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ASYMPTOTIC OF EXTREMES OF MOVING MINIMA IN ARRAYS OF INDEPENDENT RANDOM VARIABLES

1. Introduction

Let $\{X_{n,i}, i = 1, \ldots, n, n = 1, 2, \ldots\}$ be an array of independent random variables, which have identical distribution function F_n for fixed n. We define sequence of maxima of moving minima based on array $\{X_{n,i}\}$

$$M_{n,m}^{(1)} = \max_{1 \le j \le n-m+1} \min_{j \le i < j+m} X_{n,i}, \tag{1}$$

where $1 \le m \le n$.

Random variables $M_{n,m}^{(1)}$ have important interpretation in the reliability theory as lifetimes consecutive-m-out-of-n systems. A consecutive-m-out-of-n system consists of n identical and linearly ordered components. The components are independent random variables $X_{n,1}, \ldots, X_{n,n}$ with identical distribution function F_n. The system will fail if and only if a least m consecutive components fail. The lifetime of system is therefore random variable $M_{n,m}^{(1)}$ defined by (1). Consecutive-m-out-of-n systems have extensive applications. Recently they have been proposed to model telecommunication systems and oil pipelines, vacuum in accelerators, computer ring network and spacecraft relay station (see e.g. [1], [3], [4] and papers referred there). Many authors have been interested in the problem of investigation of asymptotic lifetime of $M_{n,m}^{(1)}$ system (see e.g. [3], [4]). Recently, E.R. Canfield and W.P. McCormick have studied the asymptotic of $M_{n,m}^{(1)}$ in the case, where both n and $m = m_n$ change (see [1]).

Among other things they showed, that if

$$\frac{m_n}{\ln n} \to d \ge 0, \quad \text{as } n \to \infty \tag{2}$$

then

$$P\{M_{n,m}^{(1)} \le u_n\} \to e^{-\theta\lambda}, \quad \text{as } n \to \infty \tag{3}$$

where $\theta = 1 - \exp(-\frac{1}{d})$, while $\lambda > 0$ and the sequence of real numbers $\{u_n, n = 1, 2, \ldots\}$ are defined by the equality

$$nP^{m_n}\{X_{n,1} > u_n\} = \lambda. \tag{4}$$

In the proof they have used the method of analyzing the singularities of generating functions. This result we can also obtain by applying standard methods of extremal value theory, which are used to determine extremal index (see [2]).

In this paper we extend the presented above result (3) to the case of any k-th order statistic, assuming the sequence $m = m_n$ satisfies

$$\frac{m_n}{\ln n} \to 0, \qquad \text{when } n \to \infty. \tag{5}$$

The method of the proof essentially differs from the method used by E.R. Canfield and W.P. McCormic. It is based on the investigation of asymptotic of row sums of arrays independent, zero-one valued random variables [5]. This method also allows to get estimations of rate of convergence.

2. Lemmas

Before we formulate and prove the main result of this paper, we will present some lemmas which play important role in the next part of this paper.

Let $\{X_{n,i}, i = 1, \ldots, n, n = 1, 2, \ldots\}$ be an array of independent random variables, which have identical distribution function F_n for fixed n.

Denote by

$$V_{n,j} = \min_{j \le i < j + m_n} X_{n,i}, \qquad j = 1, 2, \ldots, n - m_n + 1, \tag{6}$$

where m_n is a sequence of positive integers. Consider an array of zero-one valued random variables of the form $\{I_{n,j}, j = 1, \ldots, n - m_n + 1\}$, where $I_{n,j} = I_{\{V_{n,j} > u_n\}}$, u_n is a sequence of real numbers, I_A denotes the indicator function of a set A.

Set

$$S_n = \sum_{j=1}^{n-m_n+1} I_{n,j}.$$

Lemma 1. *For all $n = 1, 2, \ldots$ and $k = 1, \ldots, n - m_m + 1$ we have*

$$\left| P\{S_n < k\} - e^{-\lambda_n} \sum_{s=0}^{k-1} \frac{\lambda_n^s}{s!} \right| \le 2\max(T_1, T_4) \frac{(1 + T_4 e^{2\lambda_n + 1 - b_n})(2 + 0{,}6\frac{\lambda_n + 2}{b_n} + e^{2\lambda_n + 1 - b_n})}{(1 - T_1/2 - T_4(1 + 0{,}6\frac{\lambda_n + 2}{b_n}))}, \tag{7}$$

where

$$\lambda_n = (n - m_n + 1)P\{V_{n,1} > u_n\},$$
$$b_n = (e(2m_n - 1)P\{V_{n,1} > u_n\})^{-1},$$
$$T_1 = 2P^2\{V_{n,1} > u_n\} \sum_{j=1}^{m_n-1} (n - m_n + 2 - j),$$
$$T_2 = 2 \sum_{j=2}^{m_n-1} (n - m_n + 2 - j)P\{V_{n,1} > u_n, V_{n,j} > u_n\},$$
$$T_3 = 4 \sum_{i=2}^{m_n-1} \sum_{j=1}^{i-1} \sum_{l=j+m_n}^{i+m_n-1} P\{V_{n,i} > u_n, V_{n,j} > u_n, V_{n,l} > u_n\}$$
$$+ 2 \sum_{i=m_n}^{n-2m_n+2} \sum_{j=i-m_n+1}^{i-1} \sum_{l=j+m_n}^{i+m_n-1} P\{V_{n,i} > u_n, V_{n,j} > u_n, V_{n,l} > u_n\},$$
$$T_4 = T_2 + eT_3, (x)_+ = \max(0, x).$$

Proof. Note that the random variables $I_{n,j}, j = 1, \ldots, n - m_n + 1$ are $(m_n - 1)$-dependent in ever row of the array $\{I_{n,j}\}$. Then for any given $n = 1, 2, \ldots, k = 1, \ldots, n - m_n + 1$ we have

$$\left|P\{S_n < k\} - e^{-\lambda_n} \sum_{s=0}^{k-1} \frac{\lambda_n^s}{s!}\right| = \left|\sum_{s=0}^{k-1} \left(P\{S_n = s\} - e^{-\lambda_n} \frac{\lambda_n^s}{s!}\right)\right|$$
$$\leq \sum_{s=0}^{k-1} \left|P\{S_n = s\} - e^{-\lambda_n} \frac{\lambda_n^s}{s!}\right|.$$

It is easy to see, that random variables $I_{n,j}$ are identically distributed and the following relation holds

$$P\{V_{n,i_1} > u_n, \ldots, V_{n,i_r} > u_n\} = P\{V_{n,i_1+p} > u_n, \ldots, V_{n,i_r+p} > u_n\}$$

for every positive integers r, p such that

$$1 \leq i_1 < \ldots < i_r \leq n - m_n + 1$$

and

$$1 < i_r + p \leq n - m_n + 1.$$

Using Theorem 2 [5] we have the inequality (7) which ends the proof.

Denote by

$$(8) \quad \min(V_{n,j}, j = 1, \ldots, n - m_n + 1) = M_{n,m_n}^{(n-m_n+1)} \leq \ldots \leq M_{n,m_n}^{(1)}$$
$$= \max(V_{n,j}, j = 1, \ldots, n - m_n + 1)$$

order statistics of the sequence $V_{n,1}, \ldots, V_{n,n-m_n+1}$.

LEMMA 2. *Let* $\lambda_n = (n - m_n + 1)(1 - F_n(u_n))^{m_n}$, *where* $\{u_n\}$ *is a sequence of real numbers,* $\{m_n\}$ *is a sequence of positive integers. Then for each* $n = 1, 2, \ldots$ *and* $k = 1, \ldots, n - m_n + 1$ *we have*

$$\left| P\{M_{n,m_n}^{(k)} \le u_n\} - e^{-\lambda_n} \sum_{s=0}^{k-1} \frac{\lambda_n^s}{s!} \right| \tag{9}$$

$$\le 2a_n \frac{(1 + a_n e^{2\lambda_n + 1 - b_n})(2 + 0,6\frac{\lambda_n+2}{b_n} + e^{2\lambda_n+1-b_n})}{(1 - c_n/2 - a_n(1 + 0,6\frac{\lambda_n+2}{b_n}))_+},$$

where

$$\begin{aligned} b_n &= (e(2m_n - 1)(1 - F_n(u_n))^{m_n})^{-1}, \\ c_n &= (2n - 3m_n + 4)(m_n - 1)(1 - F_n(u_n))^{2m_n}, \\ a_n &= (2n - 3m_n + 3)(m_n - 2)(1 - F_n(u_n))^{m_n+1} \\ &\quad + 2e(n - m_n + 1)(m_n - 1)(1 - F_n(u_n))^{2m_n} \\ &\quad \cdot ((m_n - 1)(1 - F_n(u_n)) + 1). \end{aligned}$$

Proof. Notice that $P\{M_{n,m_n}^{(k)} \le u_n\} = P\{S_n < k\}$. Because the random variables $\{X_{n,i}, i = 1, \ldots, n, n = 1, 2, \ldots\}$ are identically distributed and independent, then for $\lambda_n, T_1, T_2, T_3, T_4$ from Lemma 1 we have

$$\begin{aligned} \lambda_n &= (n - m_n + 1)(1 - F_n(u_n))^{m_n}, \\ T_1 &= (2n - 3m_n + 4)(m_n - 1)(1 - F_n(u_n))^{2m_n}, \\ T_2 &\le (2n - 3m_n + 3)(m_n - 2)(1 - F_n(u_n))^{m_n+1}, \\ T_3 &\le 2(n - m_n + 1)(m_n - 1)(1 - F_n(u_n))^{2m_n}((m_n - 1)(1 - F_n(u_n)) + 1). \end{aligned}$$

Thus

$$\begin{aligned} \max(T_1, T_4) \le\ & \max((2n - 3m_n + 4)(m_n - 1)(1 - F_n(u_n))^{2m_n}, \\ & (2n - 3m_n + 3)(m_n - 2)(1 - F_n(u_n))^{m_n+1} \\ & + 2e(n - m_n + 1)(m_n - 1)(1 - F_n(u_n))^{2m_n} \\ & \cdot ((m_n - 1)(1 - F_n(u_n)) + 1)) \\ =\ & (2n - 3m_n + 3)(m_n - 2)(1 - F_n(u_n))^{m_n+1} \\ & + 2e(n - m_n + 1)(m_n - 1)(1 - F_n(u_n))^{2m_n} \cdot \\ & \cdot ((m_n - 1)(1 - F_n(u_n)) + 1) = a_n \end{aligned}$$

By Lemma 1 and an obvious inequality

$$\frac{1}{1-d} < \frac{1}{1-f} \quad \text{for } 0 < d, f < 1, d < f,$$

we get (9).

3. The main result

Let $\{X_{n,i}, i = 1,\dots,n, n = 1,2,\dots\}$ be an array of random variables, which are defined before and $\{V_{n,j}, j = 1,\dots,n-m_n+1, n = 1,2,\dots\}$ be an array of random variables given by (6). Consider order statistics $M^{(k)}_{n,m_n}$ defined by (8) for $k = 1,\dots,n-m_n+1$. We will present the main theorem and prove the convergence of the distributions of the random variables $M^{(k)}_{n,m_n}$ to the limits which are represented in terms of a Poisson distribution. There will also appear an estimation of the rate of convergence.

THEOREM 1. *Let m_n be a sequence of positive integers satisfying*

$$m_n = o(\ln n). \tag{10}$$

Assume that u_n is a sequence of real numbers such that

$$\lim_{n\to\infty} n(1-F_n))^{m_n} = \lambda, \lambda > 0 \tag{11}$$

and there exists a constant $K > 0$ such that

$$\ln n(1-F_n(u_n)) \le K \quad \textit{for almost all } n. \tag{12}$$

Then for each $k = 1,2,\dots$

$$P\{M^{(k)}_{n,m_n} \le u_n\} \overset{n\to\infty}{\longrightarrow} e^{-\lambda}\sum_{s=0}^{k-1}\frac{\lambda^s}{s!}. \tag{13}$$

Proof. Notice that for $m_n = o(n)$ and a_n, b_n, c_n from Lemma 2 we have

$$\begin{aligned} a_n = {} & \frac{2n-3m_n+3}{n}\frac{m_n-2}{\ln n}\ln n(1-F_n(u_n))n(1-F_n(u_n))^{m_n} \\ & + 2e\frac{n-m_n+1}{n}\frac{m_n-1}{n}n^2(1-F_n(u_n))^{2m_n} \\ & \cdot\left(\frac{(m_n-1)}{\ln n}\ln n(1-F_n(u_n))+1\right), \\ c_n = {} & \frac{(2n-3m_n+4)}{n}\frac{m_n-1}{n}n^2(1-F_n(u_n))^{2m_n}, \\ b_n = {} & (e\frac{2m_n-1}{n}n(1-F_n(u_n))^{m_n})^{-1}. \end{aligned}$$

We also have

$$\begin{aligned} & \left|P\{M^{(k)}_{n,m_n} \le u_n\} - e^{-\lambda}\sum_{s=0}^{k-1}\frac{\lambda^s}{s!}\right| \\ & \le \left|P\{M^{(k)}_{n,m_n} \le u_n\} - e^{-\lambda_n}\sum_{s=0}^{k-1}\frac{\lambda_n^s}{s!}\right| + \left|e^{-\lambda}\sum_{s=}^{k-1}\frac{\lambda^s}{s!} - e^{-\lambda_n}\sum_{s=0}^{k-1}\frac{\lambda_n^s}{s!}\right| \\ & = \left|P\{M^{(k)}_{n,m_n} \le u_n\} - e^{-\lambda_n}\sum_{s=0}^{k-1}\frac{\lambda_n^s}{s!}\right| + o(1). \end{aligned} \tag{14}$$

The last equality follows from the fact that

$$\begin{aligned}\lambda_n &= (n - m_n + 1)(1 - F_n(u_n))^{m_n} \\ &= \frac{n - m_n + 1}{n} n(1 - F_n(u_n))^{m_n} \overset{n\to\infty}{\longrightarrow} \lambda.\end{aligned}$$

The assumption (10)–(12) imply that both sequences a_n and c_n converge to zero and b_n converges to infinity (as $n \to \infty$). But this means, that the right-hand side of the inequality (9) also converges to zero. Hence and from (14) we have the relation (13).

Remark 1. The inequalities (14) and (9) give the estimation of the rate of convergence of the distributions the k-th order statistics for the array $\{V_{n,j}\}$.

Finally, we will give some examples of sequences $\{m_n\}$ and distribution functions F_n, which satisfy assumptions of Theorem 1.

EXAMPLE. Let m_n be one of the form

$$m_n = \sqrt{\ln n}, \tag{15}$$

$$m_n = \operatorname{arctg} n, \tag{16}$$

$$m_n = \ln(\ln n), \tag{17}$$

$$m_n = \frac{\ln n}{\ln(\ln n)}. \tag{18}$$

Define a sequence of distribution functions by

$$F_n(x) = \begin{cases} 0, & x < u_n - 1 \\ \left(1 - \left(\frac{\lambda}{n}\right)^{\frac{1}{m_n}}\right)(x + 1 - u_n), & u_n - 1 \le x < u_n \\ \left(\frac{\lambda}{n}\right)^{\frac{1}{m_n}}(x - u_n) + 1 - \left(\frac{\lambda}{n}\right)^{\frac{1}{m_n}}, & u_n \le x < u_n + 1 \\ 1, & x \ge u_n + 1 \end{cases} \tag{19}$$

where $\{u_n\}$ is an arbitrary sequence of real numbers. It is easy to see that $m_n = o(\ln n)$ and F_n satisfies (11). Therefore we have the relation

$$\ln(\ln n(1 - F_n(u_n))) = \ln(\ln n) - \frac{\ln n}{m_n} + \frac{\ln \lambda}{m_n}.$$

Because $\frac{\ln\lambda}{m_n}$ tends to zero as $n \to \infty$ and

$$\lim_{n\to\infty}\left(\ln(\ln n) - \frac{\ln n}{m_n}\right) = \begin{cases} -\infty, & \text{for } m_n \text{ given by (15)–(17)} \\ 0, & \text{for } m_n \text{ given by (18)} \end{cases}$$

we also have (12).

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