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AN EXTENSION OF LEWANDOWSKI'S UNIVALENCE CRITERIA

1. Introduction

In this note we obtain, by the method of subordination chains, a sufficient condition for the analyticity and the univalence of the functions defined by an integral operator. This condition involves an arbitrary function g, analytic in the unit disk. In some particular cases we find more restrictive conditions for univalence than those obtained by Lewandowski in [1] and [2]. The Corollary 3.1 resembles the criteria for starlikeness obtained by P.T.Mocanu in [4].

We denote by $U_r = \{z \in C : |z| < r\}$ the disk of z-plane, where $r \in (0,1]$, $U_1 = U, U^* = U \setminus \{0\}$, and let $I = [0, \infty)$. Let A be the class of functions f analytic in U and such that $f(0) = 0, f'(0) = 1$.

THEOREM 1.1 [1]. *Let $f \in A$. If there exists an analytic function p with positive real part in U such that $p(0) = 1$ and the inequality*

$$\left|\frac{p(z)-1}{p(z)+1}|z|^2 - (1-|z|^2)\left(\frac{zf''(z)}{f'(z)} + \frac{zp'(z)}{p(z)+1}\right)\right| \leq 1 \tag{1}$$

is true for all $z \in U$, then the function f is univalent in U.

THEOREM 1.2 [2]. *Let $a > \frac{1}{2}$, $\alpha > 0$, $\beta \in R$, $k = \frac{a}{\alpha}$ be fixed numbers and let $f \in A$ and g be analytic in U such that $f'(z) \neq 0$ and*

$$\left|\frac{zf'(z)}{f(z)g(z)} - ks\right| \leq k|s|, \qquad s = \alpha + i\beta. \tag{2}$$

If the inequality

$$\left||z|^{2k}\frac{zf'(z)}{f(z)g(z)} + (1-|z|^{2k})\left(\frac{zf'(z)}{f(z)} + s\frac{zg'(z)}{g(z)}\right) - ks\right| \leq k|s| \tag{3}$$

holds for $z \in U$, then f is univalent in U.

Let us denote by S^* the subclass of A consisting of functions which are starlike.

THEOREM 1.3 [4]. *If $f \in A$ and*

$$|f'(z) - 1| < \frac{2}{\sqrt{5}} = 0,894..., \quad z \in U, \tag{4}$$

then $f \in S^*$ and $|f(z)| < 1 + 1/\sqrt{5} = 1,447...$

2. Preliminaries

DEFINITION. A function $L : U \times I \to C$ is called a Loewner chain, if

$$L(z,t) = e^t z + a_2(t)z^2 + \cdots, \qquad |z| < 1,$$

is analytic and univalent in U for each $t \in I$ and $L(z,s) \prec L(z,t)$ for $0 \le s < t < \infty$, where by $\prec$ we denote the relation of subordination.

THEOREM 2.1 [5]. *Let $L(z,t) = a_1(t)z + a_2(t)z^2 + \cdots, a_1(t) \neq 0$ be analytic in U_r for all $t \in I$, locally absolutely continuous in I and locally uniform with respect to U_r. For almost all $t \in I$ suppose*

$$z\frac{\partial L(z,t)}{\partial z} = p(z,t)\frac{\partial L(z,t)}{\partial t}, \quad \forall z \in U_r,$$

where $p(z,t)$ is analytic in U and satisfies the condition $Re\, p(z,t) > 0$ for all $z \in U, t \in I$. If $|a_1(t)| \to \infty$ for $t \to \infty$ and $\{L(z,t)/a_1(t)\}$ forms a normal family in U_r , then for each $t \in I$ the function $L(z,t)$ has an analytic and univalent extension to the whole disk U.

3. Main results

THEOREM 3.1. *Let $f \in A$ and α, β, c be complex numbers such that $Re\ \alpha > 0$, $Re\ (\alpha + 2\beta) > 0$, $Re\ \frac{\beta}{\alpha} > -\frac{1}{2}$, $|c(\alpha+\beta) + \beta| + |\beta| \le |\alpha + \beta|$. If there exists an analytic function $g \in A$ such that*

$$\left|(1+c)\frac{f'(z)}{g'(z)} - 1\right| < 1, \quad \forall z \in U, \tag{5}$$

$$\left|\left[(1+c)\frac{f'(z)}{g'(z)} - 1\right]|z|^{2(\alpha+\beta)} + \frac{1-|z|^{2(\alpha+\beta)}}{\alpha+\beta}\left[\frac{zg''(z)}{g'(z)} - \beta\right]\right| \le 1 \tag{6}$$

for all $z \in U^*$, then the function

$$F(z) = \left(\alpha \int_0^z u^{\alpha-1} f'(u)du\right)^{1/\alpha} \tag{7}$$

is analytic and univalent in U.

Proof. Let us prove that there exists a real number $r \in (0,1]$, such that the function $L : U_r \times I \to C$ defined formally by

$$(8)\quad L(z,t) = \left[(\alpha+\beta)\int_0^{e^{-t}z} u^{\alpha-1} f'(u)du + \frac{e^{(\alpha+2\beta)t} - e^{-\alpha t}}{1+c} z^\alpha g'(e^{-t}z)\right]^{1/\alpha}$$

is analytic in U_r for all $t \in I$. Denoting

$$h_1(z,t) = (\alpha+\beta)\int_0^{e^{-t}z} u^{\alpha-1} f'(u)du,$$

we have $h_1(z,t) = z^\alpha h_2(z,t)$, where it is easy to see that the function h_2 is analytic in U for all $t \in I$ and $h_2(0,t) = \frac{\alpha+\beta}{\alpha} e^{-\alpha t}$.

From the analyticity of g' in U it follows that the function

$$h_3(z,t) = h_2(z,t) + \frac{e^{(\alpha+2\beta)t} - e^{-\alpha t}}{1+c} g'(e^{-t}z)$$

is also analytic in U and that

$$h_3(0,t) = e^{(\alpha+2\beta)t}\left[\frac{1}{1+c} + \left(\frac{c}{1+c} + \frac{\beta}{\alpha}\right) e^{-2(\alpha+\beta)t}\right].$$

Let us prove that $h_3(0,t) \neq 0, \forall t \in I$. We have $h_3(0,0) = 1 + \frac{\beta}{\alpha}$ and, since $\text{Re}\ \frac{\beta}{\alpha} > -\frac{1}{2}$, it follows that $h_3(0,0) \neq 0$. Assume now that there exists $t_0 > 0$ such that $h_3(0,t_0) = 0$. Then $e^{2(\alpha+\beta)t_0} = -[(\alpha+\beta)c + \beta]\alpha^{-1}$ and, since $|c(\alpha+\beta)+\beta| + |\beta| \le |\alpha+\beta|$ implies $|c(\alpha+\beta)+\beta| \le |\alpha|$, it follows that $e^{2Re(\alpha+\beta)t_0} \le 1$. In view of $Re(\alpha+\beta) > 0, t_0 > 0$, this inequality is imposible. Therefore, there is a disk U_r, $0 < r \le 1$, in which $h_3(z,t) \neq 0$ for all $t \in I$. Then we can choose an analytic branch of $[h_3(z,t)]^{1/\alpha}$ denoted by $h(z,t)$. We fix a determination of $(1+\frac{\beta}{\alpha})^{1/\alpha}$ denoted by γ. For $\gamma(t)$ we fix the determination equal to γ for $t = 0$, where

$$\gamma(t) = e^{(1+2\frac{\beta}{\alpha})t}\left[\frac{1}{1+c} + \left(\frac{c}{1+c} + \frac{\beta}{\alpha}\right) e^{-2(\alpha+\beta)t}\right]^{\frac{1}{\alpha}}.$$

It results that the relation (8) may be written as

$$L(z,t) = zh(z,t) = a_1(t)z + a_2(t)z^2 + \cdots, \qquad \forall z \in U_r,$$

and we obtain that the function $L(z,t)$ is analytic in U_r for all $t \in I$ and $a_1(t) = \gamma(t)$. Since $\text{Re}\ (\alpha+\beta) > 0$ and $\text{Re}\ \frac{\beta}{\alpha} > -\frac{1}{2}$, we have $\lim_{t\to\infty} |a_1(t)| = \infty$. We saw also that $a_1(t) \neq 0$ for all $t \in I$.

From the analyticity of $L(z,t)$ in U_r it follows that there is a number $r_1, 0 < r_1 < r$, and a constant $K = K(r_1)$ such that

$$|L(z,t)/a_1(t)| < K, \qquad \forall z \in U_{r_1}, \quad t \ge 0,$$

and then $\{L(z,t)/a_1(t)\}$ is a normal family in U_{r_1}. From the analyticity of $\frac{\partial L(z,t)}{\partial t}$, for all fixed numbers $T > 0$ and $r_2, 0 < r_2 < r_1$, there exists a constant $K_1 > 0$ (which depends on T and r_2) such that

$$\left|\frac{\partial L(z,t)}{\partial t}\right| < K_1, \quad \forall z \in U_{r_2}, \quad t \in [0,T].$$

It follows that the function $L(z,t)$ is locally absolutely continuous in I, locally uniform with respect to U_{r_2}. Also we have that the function

$$p(z,t) = z\frac{\partial L(z,t)}{\partial z} \Big/ \frac{\partial L(z,t)}{\partial t}$$

is analytic in $U_{r_0}, 0 < r_0 < r_2$, for all $t \geq 0$.

In order to prove that the function $p(z,t)$ has an analytic extension with positive real part in U, for all $t \geq 0$, it is sufficient to prove that the function $w(z,t)$ defined in U_{r_0} by

$$w(z,t) = \frac{p(z,t) - 1}{p(z,t) + 1}$$

can be continued analytically in U and $|w(z,t)| < 1$ for all $z \in U$ and $t \geq 0$. After computation we obtain

$$w(z,t) = \left[(1+c)\frac{f'(e^{-t}z)}{g'(e^{-t}z)} - 1\right]e^{-2(\alpha+\beta)t} + \tag{9}$$

$$+ \frac{1 - e^{-2(\alpha+\beta)t}}{\alpha+\beta}\left[\frac{e^{-t}zg''(e^{-t}z)}{g'(e^{-t}z)} - \beta\right].$$

From (5) and (6) we deduce that $g'(z) \neq 0$ for all $z \in U$ and then the function $w(z,t)$ is analytic in the unit disk U. For $t = 0$, in view of (5), we have

$$|w(z,0)| = |(1+c)\frac{f'(z)}{g'(z)} - 1| < 1. \tag{10}$$

For $z = 0$, $t > 0$, since Re $(\alpha+\beta) > 0, |c(\alpha+\beta)+\beta| + |\beta| \leq |\alpha+\beta|$ and $f, g \in A$, we get

$$|w(0,t)| = \left|\frac{[c(\alpha+\beta)+\beta]e^{-2(\alpha+\beta)t} - \beta}{\alpha+\beta}\right| < 1. \tag{11}$$

Let now be a fixed number $t > 0$ and $z \in U, z \neq 0$. In this case the function $w(z,t)$ is analytic in $\bar{U}$, because $|e^{-t}z| \leq e^{-t} < 1$ for all $z \in \bar{U}$ and it is known that

$$|w(z,t)| < \max_{|\xi|=1}|w(\xi,t)| = |w(e^{i\theta},t)|, \quad \theta = \theta(t) \in R. \tag{12}$$

Let us denote $u = e^{-t}e^{i\theta}$. Then $|u| = e^{-t}$ and from (9) we obtain

$$|w(e^{i\theta},t)| = \left|\left((1+c)\frac{f'(u)}{g'(u)} - 1\right)|u|^{2(\alpha+\beta)} + \frac{1-|u|^{2(\alpha+\beta)}}{\alpha+\beta}\left(\frac{ug''(u)}{g'(u)} - \beta\right)\right|.$$

Since $u \in U$, the relation (6) implies $|w(e^{i\theta},t)| \le 1$ and from (10), (11) and (12) we conclude that $|w(z,t)| < 1$ for all $z \in U$ and $t \ge 0$.

From Theorem 2.1 it results that the function $L(z,t)$ has an analytic and univalent extension to the whole disk U, for each $t \in I$. For $t = 0$ it results that the function

$$L(z,0) = \left[(\alpha+\beta)\int_0^z u^{\alpha-1}f'(u)du\right]^{1/\alpha}$$

is analytic and univalent in U and then the function F defined by (7) is analytic and univalent in U.

THEOREM 3.2. *Let α, β, c be complex numbers, $Re\,\alpha > 0$, $Re\,(\alpha+2\beta) > 0$, $Re\,\frac{\beta}{\alpha} > -\frac{1}{2}$, $|c(\alpha+\beta)+\beta| + |\beta| \le |\alpha+\beta|$ and let $f \in A$. If there exists an analytic function p with positive real part in U such that $p(0) = (1-c)/(1+c)$ and*

$$\left|\frac{p(z)-1}{p(z)+1}|z|^{2(\alpha+\beta)} - \frac{1-|z|^{2(\alpha+\beta)}}{\alpha+\beta}\left(\frac{zf''(z)}{f'(z)} + \frac{zp'(z)}{p(z)+1} - \beta\right)\right| \le 1 \tag{13}$$

for all $z \in U^*$, then the function F defined by (7) is analytic and univalent in the disk U.

Proof. Let p be an analytic function in U, with $\text{Re}\,p(z) > 0$ for all $z \in U$ and $p(0) = (1-c)/(1+c)$. If in Theorem 3.1 the function $g \in A$ is such that

$$g'(z) = \frac{1+c}{2}[1+p(z)]f'(z),$$

then the inequality (6) becomes (13) and the inequality (5) is true, because $\text{Re}\,p(z) > 0, \forall z \in U$.

A simple conclusion from Theorem 3.2 has a following form.

THEOREM 3.3. *Let $F \in A, F(z) \ne 0$ for all $z \in U^*$. Let α, β, c be complex numbers, such that $Re\,\alpha > 0$, $Re\,(\alpha+2\beta) > 0$, $Re\,\frac{\beta}{\alpha} > -\frac{1}{2}$, $|c(\alpha+\beta)+\beta| + |\beta| \le |\alpha+\beta|$. If there exists an analytic function p with positive real part in U such that $p(0) = (1-c)/(1+c)$ and*

$$\left|\frac{p(z)-1}{p(z)+1}|z|^{2(\alpha+\beta)} - \frac{1-|z|^{2(\alpha+\beta)}}{\alpha+\beta}\left[\frac{zF''(z)}{F'(z)} + (\alpha-1)\frac{zF'(z)}{F(z)} + \frac{zp'(z)}{p(z)+1} + 1 - (\alpha+\beta)\right]\right| \le 1 \tag{14}$$

for all $z \in U^*$, then the function F is univalent in U.

Proof. Let us consider the function $f \in A$ such that

$$f'(z) = \Big(\frac{F(z)}{z}\Big)^{\alpha-1} F'(z),$$

where we chose the uniform branch of $\big(\frac{F(z)}{z}\big)^{\alpha-1}$ equal to 1 at the origin, analytic in U. It is easy to see that the function f satisfies the assumption of Theorem 3.2, if F satisfies (14).

Remark. If in Theorem 1.2 the function g analytic in U has the form

$$g(z) = \frac{\alpha}{2as}(1 + p(z))\frac{zf'(z)}{f(z)},$$

we have the following univalence condition [6].

COROLLARY 3.1. *Let $a > \frac{1}{2}, s = \alpha + i\beta, \alpha > 0, \beta \in R, k = \frac{a}{\alpha}$ be fixed numbers and let $f \in A, f'(z) \neq 0$ in U. If there exists an analytic function p with positive real part in U such that $p(0) = \frac{2as}{\alpha} - 1$ and*

$$(15) \quad \left|\frac{p(z)-1}{p(z)+1}|z|^{2k} - \frac{1-|z|^{2k}}{k}\left[\frac{zf''(z)}{f'(z)} + \left(\frac{1}{s} - 1\right)\frac{zf'(z)}{f(z)} + \frac{zp'(z)}{p(z)+1} + 1 - k\right]\right| \le 1$$

holds in U, then f is univalent in U.

We remark that the inequalities (14) and (15) have similar form, but in (14) we have $\alpha + \beta \in C$. In the case $\alpha + \beta > 0$ we can expand the condition $|c(\alpha+\beta)+\beta| + |\beta| \le \alpha+\beta$ which derives from the study of $a_1(t)$ and $w(0,t)$.

We shall present two simple consequences of Corollary 3.1 and for this it is more useful to apply Theorem 3.2. For $c = 0, \alpha + \beta = k, k > 0$, from Theorem 3.2 we get the following one.

THEOREM 3.4. *Let $f \in A$, $k > 0$ and α be a complex number, $|\alpha - k| < k$. If there exists an analytic function p in U, such that Re $p(z) > 0, p(0) = 1$ and*

$$(16) \quad \left|\frac{p(z)-1}{p(z)+1}|z|^{2k} - \frac{1-|z|^{2k}}{k}\left(\frac{zf''(z)}{f'(z)} + \frac{zp'(z)}{p(z)+1} + \alpha - k\right)\right| \le 1$$

for all $z \in U$, then the function F defined by (7) is analytic and univalent in U.

Proof. In this case Re $(\frac{k}{\alpha} - 1) > -\frac{1}{2}$ is equivalent to $|\alpha - k| < k$ and from (11) we have

$$|w(0,t)| = \left|\frac{\alpha - k}{k}(1 - e^{-2kt})\right| < \frac{|\alpha - k|}{k} < 1.$$

COROLLARY 3.2. *Let $f \in A, k > 0, \alpha \in C, |\alpha - k| < k$. If*

$$(17) \qquad |f'(z) - 1| < 1, \qquad \forall z \in U,$$

then the function F defined by (7) *is analytic and univalent in U.*

Proof. Let us consider the function

$$(18) \qquad p(z) = \frac{2}{f'(z)} - 1$$

analytic in U with $p(0) = 1$. By (17), we have $\operatorname{Re} p(z) > 0$ in U. From (18) we get

$$-\frac{zp'(z)}{p(z)+1} = \frac{zf''(z)}{f'(z)}$$

and, by $|\alpha - k| < k$, we get immediatly that the inequality (16) holds in U.

Taking into account Theorem 1.3, we get the following results.

COROLLARY 3.3. *Let $f \in A, k > 0, \alpha \in C, |\alpha - k| < k$. If*

$$|f'(z) - 1| < \frac{2}{\sqrt{5}}, \qquad \forall z \in U,$$

then $f \in S^*$ and the function F defined by (7) *is analytic and univalent in U.*

If we take $k = 1$, from Theorem 3.4 we obtain the following corollary.

COROLLARY 3.4. *Let $f \in A$ and $\alpha \in C, |\alpha - 1| < 1$. If there exists an analytic function p with positive real part in U such that $p(0) = 1$ and*

$$(19) \left|\frac{p(z)-1}{p(z)+1}|z^2| - (1-|z|^2)\left(\frac{zf''(z)}{f'(z)} + \frac{zp'(z)}{p(z)+1}\right)\right| \le 1 - |\alpha - 1|(1-|z|^2)$$

for all $z \in U$, then the function F defined by (7) *is analytic and univalent in U.*

We observe that, if the condition (1) of Theorem 1.1 with $p(0) = 1$ will be replaced by the strong condition (19), then we have not only the univalence of f, but we obtain the univalence for a class of functions F defined by (7).

EXAMPLE. Let $k > 0, \alpha \in C, |\alpha - k| < k$. Then the function

$$F(z) = z\left[1 + \frac{2\alpha}{3(\alpha+1)}z - \frac{\alpha}{4(\alpha+2)}z^2\right]^{1/\alpha}$$

is analytic and univalent in U.

To prove it consider the function $f \in A$ of the form

$$f(z) = z + \frac{z^2}{3} - \frac{z^3}{12}.$$

So we have $|f'(z)-1| = \left|\frac{2}{3}z - \frac{1}{4}z^2\right| \le \frac{11}{12} < 1$ and from Corollary 3.2 we get that the function F defined by (20) is analytic and univalent in U.

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