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ON THE n-th ORDER ORDINARY DIFFERENTIAL EQUATION IN BANACH SPACES

This paper gives some Aronszajn-type theorems for n-th order ordinary differential equations in Banach spaces.

1. Introduction

Using the measure of noncompactness (cf.[2]) we shall give sufficient conditions for the existence of local solutions of an initial value problem for the differential equation

$$x^{(n)} = f(t, x, x', \dots, x^{(n-1)})$$

in Banach spaces. Moreover, we shall prove that the set of these solutions is a compact R_δ, i.e. it is homeomorphic to the intersection of a decreasing sequence of compact absolute retracts.

In the case of the equation

$$x^{(n)} = f(t, x)$$

we shall show that a similar theorem is true also for global solutions.

In our paper an essential role play some results from [1].

2. Assume that $I = [t_0, t_0 + a]$ is a compact interval in $\mathbb{R}$, E is a real Banach space and α is the measure of noncompactness in E. Let $B = \{y \in E^n : \|y - \eta\| \le b\}$, where $\eta = (\eta_1, \dots, \eta_n)$, $y = (y_1, \dots, y_n)$ and $\|y\| = \max(\|y_1\|, \dots, \|y_n\|)$.

Moreover, we assume that

1° $f: I \times B \to E$ is a bounded continuous function;

2° there exists an integrable function $h: I \to \mathbb{R}_+$ such that

$$\alpha\big(f(t, X_1 \times \dots \times X_n)\big) \le h(t) \max\big(\alpha(X_1), \dots, \alpha(X_n)\big)$$

for $t \in I$ and for bounded subsets $X_1, \dots, X_n$ of E.

THEOREM 1. *Under the above assumptions there exists an interval* $J = [t_0, t_0 + d]$ *such that the set of all solutions on* J *of the initial value problem*

$$x^{(n)} = f(t, x, x', \ldots, x^{(n-1)}), \tag{1}$$

$$x(t_0) = \eta_1, \ldots, x^{(n-1)}(t_0) = \eta_n \tag{2}$$

is a compact R_δ.

Obviously, the equation (1) is equivalent to the system of n first-order differential equations

$$\begin{aligned} y_1' &= y_2 \\ y_2' &= y_3 \\ &\vdots \\ y_{n-1}' &= y_n \\ y_n' &= f(t, y_1, y_2, \ldots, y_n), \end{aligned}$$

where $y_1(t) = x(t)$.

Hence Theorem 1 is a simple consequence of the following

LEMMA 1. *Consider the Cauchy problem*

$$y_i' = g_i(t, y_1, \ldots, y_n), \quad y_i(t_0) = \eta_i \qquad (i = 1, 2, \ldots, n). \tag{3}$$

Assume that for each $i = 1, 2, \ldots, n$, $g_i : I \times B \to E$ *is a bounded continuous function and there exist integrable functions* $a_{ij} : I \to \mathbb{R}_+$ *such that*

$$\alpha\big(g_i(t, A_1 \times \cdots \times A_n)\big) \le a_{i1}(t)\alpha(A_1) + \cdots + a_{in}(t)\alpha(A_n) \tag{4}$$

for $t \in I$ *and for any subsets* $A_1, \ldots, A_n$ *of* E *such that* $A_1 \times \cdots \times A_n \subset B$. *Let* $M = sup\{\|g_i(t, y)\| : t \in I,\ y \in B,\ i = 1, 2, \ldots, n\}$, $d = \min\big(a, \frac{b}{M}\big)$ *and* $J = [t_0, t_0 + d]$. *Then the set of all solutions of the problem* (3) *on* J *is a compact* R_δ.

Proof. Let $B_i = \{x \in E : \|x - \eta_i\| \le b\}$. Put

$$r_i(x) = \begin{cases} x & \text{for } x \in B_i \\ \eta_i + \dfrac{b(x - \eta_i)}{\|x - \eta_i\|} & \text{for } x \in E \setminus B_i. \end{cases}$$

Then r_i is a continuous function from E into B_i and $\alpha\big(r_i(A)\big) \le \alpha(A)$ for each bounded $A \subset E$.

Define a function $\widetilde{g}_i$ by $\widetilde{g}_i(t, y_1, \ldots, y_n) = g_i\big(t, r_1(y_1), \ldots, r_n(y_n)\big)$ for $t \in I$, $y_1, \ldots, y_n \in E$. Then $\widetilde{g}_i$ is a continuous function from $I \times E^n$ into E such that $\|\widetilde{g}_i(t, y)\| \le M$ for $t \in I$, $y \in E^n$ and

$$\alpha\big(\widetilde{g}_i(t, A_1 \times \cdots \times A_n)\big) \le a_{i1}(t)\alpha(A_1) + \cdots + a_{in}(t)\alpha(A_n)$$

for $t \in I$ and for bounded subsets $A_1, \ldots, A_n$ of E.

Denote by $C = C(J, E^n)$ the space of continuous functions $J \to E^n$ with the usual supremum norm $\|\cdot\|_C$.

Let us notice that (3) is equivalent to the equation $y = F(y)$, where $F(y) = \big(F_1(y), \ldots, F_n(y)\big)$ and

$$F_i(y)(t) = \eta_i + \int_{t_0}^{t} \widetilde{g}_i\big(s, y(s)\big)ds \quad \text{for } t \in J, \ y \in C, \ i = 1, \ldots, n.$$

Fix an index i. For each $y \in C$ and $t, \tau \in J$ we have

$$\|F_i(y)(t) - F_i(y)(\tau)\| \leq \left| \int_{\tau}^{t} \big\|\widetilde{g}_i\big(s, y(s)\big)\big\| ds \right| \leq M|t - \tau|$$

and

$$\|F_i(y)(t) - \eta_i\| \leq M|t - t_0| \leq Md \leq b.$$

Hence $F_i(C)$ is a bounded equicontinuous subset of C. Since J is compact, it follows that the set $F_i(C)$ is equiuniformly continuous.

Assume that $y^k, y \in C$ and

$$\lim_{k \to \infty} \|y^k - y\|_C = 0.$$

Then

$$\lim_{k \to \infty} \widetilde{g}_i\big(s, y^k(s)\big) = \widetilde{g}_i\big(s, y(s)\big)$$

and

$$\big\|\widetilde{g}_i\big(s, y^k(s)\big) - \widetilde{g}_i\big(s, y(s)\big)\big\| \leq 2M \quad \text{for } s \in J.$$

Now, by the Lebesgue dominated convergence theorem we get

$$\lim_{k \to \infty} \int_{t_0}^{t} \big\|\widetilde{g}_i\big(s, y^k(s)\big) - \widetilde{g}_i\big(s, y(s)\big)\big\| ds = 0,$$

$$\text{i.e.} \quad \lim_{k \to \infty} F_i(y^k)(t) = F_i(y)(t) \quad \text{for each } t \in J.$$

Because $F_i(C)$ is equicontinuous, this implies that

$$\lim_{k \to \infty} \|F_i(y^k) - F_i(y)\|_C = 0.$$

Thus $F_i : C \to C$ is continuous for $i = 1, \ldots, n$ and therefore $F : C \to C$ is continuous.

We shall prove that

(5) If $u^k \in C \quad (k = 1, 2, \ldots)$ and $\lim_{k \to \infty} \|u^k - F(u^k)\|_C = 0$, then (u^k) has a convergent subsequence.

Suppose that $u^k \in C \quad (k = 1, 2 \ldots)$ and

$$\lim_{k \to \infty} \|u^k - F(u^k)\|_C = 0. \tag{6}$$

Put $V = \{u^k : k = 1, 2, \ldots\}$, $V_i = \{u_i^k : k = 1, 2, \ldots\}$, $V(t) = \{u^k(t) : k = 1, 2, \ldots\}$ and $V_i(t) = \{u_i^k(t) : k = 1, 2, \ldots\}$ $(i = 1, \ldots, n, \ t \in J)$. By (6) we infer that $(I - F)(V)$ is an equiuniformly continuous subset of C. Since $V \subset (I - F)(V) + F(V)$ and $F(V)$ is equiuniformly continuous, the set V is also equiuniformly continuous.

Fix i and put $Z_i(t) = \{u_i^k(t) - F_i(u^k)(t) : k = 1, 2, \dots\}$. From (6) we deduce that $\alpha(Z_i(t)) = 0$. Since $V_i(t) \subset Z_i(t) + F_i(V)(t)$ and $F_i(V)(t) \subset V_i(t) - Z_i(t)$, we have

$$v_i(t) = \alpha(V_i(t)) = \alpha(F_i(V)(t)) \quad \text{for } t \in J. \tag{7}$$

Let $W_i = \{w_i^k = \tilde{g}_i(\cdot, u^k) : k \in \mathbb{N}\}$. It is clear that $w_i^k \in C$ and $\|w_i^k(t)\| \le M$ for $k \in \mathbb{N}$, $t \in J$. Since W_i satisfies the assumptions of Heinz's Theorem [4], by (4) and (7) we get

$$\alpha(V_i(t)) = \alpha(F_i(V)(t)) = \alpha\Big(\Big\{ \int_{t_0}^{t} \tilde{g}_i(s, u^k(s))ds : u^k \in V\Big\}\Big) \le$$

$$\le 2 \int_{t_0}^{t} \alpha\Big(\tilde{g}_i(s, V(s))\Big)ds \le 2 \int_{t_0}^{t} \Big(a_{i1}(s)\alpha(V_1(s)) + \dots + a_{in}(s)\alpha(V_n(s))\Big)ds.$$

Hence

$$v_i(t) \le 2 \int_{t_0}^{t} \big(a_{i1}(s)v_1(s) + \dots + a_{in}(s)v_n(s)\big)ds \quad \text{for } t \in J, \quad i = 1, \dots, n.$$

Applying now the theorem on integral inequalities, we deduce that $v_i(t) = 0$ for each $t \in J$. Since $V(t) \subset V_1(t) \times \dots \times V_n(t)$, we obtain

$$\alpha(V(t)) \le \max\Big(\alpha(V_1(t), \dots, \alpha(V_n(t))\Big) = 0.$$

Hence the set $V(t)$ is relatively compact in E^n. By Ascoli's theorem this proves that the set V is relatively compact in C. Hence the sequence (u^k) has a convergent subsequence.

By Theorem 5 of [5] from (5) we conclude that the set of all fixed points of F is a compact R_δ. Obviously, this set is identical to the set of all solutions of the problem (3) on J.

3. Consider now a Cauchy problem

$$x^{(n)} = f(t, x), \tag{8}$$

$$x^{(i)}(a) = \eta_i \qquad (i = 0, 1, \dots, n-1). \tag{9}$$

Assume that $J = [a, b]$ and

1° $f: J \times E \to E$ is a continuous function;

2° for each bounded subset Z of E there exists an integrable function $h_Z: J \to \mathbb{R}_+$ such that $\alpha(f(t, X)) \le h_Z(t)\alpha(X)$ for $t \in J$ and for $X \subset Z$;

3° there exists a continuous nondecreasing function $M:[0,\infty)\to(0,\infty)$ such that

$$\int_1^\infty \frac{dr}{\sqrt[n]{r^{n-1}M(r)}}=\infty \quad\text{and}\quad \|f(t,x)\|\le M(\|x\|)\quad (x\in E, t\in J).$$

THEOREM 2. *Under the above assumptions all solutions of the problem* (8)-(9) *exist on the interval* J *and the set of all these solutions is a compact* R_δ.

Proof. Performing the change of function

$$y(t)=x(t)-\sum_{j=0}^{n-1}\frac{\eta_j(t-a)^j}{j!}$$

we reduce the initial value problem (8)-(9) to a problem

$$y^{(n)}=\tilde f(t,y) \tag{8'}$$

$$y^{(i)}(a)=0 \qquad i=0,1,\dots,n-1, \tag{9'}$$

where the function $\tilde f$ satisfies the assumptions 1° – 3°.

The equation (8′) is equivalent to the system

$$\begin{aligned} z_1'&=z_2\\ z_2'&=z_3\\ &\;\vdots\\ z_{n-1}'&=z_n\\ z_n'&=\tilde f(t,z_1) \end{aligned} \tag{10}$$

i.e. $Z'=G(t,Z)$, where $z_1(t)=x(t)$, $Z=(z_1,\dots,z_n)$, and the mapping $G: J\times E^n\to E^n$ is defined as $G(t,Z)=\big(z_2,\dots,z_n,\tilde f_n(t,z_1)\big)$.

It is obvious that G satisfies the assumptions of Lemma 1. Hence the Cauchy problem

$$Z'=G(t,Z),\qquad Z(a)=0, \tag{11}$$

has a local solution. Let $Z(t)$ be any solution of (11) defined on some interval $J'=[a,c)$, $c<b$. By Lemma 10 of [1] from 3° it follows that the function $z_n(t)=x^{(n-1)}(t)$ is bounded on J'. Consequently, the function $Z(t)$ is bounded on J'. Therefore there exists the limit

$$\lim_{t\to c} Z(t)=p.$$

According to Lemma 1, the system (10) with the initial condition $Z(c)=p$ has a solution $Z(t)$ on an interval $[c,c+\epsilon]$. From this we conclude that all solutions of the problem (11) can be continued to the whole interval J.

From 3° it is clear that the function G is bounded on every bounded subset of $J \times E^n$.

For given $\Delta = [a, \delta] \subset J$, $k \in \mathbb{N}$ and for any bounded closed subset V of $J \times E^n$, denote by $S_k(\Delta, V)$ the set of all continuously differentiable functions $u: \Delta \to E^n$ such that $u(a) = 0$, $(t, u(t)) \in V$ and $\|u'(t) - G(t, u(t))\| \leq \frac{1}{k}$ for $t \in \Delta$.

In view of Theorem 3 of [6], in order to prove that the set of all solutions of (11) on J is a compact R_δ, it is sufficient to show that for any Δ and V the following condition holds : (D_2) Every sequence (u^k), $u^k \in S_k(\Delta, V)$ for $k = 1, 2, \ldots$, has a limit point.

Let (u^k) be a sequence such that $u^k \in S_k(\Delta, V)$ for $k = 1, 2, \ldots$. Then

$$\lim_{k \to \infty} \|u^k - H(u^k)\|_C = 0,$$

where

$$H(u^k)(t) = \int_a^t G(s, u^k(s))ds.$$

By repeating the argument from the proof of Lemma 1, we can prove that the sequence (u^k) has a convergent subsequence. This completes our proof.

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