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SMOOTH POINTS OF UNIT BALLS OF OPERATOR AND FUNCTION SPACES

0. Introduction

Let X be a Banach space and $B_1(X)$ be the unit ball of X. A point $x \in B_1(X)$ is called a smooth point if there exists a unique $f \in X^*$, the dual of X, such that $f(x) = \|f\| = 1$.

In [4], Holub characterized the smooth points of $B_1(N(l^2))$ and $B_1(K(l^2))$, where $N(X)K(X))$ denotes the nuclear (compact) operators on X. Smooth points of $B_1(K(l^p))$ and $B_1(L(l^p))1 \leq p < \infty$ were studied in [1], where $L(X)$ is the space of bounded linear operators on X.

The structure of the Hilbert space l^2, was heavily used in the study of smooth points of $B_1(N(l^2))$, in [4], So the proofs can't be generalized to the case of $l^p, p \neq 2$. Another difference between the cases of l^2 and $l^p, p \neq 2$, is the fact that the extreme points of $B_1(L(l^2))$ are completely characterized (in [5]), while the description of the extreme points of $B_1(L(l^p))$ is a very difficult (and an open) problem, [6].

In Section II of this paper, we show that the result of Holub, [4], for $N(l^2)$, is not true for $N(l^p), p \neq 2$. Further we give a characterization of smooth points in $B_1(N(l^p_n))$, $1 \leq p < \infty$, and a large class of smooth points for $B_1(N(X))$ is identified, for some Banach space X. Also an alternative proof of the Holub's result is presented.

In Section I, we present a proof of the characteriyation of smooth points of the unit ball of the completion of the injective tensor product of $X \otimes Y$ for Banach space X and Y. Our approach is different from the one presented in [13] by Heinrich, who used the equivalence of differentiability of the norm at a point and the smoothness of that point. Here we make use of the extremal structure of the dual space.

Throughout this paper the symbols $L(X), K(X), N(X)$ denote the space of all bounded linear, compact and nuclear operators on a Banach space X, respectively. The dual of X is denoted by X^*, while $S_1(X)$ is the unit sphere in X. The completion of the injective and the projective tensor products of X and Y will be denoted by $X \overset{\vee}{\otimes} Y$ and $X \overset{\wedge}{\otimes} Y$, respectively. For $z \in X \overset{\wedge}{\otimes} Y$ we write $\|z\|_v$ for the injective norm of z and for $z \in X \otimes Y$, the symbol $\|z\|_n$ stands for the projective norm of z. The nuclear norm of $T \in N(X)$ will be denoted by $\|T\|_n$. Finally for $x \in l^p$, we let $\text{supp}(x) = \{n : x(n) \neq 0\}$, and N is the set of positive integers.

We refer the reader to [2] and [8] for the basic properties of $N(X), X \overset{\wedge}{\otimes} Y$, and $X \overset{\vee}{\otimes} Y$.

1. Smooth points of $X \overset{\vee}{\otimes} Y$

In [4], Holub characterized the smooth points of $S_1(K(l^2)) = S_1(l^2 \overset{\vee}{\otimes} l^2)$. Smooth points of $S_1(K(l^p)) = S_1(l^{p^*} \overset{\vee}{\otimes} l^p)$, $l < p, \infty$, were characterized in [1]. In this section we recall characterization of the smooth points of $S_1(X \overset{\vee}{\otimes} Y)$, where X and Y are Banach spaces, and present a different way of proof from that one given in [13].

THEOREM 1.1. *Let X and Y be given Banach spaces. For $T \in S_1(X \overset{\vee}{\otimes} Y)$, the following statements are equivalent:*

(i) *T is a smooth point;*

(ii) *T (as an operator: $X^* \to Y$) attains its norm uniquely, and at a smooth point of $S_1(X^*)$.*

Note: The uniqueness in (ii) is in the following sense: there is only one element $x^* \in S_1(X^*)$ such that $\|Tx^*\| = \|T(-x^*)\| = 1 = \|T\|$.

Proof. (i) $\to$ (ii). Let $T \in S_1(X \overset{\vee}{\otimes} Y)$ be smooth. Set

$$E(T) = \{A \in S_1((X \overset{\vee}{\otimes} Y)^*) : A(T) = 1\}.$$

Since T can be considered as a continuous linear functional on $X \overset{\vee}{\otimes} Y$, then $E(T)$ is an extremal subset of $S_1((X \overset{\vee}{\otimes} Y)^*)$ (see [7]). Using the Hahn–Banach Theorem, $E(T) \neq \emptyset$. As in the proof of Theorem 1.1 in [1], $E(T)$ is w^*-closed. By Krein–Millman Theorem, $E(T)$ is the (w^*-)closed convex hull of its extreme points. But every extreme point of $E(T)$ is an extreme point of $S_1((X \overset{\vee}{\otimes} Y)^*)$. Thus, by the result of Ruess and Stegall, [10], the extreme points of $E(T)$ are of the form $A = x^* \otimes y^*$, where x^* and y^* are extreme points of $S_1(X^*)$ and $S_1(Y^*)$, respectively. Thus, if $x^* \otimes y^* \in Ext(E(T))$, then

$$(x^* \otimes y^*)(T) = \langle Tx^*, y^* \rangle = \|Tx^*\| = 1.$$

Since T is smooth, $E(T)$ can't contain more than one element, x^*. Further, x^* is smooth.

(ii) $\to$ (i). As in (i) $\to$ (ii), $E(T)$ is the (w^*-) closed convex hull of its extreme points. If $E(T)$ contains two extreme points, say $x_1^* \otimes y_1^*$ and $x_2^* \otimes y_2^*$, then:

$$\langle Tx_1^*, y*1^* \rangle = \langle Tx_2^*, y_2^* \rangle = 1.$$

Since $\|x_i^*\| = \|y_i^*\| = 1$, then

$$\|Tx_1^*\| = \|Tx_2^*\| = 1.$$

This contradicts with (ii). Hence $E(T)$ is a singleton and T is smooth. This ends the proof.

THEOREM 1.2 [1, 4]. *An operator T in $S_1(K(l^p))$ is smooth if and only if whenever $\|Tx_1\| = \|Tx_2\| = 1$, then $x_1 = \pm x_2$, for x_1, x_2 in $S_l(l^p), 1 < p < \infty$.*

Proof. It follows from Theorem 1.1 and the fact that every element of $S_1(l^p)$ is smooth.

THEOREM 1.3 [11]. *Let I be a compact set and X be arbitrary Banach space. Then a function $F \in S_1(C(I, X))$ is smooth if and only if there exists a unique t_0 in I and a smooth point x_0 in $S_1(X)$ such that $F(t_0) = x_0$.*

The proof follows from Theorem 1.1 and the fact that

$$Ext(M(F)) = \{\delta_x : x \in I\}.$$

2. Smooth points of $N(X)$

Let X be a Banach space. In this section we try to study smooth points of $N(X)$. We introduce the following definition:

DEFINITION 2.1. A set $A \subseteq S_1(X)$ is called smooth iff there is a unique $f \in B_1(X^*)$ such that $f(x) = 1$ for all $x \in A$.

It follows from the definition that a smooth point is just a smooth set consisting of that point.

Now, we give some examples of smooth sets.

LEMMA 2.2. *A set $A \subseteq S_1(l^p), 1 < p < \infty$, is smooth if and only if $A = \{z\}$ for some $z \in S_1(l^p)$.*

Proof. This follows from the fact that l^p is reflexive and every point of unit norm in l^p is smooth.

THEOREM 2.3. *Let $A \subseteq S_1(l^1)$. The following statements are equivalent:*

(i) *A is smooth;*

(ii) *A satisfies the following conditions*:

(a) $\bigcup_{x\in A}\operatorname{supp}(x) = N$

(b) *For* $x, y \in A, x$ *and* y *have the same sign on* $\operatorname{supp}(x)\cap\operatorname{supp}(y)$.

Proof. (i) $\to$ (ii). Let A be smooth, but (a) is not true. So there exists $n_0 \notin \bigcup_{x\in A}\operatorname{supp}(x)$. Take $g \in S_1(l^\infty)$ such that $g(x) = 1 = \langle g, x\rangle$ for all $x \in A$ and define $f \in S_1(l^\infty)$ by

$$f(A) = \begin{cases} g(n) & \neq n_0 \\ a & n = n_0 \end{cases}$$

where $a \neq g(n_0)$, $0 < a \leq 1$. Then $f \neq g$ and $\langle f, x\rangle = 1$ for all $x \in A$. This contradicts with (i). Hence (a) must be satisfied.

To prove (b), assume that A is smooth, but there exist x, y in A such that $\operatorname{sign}(x) \neq \operatorname{sign}(y)$ on $\operatorname{supp}(x)\cap\operatorname{supp}(y)$. Take $f \in S_1(l^\infty)$ such that $\langle f, z\rangle = 1$ for all $z \in A$. Now: $\langle f, x\rangle = \langle f, y\rangle = 1$. But for $z \in S_1(l^1)$ the equality $\langle f, z\rangle = 1$ implies that $|f(n)| = 1$ for all $n \in \operatorname{supp}(z)$. In fact $f(n) = \operatorname{sign}(z(n))$. Hence if $\operatorname{sign}(x(i_0)) \neq \operatorname{sign}(y(i_0))$, $i_0 \in \operatorname{supp}(x)\cap\operatorname{supp}(y)$, then $f(i_0) = \operatorname{sign}(x(i_0)) \neq \operatorname{sign}(y(i_0)) = f(i_0)$. This is a contradiction. Hence (b) is true.

(ii) $\to$ (i). Let A satisfy (a) and (b). Consider the function $f \in S_1(l^\infty)$ defined by

$$f(n) = \operatorname{sign}(x(n))$$

for $n \in \operatorname{supp}(x)$, and $x \in A$. Now, the condition (b) ensures that f is well defined, while the condition (a) means that f is unique and $\langle f, x\rangle = 1$ for all $x \in A$. Hence A is smooth. This ends the proof.

The arguments which can be used to prove the next theorem are similar to that in Theorem 1.3 and will be ommitted.

THEOREM 2.4. *Let* $A \subseteq S_1(c_0)$. *The following statements are equivalent:*

(i) *A is smooth.*

(ii) *There exists* $n_0 \in N$ *such that*

(a) $x(n_0) = 1$ *for all* $x \in A$

(b) $x(n) < 1$ *for all* $n \neq n_0$ *and all* $x \in A$.

Now we shall consider more complicated examples.

Let $C_1(H)$ be the class of nuclear operators on the separable Hilbert space H. This is just the trace class operators. Every $T \in C_1(H)$ has a representation of the form

$$T = \sum_{n=1}^{\infty} \lambda_n e_n \otimes f_n,$$

where $(e_n), (f_n)$ are two sequences (finite or infinite) of orthonormal vectors in H and $(\lambda_n) \in l^1$ with $\lambda_n > 0$. Further $\|T\| = \sum_{n=1}^{\infty} \lambda_n$. We refer the reader to [9] for more details on nuclear operators.

The representation of $T \in C_1(H)$ suggests the study of sets of the form $\{e_n \otimes f_n\}_{n=1}^{\infty}$ concerning the smoothness in $S_1(C_1(H))$.

THEOREM 2.5. *Let $A = \{e_n \otimes f_n\}_{n=1}^{\infty}$, where (e_n) and (f_n) are orthonormal sequences in H. Then the following statements are equivalent*:

(i) *A is a smooth set in $S_1(C_1(H))$.*

(ii) *At least one of (e_n) and (f_n) is complete.*

Proof. (i) $\rightarrow$ (ii). Let A be smooth, but neither (e_n) nor (f_n) is complete. Let

$$E_1 = \operatorname{span}\{e_1, e_2, \ldots\},$$
$$E_2 = \operatorname{span}\{f_1, f_2, \ldots\}.$$

Then $H = E_1 \oplus \widehat{E}_1 = E_2 \oplus \widehat{E}_2$, where $\widehat{E}_i$ is the orthogonal complement of E_i.

Now, since A is smooth and $C_1(H)^* = L(H)$ (see [9]), there exists a bounded linear operator $T \in S_1(H))$ such that $T(x) = 1$ for all $x \in A$. But $T(e_n \otimes f_n) = \langle e_n, Tf_n \rangle$, [9]. So $\langle e_n, Tf_n \rangle = 1$ for all n. Since (e_n) and (f_n) are incomplete then we have $\widehat{E}_1 \neq \{0\}$ and $\widehat{E}_2 \neq \{0\}$. Hence one can define a bounded linear operator $J : \widehat{E}_1 \rightarrow \widehat{E}_2$ such that $\|J\| = 1$ and $J \neq T|_{E_1}$, the restriction of T to E_1. The operator $Q : H \rightarrow H$ given by

$$Q(x) = \begin{cases} Tx & \text{if } x \in E_1 \\ Jx & \text{if } x \in \widehat{E} \end{cases}$$

is bounded and linear. Moreover $\|Q\| \leq 1$, and $Q(x) = 1 = T(x)$ for all $x \in A$. This contradicts with the smoothness of A. Hence either (e_n) or (f_n) is complete.

(ii) $\rightarrow$ (i). Let (f_n) be complete. Define $T : H \rightarrow H$ setting $Tf_n = e_n$ for all n. Then T is a bounded linear operator on H, with $\|T\| = 1$. Clearly $T(e_n \otimes f_n) = \langle e_n, Tf_n \rangle = 1$ for all n. Further if J is any other element of $S_1(L(H))$ such that $J(e_n \otimes f_n) = 1$ for all n, then $\langle e_n, Jf_n \rangle = 1$. Since H is uniformly convex, we get $Jf_n = Tf_n$. The completeness of (f_n) implies that $J = T$. Hence A is smooth. This ends the proof.

Next result gives the characterization of smooth operators in $S_1(C_1(l^2))$. Namely, we have the following

THEOREM 2.6. *Let $T \in S_1(C_1(l^2))$. Then the following statements are equivalent*:

(i) *T is smooth*;

(ii) $T = \sum_{n=1}^{\infty} \lambda_n e_n \otimes f_n$,

where $\sum \lambda_n = 1$ and $\{e_n \otimes f_n\}_{n=1}^{\infty}$ is a smooth set in $S_1(C_1(H))$.

P r o o f. (ii) $\to$ (i). Let $T = \sum_n \lambda_n e_n \otimes f_n$, with $\sum_n \lambda_n = 1$ and $\{e_n \otimes f_n\}$ is a smooth set. If T is not smooth then there exist J_1, J_2 in $S_1(L(l^2))$, $J_1 \neq J_2$ such that $J_1(T) = J_2(T) = 1$. Hence

$$\sum_n \lambda_n \langle e_n, J_1 f_n \rangle = \sum \lambda_n \langle e_n, J_2 f_n \rangle = 1.$$

But $\sum_n \lambda_n = 1$, so we get $\langle e_n, J_i f_n \rangle = 1$ for $i = 1, 2$. Since $\{e_n \otimes f_n\}$ is a smooth set we may assume, with no loss of generality, that (f_n) is complete. The uniform convexity of l^2 implies that $J_1 f_n = J_2 f_n$ for all n. Consequently $J_1 = J_2$. This clearly forces T to be smooth.

(i) $\to$ (ii). Let T be a smooth operator in $S_1(C_1(l^2))$ and $T = \sum_n \lambda_n e_n \otimes f_n$ be the Hilbert–Schmidt representation of T, [9]. So $\sum_n \lambda_n = \|T\| = 1$ and $(e_n), (f_n)$ are orthonormal sequences in l^2. If $\{e_n \otimes f_n\}$ is not smooth then this implies that there exist J_1, J_2 in $S_1(L(l^2))$ such that $J_i(e_n \otimes f_n) = 1$ for $i = 1, 2$ and all n. Hence $\sum_n \lambda_n \langle e_n, J_i, f_m \rangle = 1$ for $i = 1, 2$ and all n. This contradicts with the fact that T is smooth. Hence $\{e_n \otimes f_n\}_{n=1}^{\infty}$ is a smooth set in $S_1(C_1(l^2))$. This ends the proof.

COROLLARY 2.7 [Holub]. *$T \in S_1(C_1(l^2))$ is a smooth operator if and only if either T is* (1-1) *or T^* is* (1-1).

P r o o f. If T is smooth then, by Theorem 2.5 and Theorem 2.6, $T = \sum_n \lambda_n e_n \otimes f_n$, where either (e_n) or (f_n) is a complete set. If (e_n) is complete then T is (1-1) and while the completeness of (f_n) implies that T^* is (1-1).

For the converse: If T is (1-1) or T^* is (1-1) then T has the representation $T = \sum_n \lambda_n e_n \otimes f_n$, where either (e_n) or (f_n) is complete. Hence $\{e_n \otimes f_n\}_{n=1}^{\infty}$ is a smooth set in $S_1(C_1(l^2))$. By Theorem 2.6, T is smooth. This ends the proof.

The case of nuclear operators on l^p, $l < p \neq 2 < \infty$, is totally different. The following theorem shows that point:

THEOREM 2.8. *Let $y \in S_1(l^p)$, $l < p < 2$, be such that* $\text{supp}(y) = N$. *Then the operator $T = \delta_1 \otimes y$ is a smooth element of $S_1(N(l^p))$, where δ_1 is the first element of the natural basis of l^p.*

P r o o f. Take $y^* \in l^{p^*}$ such that $\|y^*\| = 1$ and $\langle y, y^* \rangle = 1$. Consider the bounded linear operator $J = \delta_1 \otimes y^*$ in $L(l^{p^*})$. Clearly $J(T) = \langle y, y^* \rangle = 1$.

Now let $\widehat{J}$ be any other operator in $L(l^{p^*})$ such that $\|\widehat{J}\| = 1$ and $\widehat{J}(T) = \langle J(\delta_1), y \rangle = 1$.

Since l^p is uniformly convex we get $\widehat{J}(\delta_1) = y^*$. Further, since $\text{supp}(y) = N$ we have $\text{supp}(y^*) = N$. Now, if $J(\delta_K) \neq 0$, for some k, then it follows from Lemma 2.1 of [3] that $\|\widehat{J}\| > 1$. Hence we have $J(\delta_k) = 0$ for all $k > 1$. Consequently, $\widehat{J} = J$ and $\delta_1 \otimes y$ is smooth. This ends the proof.

We have to point out that $T = \delta_1 \otimes y$ is not (1-1) nor $T^* = y \otimes \delta_1$ is (1-1).

Now we present a different type of proof to Holub's result [4] which exposes another difference between $l^p, p \neq 2$ and l^2.

THEOREM 2.9 [Holub]. *Let* $T \in S_1(N(l^2))$. *The following statements are equivalent*:

(i) T *is smooth in* $S_1(N(l^2))$;

(ii) *Either* T *or* T^* *is* (1-1).

Proof. (i) $\to$ (ii). If T is smooth then there is a unique $J \in S_1(L(l^2))$ such that $J(T) = 1$. So if $T = \sum_n \lambda_n e_n \otimes f_n$ with $\sum \lambda_n = 1$ then

$$(*) \qquad J(T) = \sum_n \lambda_n \langle e_n, J f_n \rangle.$$

Since $N(l^2) = (L(l^2))^*$ and the inclusion map is an isometry (see [12]), we can consider J as the element of $L(l^2)$, where T attains its norm at. Hence, by [7], J is an extreme point of $B_1(L(l^2))$. Consequently, J is an isometry or J^* is an isometry [5]. From (*) and the fact that l^2 is uniformly convex we get $J f_n = e_n$ for all n.

Now, if T or T^* is not (1-1), then neither (e_n) nor (f_n) is complete. Hence $J = J_1 \otimes J_2$, where $J_1 = \sum_n e_n \otimes f_n$, $\operatorname{supp}(J_2)$ is orthogonal to $\operatorname{supp}(J_1)$ and $\|J_2(x)\| = \|x\|$ for all $x \in \operatorname{supp}(J_2)$. But then J and J_1 are two elements in $S_1(L(l^2))$ such that $J(T) = J_1(T) = 1$. This contradicts with (i). Hence either T is (1-1) or T^* is (1-1).

(ii) $\to$ (i). Let $T = \sum_n \lambda_n e_n \otimes f_n$, where (e_n) is complete and $\sum_n \lambda_n = 1$. Then if $J_1, J_2 \in S_1(L(l^2))$ are such that

$$J_1(T) = J_2(T) = 1$$

we get $\langle e_n, J_1 f_n \rangle = \langle e_n, J_2 f_n \rangle$ or $\langle J_1^* e_n, f_n \rangle = \langle J_2^* e_n, f_n \rangle$. Hence $J_1^* e_n = J_2^* e_n$. Consequently $J_1^* = J_2^*$ and so $J_1 = J_2$. This ends the proof.

We have to emphasize now two facts:

(i) The extreme points of $B_1(L(l^2))$ are completely characterized in [5], while this is not the case for $B_1(L(l^p)), p \neq 2$ (see [6]).

(ii) Every element in $N(l^2)$ has a unique representation $T = \sum_n u_n \otimes \theta_n$, $\|T\|_n = \sum_n \|u_n\| \|\theta_n\|$. This is not true in $N(l^p)$ for $p \neq 2$.

Hence the following definition is invitable:

DEFINITION 2.11. Let X be a Banach space. An operator $T \in N(X)$ is called exact if and only if it admits a representation

$$T = \sum_n u_n^* \otimes \theta_n, u_n^* \in X^*, \theta_n \in X$$

such that $\|T\|_n = \sum_n \|u_n^*\| \|\theta_n\|$.

In the case X when is finite dimensional it is known that every $T \in N(X)$ has a representation $\sum_{n=1}^m u_n^* \otimes \theta_n, m \leq 2k$, where $k = dim(X)$, and $\|T\|_n = \sum_n \|u_n^*\| \|\theta_n\|$. Hence every $T \in N(X)$ is exact.

For $X = l^2$, it is well-known (see [9]) that every $T \in N(l^2)$ is exact. However, it is not known (to the author) for $N(l^p), l < p \neq 2 < \infty$ wheather every $T \in N(l^p)$ is exact or not.

Now, concerning smooth points of $S_1(L(l^p)), l < p < \infty$, we can state the following:

THEOREM 2.12. *Let $T \in S_1(N(l^p)), l < p < \infty$, be an exact operator. Then the following are equivalent*:

(i) *T is a smooth element of $S_1(N(l^p))$;*

(ii) *$T = \sum_n \lambda_n u_n^* \otimes \theta_n, \sum_n \lambda_n = 1$ and $\{u_n^* \otimes \theta_n\}$ is a smooth set in $l^{p^*} \hat{\otimes} l^p$.*

Proof. (i) → (ii). Let T be an exact smooth operator. Then

$$T = \sum_n e_n^* \otimes f_n, \sum_n \|e_n^*\| \|f_n\| = \|T\| = 1.$$

Let $\lambda_n = \|e_n^*\| \|f_n\|, u_n = \frac{e_n^*}{\|e_n^*\|}, \theta_n = \frac{f_n}{\|f_n\|}$. Then $T = \sum_n \lambda_n u_n^* \otimes \theta_n$. If $\{u_n^* \otimes \theta_n\}_n$ is not smooth then there are $J_1, J_2 \in S_1(L(l^{p^*}))$ such that

$$\langle \theta_n, J_1 u_n^* \rangle = \langle \theta_n, J_2 u_n^* \rangle = 1 \quad \text{for all } n.$$

Thus we have

$$J_1(T) = J_2(T) = 1$$

which contradicts with the smoothness of T.

(ii) → (i). Let $T = \sum_n \lambda_n u_n^* \otimes \tau e_n$ with $\|T\|_n = \sum_n \lambda_= 1$ and $\{u_n^* \otimes \theta_n\}_n$ being a smooth set in $l^{p^*} \hat{\otimes} l^p$. If $J_1, J_2 \in S_1(L(l^{p^*}))$ be such that

$$J_1(T) = J_2(T) = 1$$

we get $\langle \theta_n, J_i u_n^* \rangle = 1$ for all n and $i = 1, 2$. This contradicts with the smoothness of $\{u_n^* \otimes \theta_n\}_n$. Hence T is smooth. This ends the proof.

Remark: Theorem 2.12 is true for any Banach space X for which

$$N(X) = X^* \hat{\otimes} X.$$

Now, using the same idea as in the proof of Theorem 2.8, one can prove:

THEOREM 2.13. *Let* $T \in S_1(N(l^p))$, $l < p < 2$, *be such that*

(i) $T = \sum_i \lambda_i \delta_i \otimes y_i$, $\|y_i\| = 1$ *and* $\|T\|_n = \sum \lambda_n = 1$;

(ii) $\operatorname{supp}(y_i) \cap \operatorname{supp}(y_i) = \varphi$ *if* $i \neq j$;

(iii) $\bigcup_i \operatorname{supp}(y_i) = N$.

Then T *is a smooth element of* $S_1(N(l^p))$.

Now, for finite dimensional spaces we have:

THEOREM 2.14. *Let* X *be a finite dimensional Banach space. Then the following statements are equivalent*:

(i) T *is a smooth operator in* $S_1(N(X))$;

(ii) T *is a convex combination of the elements of some smooth set of* $S_1(X^* \hat{\otimes} X)$.

Proof. It follows from the fact that every $T \in S_1(N(X))$ is exact and the use of the same argument as in the proof of Theorem 2.12.

We end this section with the following two questions:

Q1: Is it true that if $T \in S_1(N(l^p))$, $l < p < \infty$, then T is exact?

Q2: What kind of conditions on (u_n^*) and θ_n make $\{u_n^* \otimes \theta_n\}_n$ a smooth set in $N(l^p)$?

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