tag{8a}$$

is the 2-**dimensional quadratic weighted mean** (for $m > 1$), according to the afore-mentioned Definition 2 of J.M. Rassias.

In addition, replacing $x_1 = a_1x, x_2 = a_2x$ in inequality (1) and using (6) and triangle inequality, one gets functional inequality

$$\|f(mx) + f(0) - m[f(a_1x) + f(a_2x)]\| \le c,$$

or

$$\|\overline{f}_{w^2}(x) - m^{-2}f(mx)\| \le \frac{c}{m^2} + \frac{1}{m^2}\|f(0)\| \le \frac{c}{m^2} + \frac{1}{m^2}\frac{c}{2(m-1)},$$

or

$$\|\overline{f}_{w^2}(x) - m^{-2}f(mx)\| \le \frac{2m-1}{2m^2(m-1)}c, \quad m > 1. \tag{9}$$

Functional inequalities (8), (9), and triangle inequality yield **the basic inequality**

$$\|f(x) - m^{-2}f(mx)\| \le \|f(x) - \overline{f}_{w^2}(x)\| + \|\overline{f}_{w^2}(x) - m^{-2}f(mx)\|$$
$$\le \left(\frac{3m-2}{2m(m-1)} + \frac{2m-1}{2m^2(m-1)}\right)c,$$

or

$$\|f(x) - m^{-2}f(mx)\| \le c_1(1 - m^{-2}), \quad m > 1, \tag{10}$$

where

$$c_1 = \frac{1}{2}\frac{3m^2-1}{(m-1)^2(m+1)}c, \quad m > 1. \tag{9a}$$

For instance, if $a_1 = a_2 = 1$, or $m = 2$, then $c_1 = \frac{11}{6}c$. Note that in this case **a better constant** $c_1 = \frac{1}{2}c$ may be found, if new substitution $x_1 = x_2 = x$ is applied into inequality (1) with $a_1 = a_2 = 1$. In fact, $\|f(2x) + f(0) - 4f(x)\| \le c$ with $\|f(0)\| \le \frac{c}{2}$, or

$$\|f(2x) - 4f(x)\| \le c + \|f(0)\| \le \frac{3}{2}c,$$

or

$$\|f(x) - 2^{-2}f(2x)\| \le \frac{1}{2}c(1 - 2^{-2}). \tag{11}$$

Thus $c_1 = \frac{1}{2}c$ $(< \frac{11}{6}c)$. Replacing now x with mx in (10), one concludes that

$$\|f(mx) - m^{-2}f(m^2x)\| \le c_1(1 - m^{-2}), \text{ or}$$

$$\|m^{-2}f(mx) - m^{-4}f(m^2x)\| \le c_1(m^{-2} - m^{-4}) \tag{10b}$$

holds for all $x \in X$ and any real $m > 1$.

Functional inequalities (10), (10b) and triangle inequality yield

$$\|f(x) - m^{-4}f(m^2x)\| \le \|f(x) - m^{-2}f(mx)\| + \|m^{-2}f(mx) - m^{-4}f(m^2x)\|$$
$$\le c_1[(1 - m^{-2}) + (m^{-2} - m^{-4})],$$

or

$$\|f(x) - m^{-4}f(m^2x)\| \le c_1(1 - m^{-4}), \quad m > 1, \tag{10c}$$

holds for all $x \in X$.

Similarly by induction on $n \in N$ with $x \to m^{n-1}x$ in (10) claim that **general functional inequality**

(12) $$\|f(x) - m^{-2n} f(m^n x)\| \le c_1(1 - m^{-2n}), \quad m > 1,$$

holds for all $x \in X$ and all $n \in N$.

In fact, basic inequality (10) with $x \to m^{n-1}x$ yields inequality

$$\|f(m^{n-1}x) - m^{-2} f(m^n x)\| \le c_1(1 - m^{-2}),$$

or

(12a)
$$\|m^{-2(n-1)} f(m^{n-1}x) - m^{-2n} f(m^n x)\| \le c_1(m^{-2(n-1)} - m^{-2n}), \quad m > 1,$$

for all $x \in X$. By induction hypothesis with $n \to n-1$ in (12), inequality

(12b) $$\|f(x) - m^{-2(n-1)} f(m^{n-1}x)\| \le c_1(1 - m^{-2(n-1)}), \quad m > 1,$$

holds for all $x \in X$. Thus functional inequalities (12a), (12b) and triangle inequality imply

$$\|f(x) - m^{-2n} f(m^n x)\|$$
$$\le \|f(x) - m^{-2(n-1)} f(m^{(n-1)}x)\| + \|m^{-2(n-1)} f(m^{(n-1)}x) - m^{-2n} f(m^n x)\|,$$

or

$$\|f(x) - m^{-2n} f(m^n x)\| \le c_1[(1 - m^{-2(n-1)}) + (m^{-2(n-1)} - m^{-2n})]$$
$$= c_1(1 - m^{-2n}), \quad m > 1,$$

completing the proof of the required functional inequality (12).

Claim now that the sequence $\{m^{-2n} f(m^n x)\}$ **converges**. From the general inequality (12), one proves that the above sequence is **a Cauchy sequence**. In fact, if $i > j > 0$, and $m > 1$, then

(13) $$\|m^{-2i} f(m^i x) - m^{-2j} f(m^j x)\| = m^{-2j} \|m^{-2(i-j)} f(m^i x) - f(m^j x)\|,$$

for all $x \in X$, and all $i, j \in N$. Setting $h = m^j x$ in (13) and employing general inequality (12), one concludes that

$$\|m^{-2i} f(m^i x) - m^{-2j} f(m^j x)\| = m^{-2j} \|m^{-2(i-j)} f(m^{i-j} h) - f(h)\|$$
$$\le m^{-2j} c_1(1 - m^{-2(i-j)}),$$

so

$$\|m^{-2i} f(m^i x) - m^{-2j} f(m^j x)\| \le c_1(m^{-2j} - m^{-2i}) < c_1 m^{-2j}.$$

Therefore

$$\lim_{j\to\infty} \|m^{-2i} f(m^i x) - m^{-2j} f(m^j x)\| = 0 \tag{13a}$$

completing the proof that the Cauchy sequence $\{m^{-2n} f(m^n x)\}$ converges because of the completeness property of Y.

Hence $Q_2^a = Q_2^a(x)$ is **a well-defined mapping** via the formula (4). This means that the limit (4) exists for all $x \in X$.

In addition **claim** that mapping Q_2^a satisfies the functional equation (1′) for all vectors $(x_1, x_2) \in X^2$.

In fact, it is clear from functional inequality (1) and the limit (4) that inequality

$$\begin{aligned} m^{-2n}\|f(a_1 m^n x_1 + a_2 m^n x_2) + f(a_2 m^n x_1 - a_1 m^n x_2) \\ -(a_1^2 + a_2^2)[f(m^n x_1) + f(m^n x_2)]\| \le m^{-2n} c, \quad m > 1, \end{aligned} \tag{14}$$

holds for all $x_1, x_2 \in X$, and all $n \in N$. Therefore from inequality (14) one gets

$$\begin{aligned} \| \lim_{n\to\infty} m^{-2n} f[m^n(a_1 x_1 + a_2 x_2)] + \lim_{n\to\infty} m^{-2n} f[m^n(a_2 x_1 - a_1 x_2)] \\ -(a_1^2 + a_2^2)[\lim_{n\to\infty} m^{-2n} f(m^n x_1) + \lim_{n\to\infty} m^{-2n} f(m^n x_2)]\| \\ \le c \lim_{n\to\infty} m^{-2n} = 0, \quad m > 1, \end{aligned}$$

or

$$\|Q_2^a(a_1 x_1 + a_2 x_2) + Q_2^a(a_2 x_1 - a_1 x_2) - (a_1^2 + a_2^2)[Q_2^a(x_1) + Q_2^a(x_2)]\| = 0,$$

or mapping Q_2^a satisfies the functional equation (1′) for all $x_1, x_2 \in X$, and $m > 1$. Thus Q_2^a is a 2-**dimensional quadratic mapping.** It is clear now, from general inequality (12), $n \to \infty$, and formula (4), that inequality (5) holds in X, completing the *existence* proof of Theorem 1.

Proof of uniqueness. *Let* $(Q_2^a)' : X \to Y$ *be* ***another*** *2-dimensional quadratic mapping satisfying functional equation* (1′) *such that*

$$\|f(x) - (Q_2^a)'(x)\| \le \frac{(3m^2 - 1)c}{2(m-1)^2(m+1)}, \quad m > 1, \tag{6$'$}$$

for all $x \in X$. *If there exists a 2-dimensional quadratic mapping* $Q_2^a : X \to Y$ *satisfying equation* (1′), *then*

$$Q_2^a(x) = (Q_2^a)'(x), \quad m > 1, \tag{15}$$

for all $x \in X$. To prove the above-mentioned **uniqueness** employ (5a) for Q_2^a and $(Q_2^a)'$, as well, so that

$$(6a)' \qquad (Q_2^a)'(x) = m^{-2n}(Q_2^a)'(m^n x), \quad m > 1,$$

holds for all $x \in X$, and all $n \in N$. Moreover triangle inequality and functional inequalities (5), (6)′ yield

$$\|Q_2^a(m^n x) - (Q_2^a)'(m^n x)\| \le \|Q_2^a(m^n x) - f(m^n x)\| + \|f(m^n x) - (Q_2^a)'(m^n x)\|,$$

or

$$(16) \qquad \|Q_2^a(m^n x) - (Q_2^a)'(m^n x)\| \le \frac{3m^2 - 1}{(m-1)^2(m+1)} c,$$

for all $x \in X$ and all $n \in N$. Then from (5a), (6a)′, and (16), one proves that

$$\|Q_2^a(x) - (Q_2^a)'(x)\| = \|m^{-2n} Q_2^a(m^n x) - m^{-2n}(Q_2^a)'(m^n x)\|,$$

or

$$(16a) \qquad \|Q_2^a(x) - (Q_2^a)'(x)\| \le \frac{3m^2 - 1}{(m-1)^2(m+1)} m^{-2n} c, \quad m > 1,$$

holds for all $x \in X$ and all $n \in N$.

Therefore, from (16a) and $n \to \infty$, one establishes

$$\lim_{n\to\infty} \|Q_2^a(x) - (Q_2^a)'(x)\| \le \frac{(3m^2 - 1)c}{(m-1)^2(m+1)} \lim_{n\to\infty} m^{-2n} = 0, \quad m > 1,$$

or

$$\|Q_2^a(x) - (Q_2^a)'(x)\| = 0,$$

or

$$(17) \qquad Q_2^a(x) = (Q_2^a)'(x), \quad m > 1,$$

for all $x \in X$, completing the proof of **uniqueness** and thus the **stability** of Theorem 1.

Note that an **analogous definition** to Definition 1 holds for quadratic mapping Q_2^a **for** $0 < m < 1$. Moreover functional equation

$$(2)' \qquad Q_2^a(m^{-n} x) = (m^{-n})^2 Q_2^a(x), \quad 0 < m < 1,$$

holds for all $x \in X$ and all $n \in N$. Similarly, substitution $x_1 = x_2 = 0$ in (1)′ yields

$$(1a)' \qquad Q_2^a(0) = 0, \quad 0 < m < 1.$$

Substituting $x_1 = \frac{x}{m}, x_2 = 0$, in (1)′ and employing (1a)′ one finds that

$$(2a)' \qquad Q_2^a\left(\frac{a_1}{m}x\right) + Q_2^a\left(\frac{a_2}{m}x\right) = mQ_2^a(m^{-1}x)$$

holds for all $x \in X$ and any fixed real a_1, a_2 such that $0 < m = a_1^2 + a_2^2 < 1$. In addition, substituting $x_1 = \frac{a_1}{m}x, x_2 = \frac{a_2}{m}x$, in (1)′ and employing (1a)′, one gets that

$$(2b)' \qquad Q_2^a\left(\frac{a_1}{m}x\right) + Q_2^a\left(\frac{a_2}{m}x\right) = m^{-1}Q_2^a(x)$$

holds for all $x \in X$ and any fixed real a_1, a_2 such that $0 < m = a_1^2 + a_2^2 < 1$.

Functional equations (2a)′, (2b)′ yield

$$(2c)' \qquad Q_2^a(m^{-1}x) = m^{-2}Q_2^a(x), \quad 0 < m < 1.$$

Then induction on n with $x \to m^{-(n-1)}x$ yields

$$(2d)' \qquad Q_2^a(m^{-n}x) = m^{-2n}Q_2^a(x), \quad 0 < m < 1,$$

completing the proof for equation (2)′.

DEFINITION 3. Let X be a normed linear space and Y a real complete normed linear space. Then **a non-linear mapping** $(\overline{Q}_2)_{w^2} : X \to Y$, such that

$$(3)' \qquad (\overline{Q}_2)_{w^2}(x) = m\left[Q_2^a\left(\frac{a_1}{m}x\right) + Q_2^a\left(\frac{a_2}{m}x\right)\right]$$

holds for all $x \in X$ and any fixed real $a_1 a_2$ such that $0 < m = a_1^2 + a_2^a < 1$, is called a *2-dimensional quadratic weighted mean for* $0 < m < 1$.

THEOREM 2. *Let X be a normed linear space and Y a real complete normed linear space. Assume in addition that $f : X \to Y$ is a mapping for which there exists a constant $c \geq 0$ such that the Euler–Lagrange functional inequality*

$$(4)' \quad \|f(a_1x_1 + a_2x_2) + f(a_2x_1 - a_1x_2) - (a_1^2 + a_2^2)[f(x_1) + f(x_2)]\| \leq c$$

holds for all 2-dimensional vectors $(x_1, x_2) \in X^2$, constant $c \geq 0$ independent of x_1, x_2 and any fixed reals a_1, a_2 such that $0 < m = a_1^2 + a_2^2 < 1$. Then the ***limit***

$$(5)' \qquad Q_2^a(x) = \lim_{n\to\infty} m^{2n} f(m^{-n}x), \quad 0 < m < 1,$$

exists *for all $x \in X$ and $Q_2^a : X \to Y$ is* ***the unique*** *2-dimensional* *quadratic mapping satisfying equation* $(1)'$, *such that*

$$(6)'' \qquad \|f(x) - Q_2^a(x)\| \leq \frac{(3-m^2)c}{2(1-m)^2(1+m)}, \quad 0 < m < 1,$$

and

$$(6\text{a}) \qquad Q_2^a(x) = m^{2n} Q_2^a(m^{-n}x),$$

hold for **all** $x \in X$, *all* $n \in N$ *and any fixed reals* a_1, a_2 *such that* $0 < m = a_1^2 + a_2^2 < 1$.

Proof. To prove Theorem 2 it is enough to establish **the general functional inequality**

$$(12)' \qquad \|f(x) - m^{2n} f(m^{-n}x)\| \leq c_2(1 - m^{2n}), \quad 0 < m < 1,$$

for all $x \in X$, and all $n \in N$, where $c_2 = \frac{(3-m^2)c}{2(1-m)^2(1+m)}, 0 < m < 1$. In fact, substitution $x_1 = x_2 = 0$ in inequality $(4)'$ yields that

$$(7)' \qquad \|f(0)\| \leq \frac{c}{2(1-m)}, \quad 0 < m < 1.$$

Moreover substituting $x_1 = \frac{x}{m}, x_2 = 0$ in inequality $(4)'$ and employing $(7)'$ and triangle inequality, one gets functional inequality

$$\left\| f\left(\frac{a_1}{m}x\right) + f\left(\frac{a_2}{m}x\right) - m\left[f\left(\frac{x}{m}\right) + f(0)\right]\right\| \leq c,$$

or

$$\|\overline{f}_{w^2}(x) - m^2 f(m^{-1}x)\| \leq mc + m^2\|f(0)\|,$$

or

$$(8)' \qquad \|\overline{f}_{w^2}(x) - m^2 f(m^{-1}x)\| \leq mc + m^2 \frac{c}{2(1-m)} = \frac{2m - m^2}{2(1-m)}c,$$

where

$$(8\text{a})' \qquad \overline{f}_{w^2}(x) = m\left[f\left(\frac{a_1}{m}x\right) + f\left(\frac{a_2}{m}x\right)\right], \quad 0 < m < 1,$$

is **the 2-dimensional quadratic weighted mean for** $0 < m < 1$, according to the afore-mentioned Definition 3 of J.M. Rassias. In addition, replacing $x_1 = \frac{a_1}{m}x, x_2 = \frac{a_2}{m}x$ in inequality $(4)'$, and by triangle inequality, one concludes functional inequality $\|f(x) + f(0) - \overline{f}_{w^2}(x)\| \leq c$, or

$$\|f(x) - \overline{f}_{w^2}(x)\| \leq c + \|f(0)\| \leq c + \frac{c}{2(1-m)}$$

or

$$(9)' \qquad \|f(x) - \overline{f}_{w^2}(x)\| \le \frac{3-2m}{2(1-m)}c, \ 0 < m < 1.$$

Functional inequalities (8)′, (9)′, and triangle inequality yield **the basic inequality**

$$\|f(x) - m^2 f(m^{-1}x)\| \le \|f(x) - \overline{f}_{w^2}(x)\| + \|\overline{f}_{w^2}(x) - m^2 f(m^{-1}x)\|$$
$$\le \left[\frac{3-2m}{2(1-m)} + \frac{2m-m^2}{2(1-m)}\right]c = \frac{3-m^2}{2(1-m)^2(1+m)}c(1-m^2),$$

or

$$(10)' \qquad \|f(x) - m^2 f(m^{-1}x)\| \le c_2(1-m^2), \quad 0 < m < 1,$$

where

$$(10a)' \qquad c_2 = \frac{(3-m^2)c}{2(1-m)^2(1+m)}, \quad 0 < m < 1.$$

By induction on $n \in N$ with $x \to m^{-(n-1)}x$ in (10)′, **claim** that above general inequality (12)′ holds for all $x \in X$ and all $n \in N$, for $0 < m < 1$. In fact, (10)′, (10a)′ with $x \to m^{-(n-1)}x$ yield inequality

$$\|f(m^{-(n-1)}x) - m^2 f(m^{-n}x)\| \le c_2(1-m^2),$$

or

$$(12a)' \qquad \|m^{2(n-1)} f(m^{-(n-1)}x) - m^{2n} f(m^{-n}x)\| \le c_2(m^{2(n-1)} - m^{2n}),$$

for all $x \in X$ and any fixed real m such that $0 < m < 1$.

By induction hypothesis with $n \to n-1$ in (12)' inequality

$$(12b)' \qquad \|f(x) - m^{2(n-1)} f(m^{-(n-1)}x)\| \le c_2(1 - m^{2(n-1)}),$$

holds for all $x \in X$, and any fixed real m such that $0 < m < 1$. Thus functional inequalities (12a)′, (12b)′ and triangle inequality imply

$$\|f(x) - m^{2n} f(m^{-n}x)\| \le \|f(x) - m^{2(n-1)} f(m^{-(n-1)}x)\| +$$
$$\|m^{2(n-1)} f(m^{-(n-1)}x) - m^{2n} f(m^{-n}x)\| \le c_2[(1-m^{2(n-1)}) + (m^{2(n-1)} - m^{2n})]$$
$$= c_2(1-m^{2n}), \quad 0 < m < 1,$$

completing the proof of the required functional inequality (12)′.

The rest of the proof of Theorem 2 is omitted as similar to the corresponding proof of Theorem 1.

EXAMPLE. Take $f : R \to R$ being a real function $f(x) = x^2 + k$, with constant k such that $|k| \leq \frac{c}{2(1-m)}, 0 < m < 1$.

Moreover, let a unique quadratic mapping $Q_2^a : R \to R$ exist, such that

$$Q_2^a(x) = \lim_{n\to\infty} m^{2n}[(m^{-n}x)^2 + k] = x^2, \quad 0 < m < 1.$$

Therefore inequality $(6)''$ holds, since

$$\|f(x) - Q_2^a(x)\| = \|(x^2 + k) - x^2\| = |k| \leq \frac{c}{2(1-m)}$$

and

$$\frac{1}{1-m} < \frac{3-m^2}{(1-m)^2(1+m)}, \; 0 < m < 1.$$

Note that if $m > 1$, then take any real constant k such that $|k| \leq \frac{c}{2(m-1)}$.

THEOREM 3. *Let X be a normed linear space and Y a real complete normed linear space. Assume in addition that $f : X \to Y$ is a mapping for which there exist constants $c, c' \geq 0$ such that the Euler–Lagrange functional inequality*

$$(4)'' \qquad \|f(a(x_1 + x_2)) + f(a(x_1 - x_2)) - [f(x_1) + f(x_2)]\| \leq c$$

holds for all 2-dimensional vectors $(x_1, x_2) \in X^2$, $\|f(0)\| \leq c'$, nonnegative constants c, c' independent of x_1, x_2 and $a = \frac{1}{\sqrt{2}}$ (or: $= -\frac{1}{\sqrt{2}}$). Then the ***limit***

$$(5)'' \qquad Q_2(x) = \lim_{n\to\infty} 2^{-n} f((2a)^n x), \quad m = 2a^2 = 1,$$

exists *for all $x \in X$ and $Q_2 : X \to Y$ is the* ***unique*** *2-dimensional quadratic mapping satisfying functional equation*

$$(1)'' \quad Q_2(a(x_1 + x_2)) + Q_2(a(x_1 - x_2)) = Q_2(x_1) + Q_2(x_2), \quad m = 1,$$

and $Q_2(0) = 0$, such that

$$(6)''' \qquad \|f(x) - Q_2(x)\| \leq c + c', \quad m = 1,$$

and

$$(6a)'' \qquad Q_2(x) = 2^{-n} Q_2((2a)^n x), \quad m = 1,$$

hold for all $x \in X$ and all $n \in N$.

Note that $a_1 = a_2 = a$ in Theorem 3, and thus $m = a_1^2 + a_2^2 = 2a^2 = 2(\pm\frac{1}{\sqrt{2}})^2 = 1$. Thus Theorem 3 is **a singular case** of Theorems 1, 2.

Proof. Substitution $x_1 = x_2 = x$ in inequality $(4)''$ yields that **basic inequality** $\|f(2ax) + f(0) - 2f(x)\| \le c$ or from condition: $\|f(0)\| \le c'$

$$\|f(x) - 2^{-1} f(2ax)\| \le (c + c')(1 - 2^{-1}), \tag{18}$$

holds for all $x \in X$. By induction on n with $x \to (2a)^{n-1}x$ in basic inequality (18), one concludes the **general inequality**

$$\|f(x) - 2^{-n} f((2a)^n x)\| \le (c + c')(1 - 2^{-n}), \tag{18a}$$

for all $x \in X$, all $n \in N$ and $a = \pm\frac{1}{\sqrt{2}}$.

Note that substitution $x_1 = x_2 = x$ in equation $(1)''$ yields $Q_2(2ax) + Q_2(0) = 2Q_2(x)$, or from $Q_2(0) = 0$

$$Q_2(x) = 2^{-1} Q_2(2ax). \tag{18b}$$

Then, by induction on $n \in N$ with $x \to (2a)^{n-1}x$ in (18b), one establishes $(6a)''$.

The rest of the proof of Theorem 3 is omitted as similar to the proof of Theorem 1.

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