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# INVERSE THEOREM FOR SZÁSZ-BETA OPERATORS

In the present paper, we prove an inverse theorem for the recently introduced Szász–Beta operators, using the technique of Peetre's $K$-functional.

## 1. Introduction

Durrmeyer [3] introduced the integral modification of Bernstein polynomials to approximate Lebesgue integrable functions on [0,1]. Several researches introduced and studied Durrmeyer type summation-integral operators (see e.g. [1], [7], [8], [9] and [10] etc.). Recently Gupta et al. [5] defined a new sequence $B_n(f,x)$ of linear positive operators by combining Szász and Beta operators by setting

$$B_n(f,x) = \sum_{k=0}^{\infty} p_{,k}(x) \int_0^{\infty} b_{n,k}(t) f(t) dt, x \in R^+, \tag{1.1}$$

where $P_{n,k}(x) = e^{-nx}\frac{(nx)^k}{k!}$ and $b_{n,k}(t) = \frac{1}{B(k+1,n)}\frac{t^k}{(1+t)^{n+k+1}}$ with $B(k+1,n) = k!(n-1)!/(n+k)!$ the usual Beta function.

They use them to approximate Lebesgue integrable functions $f$ on $R^+ = [0,\infty)$ and sharpened the previous direct theorems on simultaneous approximation. The paper [5] motivated us to state and prove an inverse theorem for the operators (1.1). The main tool in our considerations plays Peetre's $K$-functional.

We denote by $C_\beta[0,\infty), (\beta > 0)$ the class of continuous function on $R^+$ satisfying $|f(t)| \le Mt^\beta, M > 0$. Note that for $f \in C_\beta[0,\infty)$ the operators (1.1) are well-defined for $n > \beta$, only.

Further by $C_0$ we mean the set of continuous function on $(0,\infty)$ having a compact support and by $C_0^m$ — the subset of $C_0$ of $m$-times continuously differentiable functions.

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For given $0 < a < a' < b < +\infty$ we define Peetre's $K$-functional by

$$K(\xi, f) = \inf\{\|f - g\|_{C[a,b]} + \xi\|g''\|_{C[a,b]} : g \in G\},$$

where $0 < \xi \leq 1$ and

$$G = \{g \in C_0^2,\ \operatorname{supp} g \subset [a', b']\}.$$

The modulus of smoothness of $f$ is defined by

$$\omega_k(f, h, a, b) = \sup\{|\Delta_t^k f(x)| : |t| \leq h, x + kt \in [a, b]\},$$

where

$$\Delta_t^k f(x) = \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} f(x + jt).$$

## 2. Auxiliary results

In this section, we first mention some results which are necessary to prove the main theorem.

LEMMA 2.1. *Let*

$$U_{n,m}(x) = \sum_{k=0}^{\infty} p_{n,k}(x) \left(\frac{k}{n} - x\right)^m, \qquad m = 0, 1, 2, \ldots.$$

*Then* $U_{n,0}(x) = 1,\ U_{n,1}(x) = 0$ *and* $U_{n,2}(x) = \frac{x}{n}$ *and the recurrence relation*

$$nU_{n,m+1}(x) = x[U_{n,m}^{(1)}(x) + mU_{n,m-1}(x)]$$

*holds.*

Consequently

(i) $U_{n,m}(x)$ is a polynomial in $x$ of degree $\leq m$;

(ii) $U_{n,m}(x) = o_x(n^{-[(m+1)/2]})$,

where $[\alpha]$ stands for the integer part of $\alpha$.

LEMMA 2.2 [5]. *Let*

$$V_{n,m}(x) = \sum_{k=0}^{\infty} p_{n,k}(x) \int_0^{\infty} b_{n,k}(t)(t - x)^m dt.$$

*Then*

$$V_{n,0}(x) = 1, V_{n,1}(x) = \frac{1 + x}{n - 1}$$

*and we have the following recurrence relation*

$$(n - m - 1)V_{n,m+1}(x) = xV_{n,m}^{(1)}(x) + [(m + 1)(1 + 2x) - x]V_{n,m}(x) + mx(2 + x)V_{n,m-1}(x), \qquad n > m - 1.$$

*Consequently, for all* $x \in R^+$

$$V_{n,m}(x) = o_x(n^{-[(m+1)/2]}).$$

LEMMA 2.3. *Let*

$$\varphi_{n,m}(x) = \sum_{k=0}^{\infty} p_{n,k}(x) \int_0^{\infty} b_{n,k}(t) t^m \, dt.$$

*Then each of $\varphi_{n,m}(x)$ is a polynomial (in $x$) of degree $m$ and a rational function in $n$. Moreover for each $x \in R^+$, $\varphi_{n,m}(x) = o_x(1)$.*

Proof. Using Lemma 2.2 we have

$$\varphi_{n,0}(x) = 1$$

and

$$\begin{aligned}\varphi_{n,1}(x) &= \sum_{k=0}^{\infty} p_{n,k}(x) \int_0^{\infty} b_{n,k}(t)(t-x)\,dt + x \\ &= \frac{1+x}{n-1} + x = \frac{1+nx}{n-1}, \qquad n > 1.\end{aligned}$$

Since

$$x p_{n,k}^{(1)}(x) = (k - nx) p_{n,k}(x) \quad \text{and} \quad t(1+t) b_{n,k}^{(1)}(t) = [k - (n+1)t] b_{n,k}(t),$$

then we obtain

$$x\varphi_{n,m}^{(1)}(x) = \sum_{k=0}^{\infty} (k - nx) p_{n,k}(x) \int_0^{\infty} b_{n,k}(t) t^m \, dt$$

$$= \sum_{k=0}^{\infty} p_{n,k}(x) \int_0^{\infty} [\{k - (n+1)t\} + (n+1)t - nx] b_{n,k}(t) t^m \, dt$$

$$= \sum_{k=0}^{\infty} p_{n,k}(x) \int_0^{\infty} t(1+t) b_{n,k}^{(1)}(t) t^m \, dt + (n+1)\varphi_{n,m+1}(x) - nx\varphi_{n,m}(x)$$

and therefore for $n > m+1$ we have

$$(n - m - 1)\varphi_{n,m+1}(x) = x\varphi_{n,m}^{(1)}(x) + (nx + m + 1)\varphi_{n,m}(x).$$

From the latter recurrence relation one can easily prove the required result.

COROLLARY 2.4. *Let $\beta$ and $\delta$ be two positive numbers. Then for any $m > 2\beta$, there exists a constant $K_m$ such that*

$$\Big\| \sum_{k=0}^{\infty} p_{n,k}(x) \int_{I_x(\delta)} b_{n,k}(t) t^{\beta} \, dt \Big\|_{C[a,b]} \le K_m n^{-m},$$

*where $I_x(\delta) = (0, \infty) \setminus [x - \delta, x + \delta]$.*

Proof. Using Lemma 2.2 we have the estimations

$$\sum_{k=0}^{\infty} p_{n,k}(x) \int\limits_{|t-x|\geq\delta} b_{n,k}(t)t^{\beta}dt \leq \sum_{k=0}^{\infty} p_{n,k}(x) \int\limits_{|t-x|\geq\delta} b_{n,k}(t)\frac{(t-x)^{2m}}{\delta^{2m}}t^{\beta}dt$$

$$\leq \frac{1}{\delta^{2m}}\Big(\sum_{k=0}^{\infty} p_{n,k}(x) \int\limits_{|t-x|\geq\delta} b_{n,k}(t)(t-x)^{4m}dt\Big)^{1/2}$$

$$\times \Big(\sum_{k=0}^{\infty} p_{n,k}(x) \int\limits_{|t-x|\geq\delta} b_{n,k}(t)t^{2\beta}dt\Big)^{1/2}$$

$$\leq \frac{K_1}{\delta^{2m}} n^{-m}\Big(\sum_{k=0}^{\infty} p_{n,k}(x) \int\limits_{|t-x|\geq\delta} b_{n,k}(t)t^{2\beta}dt\Big)^{1/2}.$$

Since (in view of Lemma 2.3) for $m > 2\beta$ we have

$$\sum_{k=0}^{\infty} p_{n,k}(x) \int\limits_{|t-x|\geq\delta} b_{n,k}(t)t^{2\beta}dt$$

$$= \sum_{k=0}^{\infty} p_{n,k}(x) \int\limits_{t\leq x-\delta} b_{n,k}(t)t^{2\beta}dt + \sum_{k=0}^{\infty} p_{n,k}(x) \int\limits_{t\geq x+\delta} b_{n,k}(t)t^{2\beta}dt$$

$$\leq \sum_{k=0}^{\infty} p_{n,k}(x) \int_0^{\infty} b_{n,k}(t)(x-\delta)^{2\beta}dt + \sum_{k=0}^{\infty} p_{n,k}(x) \int_0^{\infty} b_{n,k}(t)\frac{t^m}{(x+\delta)^{m-2\beta}}dt$$

$$= (x-\delta)^{2\beta} + \frac{\varphi_{n,m}(x)}{(x+\delta)^{m-2\beta}} \leq K_2 \quad \text{for all } x \in [a,b]$$

and then the required inequality holds.

## 3. Main theorem

In this section we shall prove the following inverse theorem:

THEOREM 3.1. *Let* $0 < a_1 < a_2 < b_2 < b_1 < \infty$, $0 < \alpha < 2$ *and suppose that for* $f \in C_\beta[0,\infty)$ *we have*

(i) $\|B_n(f,.) - f(.)\|_{C[a_1,b_1]} = o(n^{-\alpha/2})$.

*Then*

(ii) $f \in \mathrm{Lip}^*(\alpha, C[a_2,b_2])$,

*where* $\mathrm{Lip}^*(\alpha, C[a,b])$ *denotes the Zygmund class of functions for which* $\omega_2(f,h,a,b) \leq Mh^\alpha$.

There are two essential steps to prove the above theorem.

(I) We first reduce the above problem to the following lemmas as a special case when $f$ has a compact support inside some interior interval $[a',b']$ of $(a_1,b_1)$.

LEMMA 3.2. *Let $f \in C_0$ with supp$f \subset [a'', b'']$, $0 < a < a' < a'' < b'' < b' < b < \infty$, and suppose that*

$$\|B_n(f,.) - f(.)\|_{C[a,b]} = o(n^{-\alpha/2}).$$

*Then*

$$K(\xi, f) \leq K_0[n^{-\alpha/2} + n\xi K(n^{-1}, f)]. \tag{3.1}$$

*Consequently $K(\xi, f) \leq K_1 \xi^{\alpha/2}$ for some constant $K_1$.*

Proof. Since supp $f \subset [a'', b'']$ then following May [6] there exists $h \in G$ such that for $i = 0$ and 2 we have

$$\|h^{(i)}(.) - B_n^{(i)}(f,.)\|_{C[a,b]} \leq K_2 n^{-1}.$$

Therefore,

$$K(\xi, f) \leq 2K_2 n^{-1} + \|f(.) - B_n(f,.)\|_{C[a,b]} + \xi \|B_n^{(2)}(f,.)\|_{C[a,b]}$$

Hence, it is sufficient to show that there exists a constant $K_3$ such that for each $g \in G$ we have

$$\|B_n^{(2)}(f,.)\|_{C[a',b']} \leq K_3 n\{\|f - g\|_{C[a',b']} + n^{-1}\|g^{(2)}\|_{C[a',b']}\}. \tag{3.2}$$

In fact

$$\|B_n^{(2)}(f,.)\|_{C[a',b']} \leq \|B_n^{(2)}(f - g,.)\|_{C[a',b']} + \|B_n^{(2)}(g,.)\|_{C[a',b']}. \tag{3.3}$$

Since

$$\frac{\partial^2}{\partial x^2}\Big[\sum_{k=0}^{\infty} p_{n,k}(x) b_{n,k}(t)\Big] = \frac{1}{x^2}\sum_{k=0}^{\infty}[(k - nx)^2 - k] p_{n,k}(x) b_{n,k}(t),$$

then by using Lemma 2.1, we have

$$\int_0^\infty \left|\frac{\partial^2}{\partial x^2}\Big(\sum_{k=0}^{\infty} p_{n,k}(x) b_{n,k}(t)\Big)\right| dt \leq \sum_{k=0}^{\infty} \frac{|(k - nx)^2 - k|}{x^2} p_{n,k}(x) \int_0^\infty b_{n,k}(t)\,dt \leq \frac{2x}{n}.$$

Therefore we obtain

$$\|B_n^{(2)}(f - g,.)\|_{C[a,b]} \leq \frac{2n}{a}\|f - g\|_{C[a,b]} = K_4 n \|f - g\|_{C[a,b]}. \tag{3.4}$$

Next, using Lemma 2.2, we see that for any $m > i$

$$\int_0^\infty \left[\frac{\partial^m}{\partial x^m}\Big(\sum_{k=0}^{\infty} p_{n,k}(x) b_{n,k}(t)\Big)\right](t - x)^i\,dt = 0. \tag{3.5}$$

Also, by Taylor's Expansion Theorem, there exists $\xi$, between $t$ and $x$, such that

$$g(t) = g(x) + g^{(1)}(x)(t - x) + g^{(2)}(\xi)(t - x)^2. \tag{3.6}$$

Using (3.5) and (3.6) we obtain

$$B_n^{(2)}(g,x) = \int_0^\infty \Big[\frac{\partial^2}{\partial x^2}\Big(\sum_{k=0}^\infty p_{n,k}(x) b_{n,k}(t)\Big)\Big] g(t)dt$$
$$= \int_0^\infty [\ldots](g(x) + g^{(1)}(x)(t-x) + g^{(2)}(\xi)(t-x)^2)dt$$
$$= \int_0^\infty [\ldots] g^{(2)}(\xi)(t-x)^2 dt.$$

Making use of Lemma 2.1, Lemma 2,2 and Schwarz inequality, we get

(3.7) $$\|B_n^{(2)}(g,.)\|_{C[a,b]}$$
$$\leq \|g^{(2)}\|_{C[a,b]} \Big\| \int_0^\infty \Big|\frac{\partial^2}{\partial x^2}\Big(\sum_{k=0}^\infty p_{n,k}(x) b_{n,k}(t)\Big)\Big|(t-x)^2 dt\Big\|_{C[a,b]}$$
$$\leq \|g^{(2)}\|_{C[a,b]} \Big\|\Big(\sum_{k=0}^\infty [(k-nx)^2 + k] p_{n,k}(x)\Big) \int_0^\infty b_{n,k}(t)(t-x)^2 dt\Big\|_{C[a,b]}$$
$$\leq K_5 \|g^{(2)}\|_{C[a,b]}.$$

Hence (3.2) follows, by combining (3.3), (3.4) and (3.7). This completes the proof of the lemma.

LEMMA 3.3. *Relation* (3.1) *implies*

$$f \in \mathrm{Lip}^*(\alpha, C[a,b]).$$

Proof. Proceeding along the lines of the proof from [2], we have

(3.8) $$K(\xi, f) \leq K_6 \xi^{\alpha/2}, \text{ for some constant } K_6 > 0.$$

Now, let $0 < |\delta| < h$. Then for any $g \in G$ we have

$$|\Delta_\delta^2 f(x)| \leq |\Delta_\delta^2 (f(x) - g(x))| + |\Delta_\delta^2 g(x)| \leq 4\|f-g\|_{C[a,b]} + \delta^2 \|g^{(2)}\|_{C[a,b]}.$$

Therefore, using (3,8), we get

$$\omega_2(f,h,a,b) \leq 4K(h^2, f) \leq 4K_6 h^\alpha$$

i.e. $f \in \mathrm{Lip}^*(\alpha, C[a,b])$.

(II) In this step we shall show that, using Lemma 3.2 and Lemma 3.3, the required result follows.

Let us choose $a', a'', b', b''$ in such a way that $a_1 < a' < a'' < a_2$ and $b_2 < b'' < b' < b_1$. Take any $g \in C_0^\infty$ such that $\operatorname{supp} g \subset [a'', b'']$ and $g(x) = 1$ on $[a_2, b_2]$.

First assume that $0 < \alpha \leq 1$. For $x \in [a', b']$, we have

$$B_n(fg,x) - f(x)g(x) = g(x)[B_n(f,x) - f(x)]$$
$$+ \int_{a_1}^{b_1} \Big(\sum_{k=0}^\infty p_{n,k}(x) b_{n,k}(t)\Big) f(t)[g(t) - g(x)]dt + o(n^{-1}) = I_1 + I_2 + o(n^{-1})$$

say, where $o(n^{-1})$ term is, by Corollary 2.4, uniform for $x \in [a', b']$.

Next, making use of the assumption $\|B_n(f,.) - f(.)\|_{C[a',b']} = o(n^{-\alpha/2})$, we get

$$\|I_1\|_{C[a',b']} \le \|g\|_\infty \|B_n(f,.) - f(.)\|_{C[a',b']} \le K_7 n^{-\alpha/2}, \tag{3.10}$$

where $\|g\|_\infty = \inf\{M : |f(x)| \le M \text{ a.e. on } [a',b']\}$, $M$ is a constant.

Also, by Mean Value Theorem, we get

$$I_2 = \int_{a_1}^{b_1} \Big(\sum_{k=0}^{\infty} p_{n,k}(x) b_{n,k}(t)\Big) f(t)[g^{(1)}(\xi)(t-x)]dt.$$

Hence, by Lemma 2.2 and Cauchy–Schwarz inequality, we see that

$$\|I_2\|_{C[a',b']} = o(n^{-1/2}) \le o(n^{-\alpha/2}). \tag{3.11}$$

Combining the estimates in (3.9), (3.10) and (3.11) we obtain

$$\|B_n(fg,.) - fg(.)\|_{C[a',b']} = o(n^{-\alpha/2}).$$

Therefore, by Lemma 3.2 and Lemma 3.3, we have $fg \in \mathrm{Lip}^*(\alpha, C[a',b'])$. Since $g(x) = 1$ on $[a_2, b_2]$ it follows that $f \in \mathrm{Lip}^*(\alpha, C[a_2, b_2])$. This proves the implication (i)$\Rightarrow$(ii), when $0 < \alpha \le 1$.

Now assume that $1 < \alpha < 2$. We also choose two points $a_1^*$ and $b_1^*$ satisfying $a_1 < a_1^* < a'$ and $b' < b_1^* < b_1$. It is sufficient to prove our assertion for $1 < \alpha < 2 - \delta$, where $\delta \in (0,1)$ is arbitrary.

We may notice, from the previous result, that the condition

$$\|B_n(f,.) - f(.)\|_{C[a_1,b_1]} = o(n^{-\alpha/2}) \text{ implies } f \in Lip(1-\delta, C[a_1^*, b_1^*]).$$

Now, for $x \in [a', b']$, we have

$$\begin{aligned} B_n(fg,x) - f(x)g(x) &= g(x)[B_n(f,x) - f(x)] + f(x)[B_n(g,x) - g(x)] \\ &\quad + \int_{a_1^*}^{b_1^*} \Big(\sum_{k=0}^{\infty} p_{n,k}(x) b_{n,k}(t)\Big)[f(t) - f(x)][g(t) \\ &\quad - g(x)]dt + o(n^{-1}) = J_1 + J_2 + J_3 + o(n^{-1}), \end{aligned}$$

where $o(n^{-1})$ term holds uniformly for $x \in [a', b']$ (by Corollary 2.4).

In fact, the relation $\|J_1\|_{C[a',b']} = o(n^{-\alpha/2})$ follows from the assumption, while $\|J_2\|_{C[a',b']} = o(n^{-1}) \le o(n^{-\alpha/2})$, from Lemma 2.2.

Also, since $|f(t) - f(x)| \le K(t - x|^{1-\delta}$ and $g(t) - g(x) = g^{(1)}(\xi)(t-x)$, then, using Jensen's inequality and Lemma 2.2, we obtain

$$\|J_3\|_{C[a',b']} = o(n^{-(2-\delta)/2}) \le o(n^{-\alpha/2}).$$

Combining the above estimates of $J_1, J_2$ and $J_3$, we get

$$\|B_n(fg,.) - fg(.)\|_{C[a',b']} = o(n^{-\alpha/2}).$$

Now the result follows, as in the first case, from Lemma 3.2 and Lemma 3.3. This completes the proof of inverse theorem.

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