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ON SOME RODRIGUES TYPE FRACTIONAL DERIVATIVE FORMULAE

The present paper contains fractional derivative formulae for hypergeometric functions of the type of some orthogonal polynomials analogous to the Rodrigues formulae for Laguerre, Jacobi, Ultraspherical, Legendre and Gegenbauer polynomials.

1. Introduction

In 1731 Euler extended the derivative formula for real or complex numbers λ and complex number z (see [1])

$$\begin{aligned} D_z^n\{z^\lambda\} &= \lambda(\lambda-1)\dots(\lambda-n+1)z^{\lambda-n} \\ &= \frac{\Gamma(\lambda+1)}{\Gamma(\lambda-n+1)}z^{\lambda-n} \quad (n=0,1,2,\dots) \end{aligned}$$

to the general form

$$D_z^\mu\{z^\lambda\} = \frac{\Gamma(\lambda+1)}{\Lambda(\lambda-\mu+1)}z^{\lambda-\mu} \tag{1.1}$$

where μ is an arbitraty complex number.

In 1974, H.L. Manocha and B.L. Sharma [1] derived the following formula for the fractional derivative of a product of two functions as a series of fractional derivatives of the individual functions

$$D_x^\lambda(UV) = \sum_{n=0}^{\infty}\binom{\lambda}{n}D_x^{\lambda-n}(U)D_x^n(V) \tag{1.2}$$

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where U and V are functions of real variable x and λ is any real or complex number.

The binomial coefficient $\binom{\lambda}{n}$ is defined by (see [2])

$$\binom{\lambda}{n} = \frac{(-1)^n(-\lambda)_n}{n!}, \tag{1.3}$$

where

$$(-\lambda)_n = (-\lambda)(-\lambda+1)(-\lambda+2)\dots(-\lambda+n-1),$$
$$(-\lambda)_o = 1.$$

Let us recall that Rodrigues formulae for Laguerre polynomials $L_n^{(\alpha)}(x)$, Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$, Ultraspherical polynomials $P_n^{(\alpha,\alpha)}(x)$, Legendre polynomials $P_n(x)$ and Gegenbauer polynomials $C_n^{\nu}(x)$ are given for arbitrary real or complex numbers α, β, ν and real number x by

$$L_n^{(\alpha)}(x) = \frac{x^{-\alpha}e^x}{n!}D^n[e^{-x}x^{n+\alpha}], \tag{1.4}$$

$$P_n^{(\alpha,\beta)}(x) = \frac{(x-1)^{-\alpha}(x+1)^{-\beta}}{2^n n!}D^n[(x-1)^{n+\alpha}(x+1)^{n+\beta}], \tag{1.5}$$

$$P_n^{(\alpha,\alpha)}(x) = \frac{(x^2-1)^{-\alpha}}{2^n n!}D^n[(x^2-1)^{n+\alpha}], \tag{1.6}$$

$$P_n(x) = \frac{1}{2^n n!}D^n[(x^2-1)^n], \tag{1.7}$$

$$C_n^{\nu}(x) = \frac{(2\nu)_n(x^2-1)^{-\nu+\frac{1}{2}}}{2^n n!(\nu+\frac{1}{2})_n}D^n[(x^2-1)^{n+\nu-\frac{1}{2}}], \tag{1.8}$$

(see cf. [2]).

2. Hypergeometric functions of the type of some well known polynomials

Here we introduce definitions of some hypergeometric functions analogous to the definitions of some well-known orthogonal polynomials such as Laguerre, Jacobi, Ultraspherical, Legendre and Gegenbauer ones (see [2])

(A) **Laguerre type function.** For an arbitrary real or complex numbers λ, α and real number x, the Laguerre type function is denoted by the symbol $L_\lambda^{(\alpha)}(x)$ and is defined as

$$L_\lambda^{(\alpha)}(x) = \frac{\Gamma(1+\alpha+\lambda)}{\Gamma(1+\alpha)\Gamma(1+\lambda)}{}_1F_1\left[\begin{matrix} -;x \\ 1+\alpha; \end{matrix}\right]. \tag{2.1}$$

The symbol ${}_1F_1$ is a confluent hypergeometric function defined by

$$(2.1a)\qquad {}_1F_1\left[\begin{matrix}-; \\ 1+\alpha;\end{matrix}\; x\right] = \sum_{n=0}^{\infty} \frac{x^n}{n!(1+\alpha)_n}, \qquad \alpha \neq -1, -2, \ldots$$

and $(1+\alpha)_n$ is defined as

$$(2.1b)\qquad \left.\begin{matrix}(a)_n = a(a+1)(a+2)\ldots(a+n-1) \\ (a)_0 = 1\end{matrix}\right\} \qquad \text{(see [2]).}$$

(B) **Jacobi type function.** For an arbitrary real or complex numbers λ, α, β the Jacobi type function is denoted by the symbol $P_\lambda^{(\alpha,\beta)}(x)$ and is defined as

$$(2.2)\quad P_\lambda^{(\alpha,\beta)}(x) = \frac{\Gamma(1+\alpha+\lambda)}{\Gamma(1+\alpha)\Gamma(1+\lambda)} \left(\frac{x+1}{2}\right)^\lambda {}_2F_1\left[\begin{matrix}-\lambda, -\beta-\lambda; \\ 1+\alpha;\end{matrix}\; \tfrac{x-1}{x+1}\right],$$

where ${}_2F_1$ is a Gauss Hypergeometric function defined by

$$(2.2a)\quad {}_2F_1\left[\begin{matrix}-\lambda, -\beta-\lambda; \\ 1+\alpha;\end{matrix}\; \tfrac{x-1}{x+1}\right] = \left(\sum_{n=0}^{\infty} \frac{(-\lambda)_n(-\beta-\lambda)_n}{n!(1+\alpha)_n}\left(\frac{x-1}{x+1}\right)^n\right).$$

In particular described below the Jacobi type functions are Ultraspherical type, Legendre type and Gegenbauer type functions.

(C) **Ultraspherical type function.** It is denoted by the symbol $P_\lambda^{(\alpha,\alpha)}(x)$ and is defined for arbitrary real or complex number α and real number x as

$$(2.3)\quad P_\lambda^{(\alpha,\alpha)}(x) = \frac{\Gamma(1+\alpha+\lambda)}{\Gamma(1+\alpha)\Gamma(1+\lambda)} \left(\frac{x+1}{2}\right)^\lambda {}_2F_1\left[\begin{matrix}-\lambda, -\alpha-\lambda; \\ 1+\alpha;\end{matrix}\; \tfrac{x-1}{x+1}\right].$$

(D) **Legendre type function.** It is denoted by the symbol $P_\lambda(x)$ and is defined for arbitrary real/complex number λ and real number x as

$$(2.4)\qquad P_\lambda(x) = \left(\frac{x+1}{2}\right)^\lambda {}_2F_1\left[\begin{matrix}-\lambda, -\lambda; \\ 1\end{matrix}\; \tfrac{x-1}{x+1}\right].$$

(E) **Gegenbauer type function.** It is denoted by the symbol $C_\lambda^\nu(x)$ and is defined for arbitrary real or complex numbers ν, λ and real number x by the following relation

$$C_\lambda^\nu(x) = \frac{\Gamma(2\nu+\lambda)\Gamma(\nu+\frac{1}{2})}{\Gamma(2\nu)\Gamma(\lambda+\nu+\frac{1}{2})} P_\lambda^{(\nu-\frac{1}{2},\nu-\frac{1}{2})}(x). \tag{2.5}$$

We also need to extend the symbol $(a)_n$ to an arbitrary real or complex number λ. We shall denote it by $(a)_\lambda$ and define by

$$(a)_\lambda = \frac{\Gamma(a+\lambda)}{\Gamma(a)}. \tag{2.6}$$

3. Rodrigues type formulae

A. Rodrigues type fractional derivative formula for Laguerre type function $L_\lambda^{(\alpha)}(x)$ is given below by

$$L_\lambda^{(\alpha)}(x) = \frac{x^{-\alpha}e^x}{\Gamma(1+\lambda)} D^\lambda[e^{-x}x^{\lambda+\alpha}]. \tag{3.1}$$

Proof of (3.1). Using (1.2) the R.H.S. of (3.1) is equal to

$$\begin{aligned}
&\frac{x^{-\alpha}e^x}{\Gamma(1+\lambda)}\left(\sum_{n=0}^{\infty}\binom{\lambda}{n}[D^{\lambda-n}x^{\lambda+\alpha}]D^n e^{-x}\right)\\
&= \frac{x^{-\alpha}e^x}{\Gamma(1+\lambda)}\left(\sum_{n=0}^{\infty}\frac{(-1)^n(-\lambda)_n(1+\alpha)_\lambda x^{n+\alpha}(-1)^n e^{-x}}{n!(1+\alpha)_n}\right)\\
&= \frac{\Gamma(1+\alpha+\lambda)}{\Gamma(1+\lambda)\Gamma(1+\alpha)}\,{}_1F_1\left[\begin{matrix}-\lambda; & x\\ 1+\alpha; & \end{matrix}\right] = L_\lambda^{(\alpha)}(x)
\end{aligned}$$

in view of (2.1).

This completes the proof of (3.1). If $\lambda = n$, a positive integer, (3.1) reduces to Rodrigues formula (1.4) for Laguerre polynomials.

B. Similarly, Rodrigues type fractional derivative formula for the Jacobi type function is as follows

$$P_\lambda^{(\alpha,\beta)}(x) = \frac{(x-1)^{-\alpha}(x+1)^\beta}{2^\lambda\Gamma(1+\lambda)} D^\lambda[(x-1)^{\lambda+\alpha}(x+1)^{\lambda+\beta}]. \tag{3.2}$$

Proof of (3.2). Using (1.2) the R.H.S. of (3.2) is equal to

$$\frac{(x-1)^{-\alpha}(x+1)^{-\beta}}{2^{\lambda}\Gamma(1+\lambda)}\left(\sum_{n=0}^{\infty}\binom{\lambda}{n}[D^{\lambda-n}(x-1)^{\lambda+\alpha}]D^{n}(x+1)^{\lambda+\beta}\right)$$

$$=\frac{(x-1)^{-\alpha}(x+1)^{-\beta}}{2^{\lambda}\Gamma(1+\lambda)}$$

$$\times\left(\sum_{n=0}^{\infty}\binom{\lambda}{n}\frac{(1+\alpha)_{\lambda}(x-1)^{n+\alpha}(1+\beta)_{\lambda}(x+1)^{\lambda-n+\beta}}{(1+\alpha)_{n}(1+\beta)_{\lambda-n}}\right)$$

$$=\frac{\Gamma(1+\alpha+\lambda)(x+1)^{\lambda}}{2^{\lambda}\Gamma(1+\lambda)\Gamma(1+\alpha)}\left(\sum_{n=0}^{\infty}\frac{(-1)^{n}(-\lambda)_{n}(-\beta-\lambda)_{n}}{n!(1+\alpha)_{n}(-1)^{n}}\left(\frac{x-1}{x+1}\right)^{n}\right).$$

Using (1.3) the latter equals

$$\frac{\Gamma(1+\alpha+\lambda)}{\Gamma(1+\alpha)\Gamma(1+\lambda)}\left(\frac{x+1}{2}\right)^{\lambda}{}_{2}F_{1}\left[\begin{matrix}-\lambda,-\lambda-\beta; & \\ & \frac{x-1}{x+1}\\ 1+\alpha; & \end{matrix}\right]=P_{\lambda}^{(\alpha,\beta)}(x)$$

in view of (2.2).

This completes the proof of (3.2). For $\lambda = n$, a positive integer, (3.2) reduces to the Rodrigues formula (1.5) for Jacobi polynomials.

C. For $\beta = \alpha$, (3.2) reduces to the following fractional derivative formula for Ultraspherical type function

$$P_{\lambda}^{(\alpha,\alpha)}(x)=\frac{(x^{2}-1)^{-\alpha}}{2^{\lambda}\Gamma(1+\lambda)}D^{\lambda}[(x^{2}-1)^{\lambda+\alpha}] \tag{3.3}$$

which for $\lambda = n$, a positive integer, reduces to the Rodrigues formula (1.6) for Ultraspherical polynomials.

D. For $\alpha = 0$, (3.3) reduces to the following Rodrigues type fractional derivative formula for Legendre type function

$$P_{\lambda}(x)=\frac{1}{2^{\lambda}\Gamma(1+\lambda)}D^{\lambda}[(x^{2}-1)^{\lambda}]. \tag{3.4}$$

For $\lambda = n$, a positive integer, (3.4) becomes the Rodrigues formula (1.7) for Legendre polynomials.

E. Now replacing α by $\nu - \frac{1}{2}$ in (3.3) and making use of the relation (2.5) we get the following Rodrigues type fractional derivative formula for Gegenbauer type function

$$C_{\lambda}^{\nu}(x)=\frac{\Gamma(2\nu+\lambda)\Gamma(\nu+\frac{1}{2})(x^{2}-1)^{-\nu+\frac{1}{2}}}{2^{\lambda}\Gamma(2\nu)\Gamma(\nu+\lambda+\frac{1}{2})\Gamma(1+\lambda)}D^{\lambda}[(x^{2}-1)^{\lambda+\nu-\frac{1}{2}}]. \tag{3.5}$$

A similar formula for Hermite polynomial is not apparent.

The Rodrigues type fractional derivative formulae (3.1-5) are expected to be of great importance in the study of special functions and fractional calculus just like the generalization of factorial to Gamma function has gained much importance due to its significant contributions in these and other related areas.

References

[1] H. L. Manocha and B. L. Sharma, *Fractional derivatives and summations*, J. Indian Math. Soc. (N.S.) 38 (1974), 371–382.

[2] E. D. Rainville, *Special Functions*. Macmillan, New York; Reprinted by Chelsea Publ. Co., Bronx, New York, 1971.

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