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Satendra Kumar Vaish

REARRANGEMENTS OF THE COEFFICIENTS OF ENTIRE FUNCTIONS

1. Introduction

Let $\pi(n' \to n)$ denote a rearrangement of all non-negative integers by which the integer n is replaced by the integer n'. Guha in [1] proved the following:

THEOREM A. *A necessary and sufficient condition that a rearrangement $\pi(n' \to n)$ keeps unaltered the radius of convergence of any power series $\sum_{n=0}^{\infty} a_n z^n$ is that $n' = n + o(n)$.*

In this paper we use this theorem to show that the same characterization of rearrangements keeps unaltered the (p,q)-order of any entire function $f(z) = \sum_{n=0}^{\infty} a_n z^n$ having an index-pair (p,q), $(p \geq 2,\ q \geq 1,\ p \geq q)$. However, we see by an example that the characterization fails in the case of (p,q)-type.

The following notations are frequently used in the sequal:

NOTATION 1. $\exp^{[0]} x = \log^{[0]} x = x$;

$$\exp^{[m]} x = \log^{[-m]} x = \exp(\exp^{[m-1]} x) = \log(\log^{[-m-1]} x),$$

$m = 0, \pm 1, \pm 2, \ldots$. Throughout this paper whenever $(\log^{[m]} x)^{\alpha} (0 < \alpha < \infty)$ occurs, it is understood that x is such that this expression is a real number.

NOTATION 2.

$$P_t(\alpha) \equiv P_t(\alpha, p, q) = \begin{cases} \alpha & \text{if } p > q \\ t + \alpha & \text{if } p = q = 2 \\ \max(1, \alpha) & \text{if } 3 \leq p = q < \infty \\ \infty & \text{if } p = q = \infty, \end{cases}$$

where $0 \leq \alpha \leq \infty$ and $0 \leq t \leq 1$.

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The main result in this paper is the following:

2. THEOREM. *A necessary and sufficient condition that a rearrangement $\pi(n' \to n)$ keeps unaltered the (p,q)-order of an entire function $f(z) = \sum_{n=0}^{\infty} a_n z^n$ having an index-pair (p,q), $(p \geq 2,\ q \geq 1,\ p \geq q)$ is that $n' = n + o(n)$.*

Proof. If $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is an entire function then, by Theorem A, the rearranged series $\sum_{n=0}^{\infty} a_{n'} z^n$ also represents an entire function $f_1(z)$, say. Let $\rho(p,q)$ and $\rho_1(p,q)$ be the (p,q)-orders of $f(z)$ and $f_1(z)$, respectively. Then by [2]:

$$\rho \equiv \rho(p,q) = P_1(L(p,q)), \tag{2.1}$$

where

$$L(p,q) = \limsup_{n\to\infty} \frac{\log^{[p-1]} n}{\log^{[q-1]}(n^{-1}\log|a_n|^{-1})} \tag{2.2}$$

and

$$\rho_1 \equiv \rho_1(p,q) = P_1(L_1(p,q)), \tag{2.3}$$

where

$$L_1(p,q) = \limsup_{n\to\infty} \frac{\log^{[p-1]} n}{\log^{[q-1]}(n^{-1}\log|a_{n'}|^{-1})}. \tag{2.4}$$

Sufficiency. Let $n' = n + o(n)$. Firstly, we suppose that $0 < \rho < \infty$. Then for any $\varepsilon > 0$ and for all large n, we have

$$|a_n| < \exp\{-n\exp^{[q-2]}(\log^{[p-2]} n')^{1/(L+\varepsilon)}\}. \tag{2.5}$$

Since $n' \to \infty$ with n, and distinct values of n correspond to distinct values of n' and vice versa, we have for all large n,

$$|a_{n'}| < \exp\{-n'\exp^{[q-2]}(\log^{[p-2]} n')^{1/(L+\varepsilon)}\}. \tag{2.6}$$

Since $n' = n + o(n)$, therefore for suitably chosen $\varepsilon > 0$ and for any given number $d > 0$, we have, for all large n,

$$|a_{n'}| < \exp\{-n\exp^{[q-2]}(\log^{[p-2]} n)^{1/(L+d)}\}. \tag{2.7}$$

Further, since n' tends to infinity with n, it follows that for infinitely many n,

$$|a_{n'}| > \exp\{-n'\exp^{[q-2]}(\log^{[p-2]} n')^{1/(L-\varepsilon)}\}. \tag{2.8}$$

Choosing ε suitably, we find, for infinitely many n, that

$$|a_{n'}| > \exp\{-n\exp^{[q-2]}(\log^{[p-2]} n)^{1/(L-d)}\}. \tag{2.9}$$

Notice that (2.7) and (2.9) give

$$\limsup_{n\to\infty} \frac{\log^{[p-1]} n}{\log^{[q-1]}(n^{-1}\log|a_{n'}|^{-1})} = L(p,q) \quad \text{or} \quad L_1(p,q) = L(p,q).$$

So, $\rho_1 = \rho$.

The above argument with suitable alternation solves the cases $\rho = 0$ and $\rho = \infty$.

Necessity. Let $\rho_1 = \rho$. Then, for any given $\varepsilon > 0$

$$n \exp^{[q-2]}(\log^{[p-2]} n)^{1/(L+\varepsilon)} \leq \log |a_{n'}|^{-1} \tag{2.10}$$

for all large n so for n', while

$$n' \exp^{[q-2]}(\log^{[p-2]} n')^{1/(L-\varepsilon)} \geq \log |a_{n'}|^{-1} \tag{2.11}$$

for an infinity of n'.

Observe that (2.10) and (2.11) give

$$\liminf_{n\to\infty} \frac{n' \exp^{[q-2]}(\log^{[p-2]} n')^{1/L}}{n \exp^{[q-2]}(\log^{[p-2]} n)^{1/L}} \geq 1. \tag{2.12}$$

Interchanging n and n', we get

$$\liminf_{n\to\infty} \frac{n \exp^{[q-2]}(\log^{[p-2]} n)^{1/L}}{n' \exp^{[q-2]}(\log^{p-2]} n')^{1/L}} \geq 1,$$

or

$$\limsup_{n\to\infty} \frac{n' \exp^{[q-2]}(\log^{[p-2]} n')^{1/L}}{n \exp^{[q-2]}(\log^{[p-2]} n)^{1/L}} \leq 1. \tag{2.13}$$

Combining (2.12) and (2.13), we see that

$$\frac{n' \exp^{[q-2]}(\log^{[p-2]} n')^{1/L}}{n \exp^{[q-2]}(\log^{[p-2]} n)^{1/L}} \to 1 \quad \text{as } n \to \infty.$$

Taking logarithms, we get

$$\frac{\exp^{[q-2]}(\log^{[p-2]} n')^{1/L}}{\exp^{[q-2]}(\log^{[p-2]} n)^{1/L}} \to 1 \quad \text{as } n \to \infty.$$

Hence, $\frac{n'}{n} \to 1$ as requires.

This completes the proof of our theorem.

3. If $0 < V < \infty$, then the (p,q)-type T of an entire function $f(z)$ having (p,q)-order $\rho(b < \rho < \infty)$ is defined [3] as $T = MV$, where

$$b = 1 \text{ if } p = q, \ b = 0 \text{ if } p > q;$$

$$V \equiv V(p,q) = \limsup_{n\to\infty} \frac{\log^{[p-2]} n}{(\log^{[q-1]} |a_n|^{-1/n})^{\rho - A}},$$

$$A = 1 \text{ if } (p,q) = (2,2), \ A = 0 \text{ if } (p,q) \neq (2,2);$$

and

$$M \equiv M(p,q) = \begin{cases} (\rho-1)^{(\rho-1)}/\rho^{\rho}, & \text{if } (p,q) = (2,2) \\ 1/(e^{\rho}), & \text{if } (p,q) = (2,1) \\ 1, & \text{if } p \geq 3. \end{cases}$$

Now, we present an example that the type of an entire function need not remain unaltered under the characterisation of the rearrangements $n' = n + o(n)$. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be an entire function defined as follows

$$a_n = \begin{cases} (\log^{[p-q-1]} n)^{-n} & \text{for } n \geq 2 \\ 0 & \text{for } n = 0, 1, \end{cases}$$

where $p \geq 2$, $q \geq 1$, $p \geq q$.

Then, the (p, q)-order ρ of $f(z)$ is given by

$$\rho = \begin{cases} 1 & \text{for } p > q \\ 2 & \text{for } p = q = 2 \\ 1 & \text{for } 3 \leq p = q < \infty \\ \infty & \text{for } p = q = \infty, \end{cases}$$

and for $b < \rho < \infty$, the (p, q)-type T having the value

$$T = \begin{cases} 1/4 & \text{for } (p, q) = (2, 2) \\ 1/e & \text{for } (p, q) = (2, 1) \\ 1 & \text{for } p \geq 3. \end{cases}$$

Set, $n' = t(n) = [(1 + \frac{1}{\log n})n]$ (integral part) for $n = 2, 3, 4, \ldots$ and $n' = n$ for $n = 0, 1$.

It can be easily seen that $n' = n + o(n)$.

For simplicity, we consider the case $(p, q) = (2, 1)$. Then, the (2,1)-type of rearranged series

$$f_1(z) = \sum_{n=2}^{\infty} z^n / t(n)^{t(n)}$$

is equal to $1/e^2$ and thus is different from the (2,1)-type of $f(z)$.

References

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I/43A-8, LAI BAGH,
G.B. PANT UNIVERSITY,
PANTNAGAR-263145
DISTT. NAINITAL (U.P.), INDIA

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