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STRICT CONVEXITY IN LINEAR n-NORMED SPACES

1. Let n be a positive integer, X be a linear space of dimension $\geq n$ and $\|\cdot,\ldots,\cdot\|$ be a real-valued function on X^n satisfying the following conditions:

(nN$_1$) $\|a_1,\ldots,a_n\| = 0$ if and only if $a_1,\ldots,a_n$ are linearly dependent,

(nN$_2$) $\|a_1,\ldots,a_n\| = \|a_{i_1},\ldots,a_{i_n}\|$ for every permutation $(i_1,\ldots,i_n)$ of $(1,\ldots,n)$,

(nN$_3$) $\|\alpha a_1,a_2,\ldots,a_n\| = |\alpha|\|a_1,a_2,\ldots,a_n\|$ for any real α,

(nN$_4$) $\|a_1+a_1',a_2,\ldots,a_n\| \leq \|a_1,a_2,\ldots,a_n\| + \|a_1',a_2,\ldots,a_n\|$.

$\|\cdot,\ldots,\cdot\|$ is called an *n-norm* on X and $(X,\|\cdot,\ldots,\cdot\|)$ is called a *linear n-normed space* ([3]). Since $2\|a_1,a_2,\ldots,a_n\| \geq \|a_1-a_1,a_2,\ldots,a_n\| = 0$, the n-norm is not negative.

The concept of a linear n-normed space is a generalization of the concepts of a normed linear space ($n=1$) and of a linear 2-normed space ([2]). For more details on linear n-normed spaces, we refer to [1] and [3].

For $a,b\in X$, let $L(\{a,b\})$ denote the subspace of X generated by a,b. Whenever the notation $L(\{a,b\})$ is used, a and b are assumed to be linearly independent.

For $c_2,\ldots,c_n \in X$ being linearly independent, let $C=\{c_2,\ldots,c_n\}$. Let $X(C)$ denote the subspace of X generated by C and let X_C be the quotient space $X/X(C)$. For every $a\in X$, denote the equivalent class of a with respect to $X(C)$ by $(a)_C$. Then X_C is a linear space with addition $(a)_C+(b)_C=(a+b)_C$ and scalar multiplication $(\alpha a)_C=\alpha(a)_C$. For any $a,b\in X$ with $(a)_C=(b)_C$, since $a-b,c_2,c_3,\ldots,c_n$ are linearly

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dependent,

$$\big| \|a, c_2, \ldots, c_n\| - \|b, c_2, \ldots, c_n\| \big| \leq \|a - b, c_2, \ldots, c_n\| = 0$$

and hence we have

$$\|a, c_2, \ldots, c_n\| = \|b, c_2, \ldots, c_n\|.$$

Therefore, the function $\|\cdot\|_C$ defined on X_C by $\|(a)_C\|_C = \|a, c_2, \ldots, c_n\|$ is independent on the special representative a of $(a)_C$.

In this paper, we give new characterizations of strict convexity in linear n-normed spaces in terms of the quotient spaces mentioned above.

2. Now, we derive some elementary n-norm results:

THEOREM 1. *Let $(X, \|\cdot, \ldots, \cdot\|)$ be a linear n-normed space. If $\|a + b, c_2, \ldots, c_n\| = \|a, c_2, \ldots, c_n\| + \|b, c_2, \ldots, c_n\|$, then*

$$\|\alpha a + \beta b, c_2, \ldots, c_n\| = \alpha\|a, c_2, \ldots, c_n\| + \beta\|b, c_2, \ldots, c_n\|$$

for all $\alpha, \beta \geq 0$.

Proof. Note that

$$\|\alpha a + \beta b, c_2, \ldots, c_n\| \leq \alpha\|a, c_2, \ldots, c_n\| + \beta\|b, c_2, \ldots, c_n\|.$$

In order to show the opposite inequality we may assume, without loss of generality, that $0 \leq \alpha \leq \beta$. Then we have

$$\begin{aligned} \|\alpha a + \beta b, c_2, \ldots, c_n\| &= \|\beta(a + b) - (\beta - \alpha)a, c_2, \ldots, c_n\| \\ &\geq \beta\|a + b, c_2, \ldots, c_n\| - (\beta - \alpha)\|a, c_2, \ldots, c_n\| \\ &= \alpha\|a, c_2 \ldots, c_n\| + \beta\|b, c_2, \ldots, c_n\|. \end{aligned}$$

This completes the proof.

THEOREM 2. *Let $(X, \|\cdot, \ldots, \cdot\|)$ be a linear n-normed space and let $c_2, \ldots, c_n \notin L(\{a, b\})$. Then the following statements are equivalent:*

(1) *If $\|a + b, c_2, \ldots, c_n\| = \|a, c_2, \ldots, c_n\| + \|b, c_2, \ldots, c_n\|$ and $\|a, c_2, \ldots, c_n\| = \|b, c_2, \ldots, c_n\| = 1$, then $a = b$.*

(2) *If $\frac{1}{2}\|a + b, c_2, \ldots, c_n\| = \|a, c_2, \ldots, c_n\| = \|b, c_2, \ldots, c_n\|$, then $a = b$.*

(3) *If $\|a + \alpha b, c_2, \ldots, c_n\| = 2\|a, c_2, \ldots, c_n\|$, then $a = \alpha b$ for $\alpha = \|a, c_2, \ldots, c_n\| / \|b, c_2, \ldots, c_n\|$.*

(4) *If $\|a+b, c_2, \ldots, c_n\| = \|a, c_2, \ldots, c_n\| + \|b, c_2, \ldots, c_n\|$, then $\|b, c_2, \ldots, c_n\|a = \|a, c_2, \ldots, c_n\|b$.*

Proof. (1)$\Rightarrow$(2): Observe that if $\frac{1}{2}\|a + b, c_2, \ldots, c_n\| = \|a, c_2, \ldots, c_n\| = \|b, c_2, \ldots, c_n\| = \gamma$, then $\gamma \neq 0$ since $c_2, \ldots, c_n \notin L(\{a, b\})$.

Now we have

$$\frac{1}{2}\left\|a+b, c_2, \ldots, \frac{c_i}{\gamma}, \ldots, c_n\right\| = \left\|a, c_2, \ldots, \frac{c_i}{\gamma}, \ldots, c_n\right\| = \left\|b, c_2, \ldots, \frac{c_i}{\gamma}, \ldots, c_n\right\| = 1$$

and therefore, by the condition (1), $a = b$.

(2)⇒(3): Since $\|\alpha b, c_2, \ldots, c_n\| = \|a, c_2, \ldots, c_n\| = \frac{1}{2}\|a + \alpha b, c_2, \ldots, c_n\|$, then $a = \alpha b$, by the condition (2).

(3)⇒(4): Suppose that $\|a, c_2, \ldots, c_n\| \leq \|b, c_2, \ldots, c_n\|$, so

$$\alpha = \|a, c_2, \ldots, c_n\| / \|b, c_2, \ldots, c_n\| \leq 1.$$

Then we have

$$\|a+b, c_2, \ldots, c_n\| \leq \|a + \alpha b, c_2, \ldots, c_n\| + \|b, c_2, \ldots, c_n\| - \|a, c_2, \ldots, c_n\| = \|a, c_2, \ldots, c_n\| + \|b, c_2, \ldots, c_n\|.$$

If the identity in the condition (4) is satisfied, then the latter means that

$$\|a + \alpha b, c_2, \ldots, c_n\| = 2\|a, c_2, \ldots, c_n\|.$$

Thus, from (3), we have

$$\|a, c_2, \ldots, c_n\| b = \|b, c_2, \ldots, c_n\| a.$$

(4)⇒(1): It is evident.

This completes the proof.

3. Finally, we give some characterizations of strict convexity in a linear n-normed space $(X, \|\cdot, \ldots, \cdot\|)$.

DEFINITION 1 ([2]). A linear n-normed space $(X, \|\cdot, \ldots, \cdot\|)$ is said to be *strictly convex* if the condition $\|a, c_2, \ldots, c_n\| = \|b, c_2, \ldots, c_n\| = \frac{1}{2}\|a + b, c_2, \ldots, c_n\|$ for $c_2, \ldots, c_n \notin L(\{a, b\})$ implies that $a = b$.

Note that, by Theorem 2, a linear n-normed space $(X, \|\cdot, \ldots, \cdot\|)$ is strictly convex if and only if the condition

$$\|a+b, c_2, \ldots, c_n\| = \|a, c_2, \ldots, c_n\| + \|b, c_2, \ldots, c_n\|$$

for $c_2, \ldots, c_n \notin L(\{a, b\})$ and

$$\|a, c_2, \ldots, c_n\| = \|b, c_2, \ldots, c_n\| = 1$$

imply that $a = b$.

THEOREM 3. *The following statements are equivalent*:

(1) $(X, \|\cdot, \dots, \cdot\|)$ *is strictly convex.*

(2) *For every linearly independent set* $C = \{c_2, \dots, c_n\} \subset X$, $(X_C, \|\cdot\|_C)$ *is strictly convex.*

(3) *The condition* $\|a+b, c_2, \dots, c_n\| = \|a, c_2, \dots, c_n\| + \|b, c_2, \dots, c_n\|$ *for* $c_2, \dots, c_n \notin L(\{a, b\})$ *implies that* $b = \alpha a$ *for some* $\alpha > 0$.

Proof. (1)⇒(2): Let $(X, \|\cdot, \dots, \cdot\|)$ be strictly convex. Take arbitrary $C = \{c_2, \dots, c_n\}$, where $c_2, \dots, c_n$ are linearly independent points in X, and assume that $\|(a)_C\|_C = \|(b)_C\|_C = \frac{1}{2}\|(a)_C + (b)_C\|_C = 1$. Then

$$\|a, c_2, \dots, c_n\| = \|b, c_2, \dots, c_n\| = \frac{1}{2}\|a + b, c_2, \dots, c_n\| = 1.$$

We need to show that $(a)_c = (b)_c$.

It is obvious if $c_2, \dots, c_n \notin \alpha(\{a, b\})$.

If $c_i \in L(\{a, b\})$ for some $i \in \{2, 3, \dots, n\}$, then $c_i = \alpha a + \beta b$ for some real α, β, and so

$$(0)_C = \alpha(a)_C + \beta(b)_C.$$

Since $\|(a)_C\|_C = \|(b)_C\|_C = 1$, the latter implies that $\alpha = \pm\beta$.

If $\alpha = \beta$, then $c_i = \alpha(a + b)$, which contradicts with

$$\begin{aligned} 0 = \|a + b, c_2, \dots, c_n\| &= \|a, c_2, \dots, c_n\| + \|b, c_2, \dots, c_n\| \\ &= \|(a)_C\|_C + \|(b)_C\|_C \\ &= 2. \end{aligned}$$

Therefore, $\alpha = -\beta$ and hence $(a)_C = (b)_C$. Thus, $(X_C, \|\cdot\|_C)$ is strictly convex.

(2)⇒(3): Assume that the condition (2) holds. Take arbitrary $c_2, \dots, c_n \notin L(\{a, b\})$ such that

$$\|a + b, c_2, \dots, c_n\| = \|a, c_2, \dots, c_n\| + \|b, c_2, \dots, c_n\|.$$

Then we have

$$\|(a)_C + (b)_C\|_C = \|(a)_C\|_C + \|(b)_C\|_C.$$

Since $(X_C, \|\cdot\|_C)$ is strictly convex, the latter means that $(b)_C = \alpha(a)_C$ for some $\alpha > 0$. Therefore, $b = \alpha a$ for some $\alpha > 0$.

(3)⇒(1): See Theorem 2.

This completes the proof.

COROLLARY 4. *The following statements are equivalent:*

(1) $(X, \|\cdot, \ldots, \cdot\|)$ *is strictly convex.*

(2) *The condition* $\|a, c_2, \ldots, c_n\| = \|b, c_2, \ldots, c_n\| = 1$, $a \neq b$, *for given* $c_2, \ldots, c_n \notin L(\{a, b\})$ *implies that* $\|\frac{1}{2}(a+b), c_2, \ldots, c_n\| < 1$.

Proof. (1)⇒(2): Suppose that $\|a, c_2, \ldots, c_n\| = \|b, c_2, \ldots, c_n\| = 1$, $a \neq b$ and $c_2, \ldots, c_n \notin L(\{a, b\})$. By the condition (nN_4), we have

$$\|a+b, c_2, \ldots, c_n\| \leq 2.$$

If $\|a, c_2, \ldots, c_n\| = \|b, c_2, \ldots, c_n\| = \|\frac{1}{2}(a+b), c_2, \ldots, c_n\| = 1$, then

$$\|a, c_2, \ldots, c_n\| + \|b, c_2, \ldots, c_n\| = \|a+b, c_2, \ldots, c_n\|.$$

Since $c_2, \ldots, c_n \notin L(\{a, b\})$, by the condition (1), we have $b = \alpha a$ for some $\alpha > 0$. Thus $\|b, c_2, \ldots, c_n\| = \alpha\|a, c_2, \ldots, c_n\| = 1$ and therefore $\alpha = 1$, which contradicts with $a \neq b$. So, we have

$$\left\|\frac{1}{2}(a+b), c_2, \ldots, c_n\right\| < 1.$$

(2)⇒(1): Let $\|a+b, c_2, \ldots, c_n\| = \|a, c_2, \ldots, c_n\| + \|b, c_2, \ldots, c_n\|$ and $a, b \neq 0$. We need to consider only the case $c_2, \ldots, c_n \notin L(\{a, b\})$. If so, then

$$\left\|\frac{a}{\|a, c_2, \ldots, c_n\|} + \frac{b}{\|b, c_2, \ldots, c_n\|}, c_2, \ldots, c_n\right\| \geq 2.$$

Hence, since $\|a/\|a, c_2, \ldots, c_n\|, c_2, \ldots, c_n\| = \|b/\|b, c_2, \ldots, c_n\|, c_2, \ldots, c_n\| = 1$, by the condition (2), we have

$$\frac{a}{\|a, c_2, \ldots, c_n\|} = \frac{b}{\|b, c_2, \ldots, c_n\|}.$$

Therefore, $b = \alpha a$ for some $\alpha > 0$.

This completes the proof.

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