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ON THE SECOND BOUNDARY-VALUE PROBLEM FOR THE AIRY EQUATION

1. Introduction

Consider the equation

$$D_x^3 u(x,t) - D_t u(x,t) = 0. \tag{1}$$

In [3] there has been examined the equation $D_t u = m D_x^3 u$ which is called the Airy equation and is a linear version of the Korteweg–de Vries (KdV) equation. It arises in the description of the slow variation of a wave front in coordinates moving with the wave. It also describes the propagation of oscillatory wave packets. In [5], [6] it is proved that equation $D_t u = D_x^3 u$ is one of the canonical forms of third order partial differential equations and it is called the equation with characteristics multiple (see [4], p. 132).

The first boundary value problem (or also called the Cattabriga problem) for Airy equation has been examined in [2], [4]; moreover, in [3] the Cauchy problem for this equation has been considered. Papers [9], [10] were devoted to solve contact problems for the said equation.

This paper concerns the second boundary–value problem for equation (1) in the domain

$$\mathfrak{D} = \{(x,t) \in \mathbb{R}^2 : 0 < x < 1,\ 0 < t \le T\}, \quad T = \text{const.} > 0.$$

First, we shall examine properties of some integrals related to equation (1). Next, we introduce the operator (see [1])

$$\mathfrak{R}_\sigma[f(t)] = \frac{d}{dt}\int_0^t (t-\tau)^{-\sigma} f(\tau)d\tau, \quad 0 < \sigma < 1 \tag{2}$$

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and study some properties of this operator. Then, we reduce the problem considered to the system of Volterra integral equations and solve it. The general idea of our reasoning is similar to that in paper [1].

Let us note that, to the best of our knowledge, the second boundary-value problem for equation (1) has not been examined so far.

2. Fundamental solutions

The fundamental solutions of equation (1) are of the form (see [4], p. 133)

$$\mathfrak{U}(x,t;y,s) = \begin{cases} (t-s)^{-\frac{1}{3}}\mathbf{Ai}[(x-y)(t-s)^{-\frac{1}{3}}], & t > s, \\ 0, & t \le s \end{cases} \tag{3}$$

$$\mathcal{V}(x,t;y,s) = \begin{cases} (t-s)^{-\frac{1}{3}}\mathbf{Bi}[(x-y)(t-s)^{-\frac{1}{3}}], & t > s, \\ 0, & t \le s \end{cases} \tag{4}$$

where

$$\mathbf{Ai}(\xi) = \frac{\pi\sqrt{\xi}}{3\sqrt{3}}\left[J_{\frac{1}{3}}\left(\frac{2}{3\sqrt{3}}\xi^{\frac{3}{2}}\right) + J_{-\frac{1}{3}}\left(\frac{2}{3\sqrt{3}}\xi^{\frac{3}{2}}\right)\right], \tag{5}$$

$$\mathbf{Bi}(\xi) = \frac{\pi\sqrt{\xi}}{3\sqrt{3}}\left[J_{\frac{1}{3}}\left(\frac{2}{3\sqrt{3}}\xi^{\frac{3}{2}}\right) - J_{-\frac{1}{3}}\left(\frac{2}{3\sqrt{3}}\xi^{\frac{3}{2}}\right)\right] \tag{6}$$

and J_μ is the Bessel function of the μ-th order; **Ai** is called the Airy function and **Bi** the associated Airy function. The functions **Ai** and **Bi** are solutions of the ordinary differential equation (see [4], p. 133)

$$z''(\xi) + \tfrac{\xi}{3}z(\xi) = 0; \tag{7}$$

moreover, for these functions the following relations

$$\int_0^{+\infty} \mathbf{Ai}(\xi)d\xi = \frac{2\pi}{3}, \quad \int_{-\infty}^{0} \mathbf{Ai}(\xi)d\xi = \frac{\pi}{3}, \quad \int_0^{+\infty} \mathbf{Bi}(\xi)d\xi = 0 \tag{8}$$

hold (see [4], p. 139).

The functions $\mathfrak{U}$ and $\mathcal{V}$ and their derivatives satisfy the following inequalities (see [2], [4])

$$|D_x^k\mathfrak{U}(x,t;y,s)| \le c_1|x-y|^{\frac{2k-1}{4}}(t-s)^{-\frac{2k+1}{4}}, \tag{9}$$

$$|D_x^k\mathcal{V}(x,t;y,s)| \le c_2|x-y|^{\frac{2k-1}{4}}(t-s)^{-\frac{2k+1}{4}}, \tag{10}$$

for $x-y \ge 0$, $k = 0,1,\ldots$ and $c_1, c_2 = \text{const.} > 0$, and the inequality

$$|D_x^k\mathfrak{U}(x,t;y,s)| \le c_3(t-s)^{-\frac{k+1}{3}}\exp[-c_4|x-y|^{\frac{3}{2}}(t-s)^{-\frac{1}{2}}] \tag{11}$$

for $x-y \le 0$, $k = 0,1,\ldots$, and $c_3, c_4 = \text{const.} > 0$.

3. Airy potentials of first kind

Let us consider the integrals

$$(12) \qquad \mathfrak{I}_1(x,t;\varphi_1) = \int_0^t \mathfrak{U}(x,t;0,s)\varphi_1(s)ds,$$

$$(13) \qquad \mathfrak{I}_2(x,t;\varphi_2) = \int_0^t \mathfrak{V}(x,t;0,s)\varphi_2(s)ds,$$

where functions $\varphi_1(t)$, $\varphi_2(t)$ are continuous for $0 \le t \le T$. The integrals $\mathfrak{I}_1$, $\mathfrak{I}_2$, have properties similar to those of heat potentials of first kind. We shall call them the **Airy potentials of first kind** of the straight line $x = 0$, $0 \le t \le T$, with the density φ_1, φ_2, respectively. We have the following result.

THEOREM 1. *The Airy potential of first kind* $\mathfrak{I}_1$ *is of class* C^∞ *for* $x \neq 0$ *and* $0 < t \le T$ *and satisfies equation* (1); *moreover, the relations*

$$(14) \qquad \lim_{x\to 0, x>0} D_x^2 \mathfrak{I}_1(x,t;\varphi_1) = -\frac{2\pi}{3}\varphi_1(t),$$

$$(15) \qquad \lim_{x\to 0, x<0} D_x^2 \mathfrak{I}_1(x,t;\varphi_1) = \frac{\pi}{3}\varphi_1(t)$$

hold.

Proof. The first part of Theorem 1 is a consequence of estimates (9) and (11). We prove relation (14). The integral $\mathfrak{I}_1$ can be represented in the form

$$(16) \qquad D_x^2 \mathfrak{I}_1(x,t;\varphi_1) = \varphi_1(t)\cdot \mathrm{I}_1(x,t) + \mathrm{I}_2(x,t),$$

where

$$\mathrm{I}_1(x,t) = \int_0^t D_x^2 \mathfrak{U}(x,t;0,s)ds,$$

$$\mathrm{I}_2(x,t) = \int_0^t D_x^2 \mathfrak{U}(x,t;0,s)[\varphi_1(s) - \varphi_1(t)]ds.$$

Making use of (3), we have

$$\mathrm{I}_1(x,t) = \int_0^t (t-s)^{-1}\mathbf{Ai}''[x(t-s)^{-\frac{1}{3}}]ds,$$

hence, in virtue of (7), we get

$$\mathrm{I}_1(x,t) = -\frac{1}{3}\int_0^t x(t-s)^{-\frac{4}{3}}\mathbf{Ai}[x(t-s)^{-\frac{1}{3}}]ds.$$

Setting $x(t-s)^{-1/3}=\xi$, we have

$$\mathrm{I}_1(x,t) = -\int\limits_{xt^{-\frac{1}{3}}}^{+\infty} \mathbf{Ai}(\xi)d\xi,$$

hence, using the relation (8), we obtain

$$\lim_{x\to 0, x>0} \mathrm{I}_1(x,t) = -\frac{2\pi}{3}. \tag{17}$$

Due to the continuity of the function φ_1, we can choose the number δ in such a way that

$$|\varphi_1(s)-\varphi_1(t)| < \frac{\varepsilon}{2} \quad \text{for} \quad |s-t|<\delta. \tag{18}$$

Now, we investigate the behaviour of the integral I_2. We have

$$\mathrm{I}_2(x,t;\varphi_1) = \mathrm{I}_{21}(x,t) + \mathrm{I}_{22}(x,t), \tag{19}$$

where

$$\mathrm{I}_{21}(x,t) = \int\limits_0^{t-\delta} D_x^2 \mathfrak{U}(x,t;0,s)[\varphi_1(s)-\varphi_1(t)]ds,$$

$$\mathrm{I}_{22}(x,t) = \int\limits_{t-\delta}^{t} D_x^2 \mathfrak{U}(x,t;0,s)[\varphi_1(s)-\varphi_1(t)]ds.$$

Making use of estimate (9), we get

$$|\mathrm{I}_{21}(x,t)| \le 2M_1c_1 \int\limits_0^{t-\delta} x^{\frac{3}{4}}(t-s)^{-\frac{5}{4}}ds,$$

where $M_1 = \sup|\varphi_1(t)|$ for $0 \le t \le T$. Then, for $s \in [0, t-\delta]$ we have

$$|\mathrm{I}_{21}(x,t)| \le 2M_1c_1Tx^{\frac{3}{4}}\delta^{-\frac{5}{4}};$$

therefore the positive number $\rho_1 = \rho_1(\delta)$ can be chosen in such a way that for $x < \rho_1$ the inequality

$$|\mathrm{I}_{21}(x,t)| \le \frac{\varepsilon}{2} \tag{20}$$

holds. The integral I_{22} may be written in the form

$$\mathrm{I}_{22}(x,t) = -\frac{1}{3}\int\limits_{t-\delta}^{t} x(t-s)^{-\frac{4}{3}}\mathbf{Ai}[x(t-s)^{-\frac{1}{3}}][\varphi_1(s)-\varphi_1(t)]ds.$$

Applying the mean-value theorem, we have

$$\mathrm{I}_{22}(x,t) = \frac{-\delta x}{3(\theta\delta)^{\frac{4}{3}}}\mathbf{Ai}[x(\theta\delta)^{-\frac{1}{3}}][\varphi_1(t-\theta\delta)-\varphi_1(t)]ds,$$

where $0 < \theta < 1$. Hence, in accordance with (9) and (18), we get

$$|\mathrm{I}_{22}(x,t)| = \frac{x^{\frac{3}{4}}}{3\theta^{\frac{5}{4}}\delta^{\frac{1}{4}}} \cdot \frac{\varepsilon}{2};$$

therefore, the positive number $\rho_2 = \rho_2(\delta)$ can be chosen in such a way that for $x < \rho_2$ the inequality

$$|\mathrm{I}_{22}(x,t)| \leq \frac{\varepsilon}{2} \tag{21}$$

holds. If we now denote $\rho = \min(\rho_1, \rho_2)$, then, joinning on (19), (20) and (21), we obtain

$$|\mathrm{I}_2(x,t)| \leq \varepsilon, \quad \text{for} \quad x < \rho.$$

This means that the integral I_2 tends to zero, when $x \to 0, x > 0$. Finally, bearing in mind this result and relations (16), (17), we arrive at equality (14).

Now, we prove (15). The integral $\mathfrak{I}_1$ can be represented in the form

$$D_x^2\mathfrak{I}_1(x,t;\varphi_1) = \varphi_1(t)\cdot \mathrm{I}_3(x,t) + \mathrm{I}_4(x,t), \tag{22}$$

where

$$\mathrm{I}_3(x,t) = \int_0^t D_x^2\mathfrak{U}(x,t;0,s)ds, \quad \mathrm{I}_4(x,t) = \int_0^t D_x^2\mathfrak{U}(x,t;0,s)[\varphi_1(s)-\varphi_1(t)]ds.$$

Making use of (3), we have

$$\mathrm{I}_3(x,t) = \int_0^t (t-s)^{-1}\mathbf{Ai}''[x(t-s)^{-\frac{1}{3}}]ds,$$

hence, in virtue of (7), we get

$$\mathrm{I}_3(x,t) = -\frac{1}{3}\int_0^t x(t-s)^{-\frac{4}{3}}\mathbf{Ai}[x(t-s)^{-\frac{1}{3}}]ds.$$

Setting $x(t-s)^{-\frac{1}{3}} = \xi$, we have

$$\mathrm{I}_3(x,t) = \int_{-\infty}^{xt^{-\frac{1}{3}}} \mathbf{Ai}(\xi)d\xi;$$

hence, using the relation (8), we obtain

$$\lim_{x\to 0, x<0} \mathrm{I}_3(x,t) = \frac{\pi}{3}. \tag{23}$$

Now, we examine the behaviour of the integral I_4. We have

$$\mathrm{I}_4(x,t;\varphi_1) = \mathrm{I}_{41}(x,t) + \mathrm{I}_{42}(x,t), \tag{24}$$

where

$$\mathrm{I}_{41}(x,t) = \int\limits_{0}^{t-\delta} D_x^2\mathfrak{U}(x,t;0,s)[\varphi_1(s) - \varphi_1(t)]ds,$$

$$\mathrm{I}_{42}(x,t) = \int\limits_{t-\delta}^{t} D_x^2\mathfrak{U}(x,t;0,s)[\varphi_1(s) - \varphi_1(t)]ds.$$

Making use of estimate (11), we get

$$|\mathrm{I}_{41}(x,t)| \le 2M_1c_3 \int\limits_{0}^{t-\delta} |x|(t-s)^{-\frac{4}{3}} \exp[-c_4|x|^{\frac{3}{2}}(t-s)^{-\frac{1}{2}}]ds,$$

where $M_1 = \sup|\varphi_1(t)|$ for $0 \le t \le T$, then setting $|x|^{1/2} \cdot (t-s)^{-1/6} = \xi$, we have

$$|\mathrm{I}_{41}(x,t)| \le 12M_1c_3 \int\limits_{|x|^{1/2}\cdot t^{-1/6}}^{|x|^{1/2}\cdot\delta^{-1/6}} \xi\exp[-c_4\xi^3]d\xi.$$

As $x \to 0, x < 0$ both integration limits in the last integral tend to zero; therefore the positive number $\rho_3 = \rho_3(\delta)$ can be chosen in such a way that for $|x| < \rho_3$ the inequality

$$|\mathrm{I}_{41}(x,t)| \le \frac{\varepsilon}{2} \tag{25}$$

holds. The integral I_{42} may be written in the form

$$\mathrm{I}_{42}(x,t) = -\frac{1}{3}\int\limits_{t-\delta}^{t} x(t-s)^{-\frac{4}{3}}\mathbf{Ai}[x(t-s)^{-\frac{1}{3}}][\varphi_1(s) - \varphi_1(t)]ds.$$

Hence, in accordance with (11) and (18), we get

$$|\mathrm{I}_{42}(x,t)| \le c_3\frac{\varepsilon}{2}\int\limits_{t-\delta}^{t} |x|(t-s)^{-\frac{4}{3}} \exp[-c_4|x|^{\frac{3}{2}}(t-s)^{-\frac{1}{2}}]ds.$$

Putting $|x|^{1/2}(t-s)^{-1/6} = \xi$, we have

$$|\mathrm{I}_{42}(x,t)| \le 6c_3\frac{\varepsilon}{2}\int\limits_{|x|^{1/2}\cdot\delta^{-1/6}}^{+\infty} \xi\exp[-c_4\xi^3]d\xi.$$

Because of the relation

$$\int\limits_{|x|^{1/2}\cdot\delta^{-1/6}}^{+\infty} \xi\exp[-c_4\xi^3]d\xi \le \int\limits_{0}^{+\infty} \xi\exp[-c_4\xi^3]d\xi =$$

$$= c_4^{-2/3} \int_0^{+\infty} \eta \exp[-\eta^3] d\eta = \tfrac{1}{3} c_4^{-2/3} \cdot \Gamma(\tfrac{2}{3}),$$

the positive number $\rho_4 = \rho_4(\delta)$ can be chosen in such a way that for $|x| < \rho_4$ the inequality

$$|\mathrm{I}_{42}(x,t)| \leq \frac{\epsilon}{2} \tag{26}$$

holds. If we denote $\overline{\rho} = \min(\rho_3, \rho_4)$, then joinning on (25), (26) and (27), we obtain

$$|\mathrm{I}_4(x,t)| \leq \varepsilon, \quad \text{for} \quad x < \overline{\rho}.$$

This means that the integral I_4 tends to zero, when $x \to 0, x < 0$. Finally, bearing in mind this result and relations (22), (23) we arrive at equality (15), and the Theorem 1 is proved.

Remark 1. If the function φ_1 is continuous, then the relations

$$\lim_{x \to 0} D_x^k \mathfrak{J}_1(x,t;\varphi_1) = D_x^k \mathfrak{J}_1(0,t;\varphi_1), \quad k = 0, 1, \tag{27}$$

hold.

The proof of relations (27) does not cause any difficulty, due to the weak singularity of the intergrand of the integral $\mathfrak{J}_1$.

THEOREM 2. *The Airy potential of first kind* $\mathfrak{J}_2$ *is of class* C^∞ *for* $x > 0$, $0 < t \leq T$ *and satisfies equation (1); moreover, the relations*

$$\lim_{x \to 0, x > 0} D_x^2 \mathfrak{J}_2(x,t;\varphi_2) = 0, \tag{28}$$

$$\lim_{x \to 0, x > 0} D_x^k \mathfrak{J}_2(x,t;\varphi_2) = D_x^k \mathfrak{J}_2(0,t;\varphi_2), \quad k = 0, 1, \tag{29}$$

hold.

Proof. The first part of the theorem is a consequence of estimate (10). We prove relation (28). The integral $\mathfrak{J}_2$ can be represented in the form

$$D_x^2 \mathfrak{J}_2(x,t;\varphi_2) = \varphi_2(t) \cdot \mathrm{I}_5(x,t) + \mathrm{I}_6(x,t), \tag{30}$$

where

$$\mathrm{I}_5(x,t) = \int_0^t D_x^2 V(x,t;0,s) ds, \quad \mathrm{I}_6(x,t) = \int_0^t D_x^2 V(x,t;0,s)[\varphi_2(s) - \varphi_2(t)] ds.$$

Making use of (4), we have

$$\mathrm{I}_5(x,t) = \int_0^t (t-s)^{-1} \mathbf{Bi}''[x(t-s)^{-\frac{1}{3}}] ds,$$

hence, in virtue of (7), we get

$$\mathrm{I}_5(x,t) = -\frac{1}{3}\int_0^t x(t-s)^{-\frac{4}{3}}\mathbf{Bi}[x(t-s)^{-\frac{1}{3}}]ds.$$

Setting $x(t-s)^{-1/3} = \xi$, we have

$$\mathrm{I}_5(x,t) = -\int_{xt^{-\frac{1}{3}}}^{+\infty} \mathbf{Bi}(\xi)d\xi,$$

hence, using the relation (8), we obtain

$$\lim_{x\to 0, x>0} \mathrm{I}_5(x,t) = 0. \tag{31}$$

By similar arguments to those in the proof of Theorem 1, one can show that

$$\lim_{x\to 0, x>0} \mathrm{I}_6(x,t) = 0. \tag{32}$$

The formulae (30), (31) and (32) directly imply relation (28).

The proof of relations (29) does not cause any difficulty, due to the weak singularity of the intergrand of the integral $\mathfrak{I}_2$. The proof of Theorem 2 is completed.

4. The operator $\mathcal{R}_\sigma$

In this section we prove some lemmas concerning the properties of the operator $\mathcal{R}_\sigma$ defined by (2).

LEMMA 1. *If the function* φ_1 *is continuous in* $[0,T]$, *then*

$$\mathrm{Q}_1 \equiv \mathcal{R}_{2/3}[\mathfrak{I}_1(0,t;\varphi_1)] = \frac{2\pi}{\sqrt{3}}\mathbf{Ai}(0)\varphi_1(t). \tag{33}$$

Proof. By definition (2), we have

$$\mathrm{Q}_1 = \frac{d}{dt}\int_0^t (t-\tau)^{-\frac{2}{3}}\mathfrak{I}_1(0,\tau;\varphi_1)d\tau.$$

In accordance with (12), we get

$$\mathrm{Q}_1 = \frac{d}{dt}\int_0^t \int_0^\tau (t-\tau)^{-\frac{2}{3}}\mathfrak{U}(0,\tau;0,s)\varphi_1(s)dsd\tau,$$

and, by (3), we can write

$$\mathrm{Q}_1 = \mathbf{Ai}(0)\frac{d}{dt}\int_0^t \int_0^\tau (t-\tau)^{-\frac{2}{3}}(\tau-s)^{-\frac{1}{3}}\varphi_1(s)dsd\tau.$$

Changing the order of integration, we obtain

$$Q_1 = \mathbf{Ai}(0)\frac{d}{dt}\int\limits_0^t \Big[\int\limits_s^t (t-\tau)^{-\frac{2}{3}}(\tau - s)^{-\frac{1}{3}}d\tau\Big]\varphi_1(s)ds.$$

Since (see [4], p. 149)

$$\int\limits_s^t (t-\tau)^{-\frac{2}{3}}(\tau - s)^{-\frac{1}{3}}d\tau = \frac{2\pi}{\sqrt{3}},$$

it follows that

$$Q_1 = \frac{2\pi}{\sqrt{3}}\mathbf{Ai}(0)\frac{d}{dt}\int\limits_0^t \varphi_1(s)ds,$$

which proves relation (33).

In an entirely similar way we can prove the following lemma.

LEMMA 2. *If the function φ_2 is continuous in $[0,T]$, then*

$$\mathcal{R}_{2/3}[\mathfrak{I}_2(0,t;\varphi_1)] = \frac{2\pi}{\sqrt{3}}\mathbf{Bi}(0)\varphi_2(t). \tag{34}$$

Consider now an integral $\mathfrak{I}_3$ defined by

$$\mathfrak{I}_3(x,t;\varphi_1) = \int\limits_0^t \mathfrak{U}(x,t;1,s)\varphi_3(s)ds, \tag{35}$$

where the function $\varphi_3(t)$ is continuous for $0 \le t \le T$.

LEMMA 3. *If the function φ_3 is continuous in $[0,T]$, then*

$$Q_3 \equiv \mathcal{R}_{2/3}[\mathfrak{I}_3(0,t;\varphi_1)] = \int\limits_0^t \mathsf{K}_3(t,s)\varphi_3(s)ds, \tag{36}$$

where

$$\mathsf{K}_3(t,s) = \frac{\partial}{\partial t}\int\limits_s^t (t-\tau)^{-\frac{2}{3}}\mathfrak{U}(0,\tau;1,s)d\tau.$$

Moreover, we have

$$|\mathsf{K}_3(t,s)| \le C_3(t-s)^{-\frac{2}{3}}, \tag{37}$$

where $C_3 = const. > 0$.

Proof. According to the definition (2), we have

$$Q_3 = \frac{d}{dt}\int\limits_0^t (t-\tau)^{-\frac{2}{3}}\mathfrak{I}_3(0,\tau;\varphi_3)d\tau.$$

By definition (35), we obtain

$$Q_3 = \frac{d}{dt}\int_0^t \int_0^\tau (t-\tau)^{-\frac{2}{3}}\mathfrak{U}(0,\tau;1,s)\varphi_3(s)dsd\tau,$$

hence changing the order of integration, we get

$$Q_3 = \frac{d}{dt}\int_0^t \left[\int_s^t (t-\tau)^{-\frac{2}{3}}\mathfrak{U}(0,\tau;1,s)d\tau\right]\varphi_3(s)ds.$$

Thus, we arrive at the relation

$$Q_3 = \varphi_3(t)\lim_{s\to t}\int_s^t (t-\tau)^{-\frac{2}{3}}\mathfrak{U}(1,\tau;1,s)d\tau + \int_0^t K_3(t,s)\varphi_3(s)ds.$$

To obtain the formula (36) it is sufficient to show that

(38) $$\lim_{s\to t}\int_s^t (t-\tau)^{-\frac{2}{3}}\mathfrak{U}(0,\tau;1,s)d\tau = 0.$$

Let us denote

$$I_7 \equiv \int_s^t (t-\tau)^{-\frac{2}{3}}\mathfrak{U}(0,\tau;1,s)d\tau.$$

Applying the estimate (11), we find

$$|I_7| \le c_3\int_s^t (t-\tau)^{-\frac{2}{3}}(\tau-s)^{-\frac{1}{3}}\exp[\frac{-c_4}{(\tau-s)^{\frac{1}{3}}}]d\tau.$$

Using the inequality (see [7], p. 476)

(39) $$s^\lambda e^{-\lambda} \le (\lambda e^{-1})^\lambda \qquad \lambda > 0, \quad 0 \le s < \infty,$$

we obtain

$$|I_7| \le \text{const.}\int_s^t (t-\tau)^{-\frac{2}{3}}d\tau,$$

hence $|I_7| \le \text{const.}(t-s)^{\frac{1}{3}}$ which proves the validity of (38). Basing on the estimate (11), one can find

$$|\mathfrak{U}(0,t;1,s)| \le \text{const.}, \quad |\Delta_t\mathfrak{U}(0,t;1,s)| \le \text{const.}|\Delta t|.$$

Taking into account the above estimates and making use of lemma of Baderko (see [1], p. 1785) we obtain the inequality (37). Thus, the proof of Lemma 3 is completed.

In the further considerations we will also need the lemma as follows.

LEMMA 4. *If the function φ_1 is continuous in $[0,T]$, then*

$$Q_4 \equiv \mathcal{R}_{2/3}[\mathfrak{I}_1(1,t;\varphi_1)] = \int_0^t \mathsf{K}_1(t,s)\varphi_1(s)ds, \tag{40}$$

where

$$\mathsf{K}_1(t,s) = \frac{\partial}{\partial t}\int_s^t (t-\tau)^{-\frac{2}{3}}\mathfrak{U}(1,\tau;0,s)d\tau.$$

Moreover, we have

$$|\mathsf{K}_1(t,s)| \le C_1(t-s)^{-\frac{11}{12}}, \tag{41}$$

where $C_1 = const. > 0$.

Proof. By definition (2), we have

$$Q_4 = \frac{d}{dt}\int_0^t (t-\tau)^{-\frac{2}{3}}\mathfrak{I}_1(1,\tau;\varphi_1)d\tau.$$

In accordance with (12), we get

$$Q_4 = \frac{d}{dt}\int_0^t \int_0^\tau (t-\tau)^{-\frac{2}{3}}\mathfrak{U}(1,\tau;0,s)\varphi_1(s)dsd\tau.$$

Changing the order of integration, we obtain

$$Q_4 = \frac{d}{dt}\int_0^t \left[\int_s^t (t-\tau)^{-\frac{2}{3}}\mathfrak{U}(1,\tau;0,s)d\tau\right]\varphi_1(s)ds.$$

Thus, we arrive at the relation

$$Q_4 = \varphi_1(t)\lim_{s\to t}\int_s^t (t-\tau)^{-\frac{2}{3}}\mathfrak{U}(1,\tau;0,s)d\tau + \int_0^t \mathsf{K}_1(t,s)\varphi_1(s)ds.$$

To obtain the formula (40) it is sufficient to show that

$$\lim_{s\to t}\int_s^t (t-\tau)^{-\frac{2}{3}}\mathfrak{U}(1,\tau;0,s)d\tau = 0. \tag{42}$$

Let us denote

$$\mathrm{I}_8 \equiv \int_s^t (t-\tau)^{-\frac{2}{3}}\mathfrak{U}(1,\tau;0,s)d\tau.$$

Applying the estimate (9), we find

$$|\mathrm{I}_8| \le C\int_s^t (t-\tau)^{-\frac{2}{3}}(\tau-s)^{-\frac{1}{4}}d\tau.$$

Using the equality

$$\int\limits_s^t (t-\tau)^{-\frac{2}{3}}(\tau-s)^{-\frac{1}{4}}d\tau = C(t-s)^{\frac{1}{12}},$$

we obtain $|\mathrm{I}_8| \leq C(t-s)^{\frac{1}{12}}$ which proves the validity of (42).

In accordance with (9), we have

$$|\mathfrak{U}(1,t;0,s)| \leq \text{const.}(t-s)^{-\frac{1}{4}}, \quad |\Delta_t\mathfrak{U}(1,t;0,s)| \leq \text{const.}|\Delta t|^{\frac{3}{4}}(t-s)^{-1}.$$

Taking into account the above estimates and making use of lemma of Baderko (see [1], p. 1785) we obtain the inequality (41).

Thus, the proof of Lemma 4 is completed.

In an entirely similar way we can prove the following lemma.

LEMMA 5. *If the function φ_2 is continuous in $[0,T]$, then*

$$\mathcal{R}_{2/3}[\mathfrak{I}_2(1,t;\varphi_2)] = \int\limits_0^t \mathsf{K}_2(t,s)\varphi_2(s)ds, \tag{43}$$

where

$$\mathsf{K}_2(t,s) = \frac{\partial}{\partial t}\int\limits_s^t (t-\tau)^{-\frac{2}{3}}\mathcal{V}(1,\tau;0,s)d\tau.$$

Moreover, we have

$$|\mathsf{K}_2(t,s)| \leq C_2(t-s)^{-\frac{11}{12}}, \tag{44}$$

where $C_2 = const. > 0$.

5. Properties of the function $\mathcal{W}$

Now, we proceed to examine the function $\mathcal{W}$ given by formula

$$\mathcal{W}(x,t;g) = \int\limits_0^t \int\limits_0^1 \mathfrak{U}(x,t;y,s)g(y,s)dyds.$$

LEMMA 6. *If the function g is continuous and satisfies the inequality $|g(x,t)| \leq M_g$, where $M_g = const. > 0$, $(x,t) \in \mathfrak{D}$, then*

$$\mathcal{R}_{2/3}[\mathcal{W}(0,t;g)] = \int\limits_0^t \int\limits_0^1 \left[\frac{\partial}{\partial t}\int\limits_s^t (t-\tau)^{-\frac{2}{3}}\mathfrak{U}(0,\tau;y,s)d\tau\right]g(y,s)dyds \equiv w_0(t). \tag{45}$$

Moreover, we have

$$|w_0(t)| \leq const.\ M_g t^{\frac{\sigma}{2}}, \tag{46}$$

where $\frac{1}{3} < \sigma < \frac{2}{3}$.

P r o o f. The function $w_0(t)$ can be represented in the form

$$w_0(t) = \frac{\partial}{\partial t} \int_0^t \int_0^\tau \int_0^1 (t-\tau)^{-\frac{2}{3}} \mathfrak{U}(0,\tau;y,s) g(y,s)\,dy\,ds\,d\tau,$$

hence, changing the order of integration, we find

$$w_0(t) = \frac{\partial}{\partial t} \int_0^t \int_s^t \int_0^1 (t-\tau)^{-\frac{2}{3}} \mathfrak{U}(0,\tau;y,s) g(y,s)\,dy\,d\tau\,ds.$$

Let us observe that to derive (45), it is sufficient to prove that $\lim_{s\to t} \mathbf{I}_0(t,s) = 0$, where

$$\mathbf{I}_0(t,s) = \int_s^t \int_0^1 (t-\tau)^{-\frac{2}{3}} \mathfrak{U}(0,\tau;y,s) g(y,s)\,dy\,d\tau.$$

In view of (11) and of the properties of the function g, we have

$$|\mathbf{I}_0(t,s)| \le \text{const.} M_g \int_s^t \int_0^1 (t-\tau)^{-\frac{2}{3}} (\tau-s)^{-\frac{1}{3}} \times \exp[-c_2 y^{\frac{3}{2}} (\tau-s)^{-\frac{1}{2}}]\,dy\,d\tau.$$

Using the inequality (39), we get

$$(\tau-s)^{-\frac{1}{3}} \exp[-c_2 y^{\frac{3}{2}} (\tau-s)^{-\frac{1}{2}}] \le \text{const.} y^{-\frac{3}{2}\sigma} (\tau-s)^{\frac{\sigma}{2}-\frac{1}{3}},$$

where $0 < \sigma < \frac{2}{3}$. This implies the inequality

$$\begin{aligned}|\mathbf{I}_0(t,s)| &\le \text{const.} M_g \int_s^t (t-\tau)^{-\frac{2}{3}} (\tau-s)^{\frac{\sigma}{2}-\frac{1}{3}}\,d\tau \int_0^1 y^{-\frac{3}{2}\sigma}\,dy \\ &\le \text{const.} M_g (t-s)^{\frac{\sigma}{2}},\end{aligned}$$

which proves that the integral $\mathbf{I}_0(t,s)$ tends to zero as $s \to t$.

Let us consider the function

$$q_0(\tau,s) = \int_0^1 \mathbf{U}(0,\tau;y,s) g(y,s)\,dy.$$

On the basis of the inequality (11), we have

$$|q_0(\tau,s)| \le \text{const.} M_g \int_0^1 (\tau-s)^{-\frac{1}{3}} \exp[-c_2 y^{\frac{3}{2}} \cdot (\tau-s)^{-\frac{1}{2}}]\,dy,$$

hence, applying inequality (39), we find

(47) $|q_0(\tau,s)| \le \text{const.} M_g \int\limits_0^1 y^{-\frac{3}{2}\sigma}(\tau-s)^{-\frac{1}{3}+\frac{\sigma}{2}}dy \le \text{const.} M_g(\tau-s)^{-\frac{1}{3}+\frac{\sigma}{2}}$,

where $0<\sigma<\frac{2}{3}$.

Now, we proceed to estimate the function $\Delta_\tau q_0(\tau,s)$. It follows from the mean value theorem that

$$|\Delta_\tau q_0(\tau,s)| \le \text{const.} M_g|\Delta\tau| \int\limits_0^1 \left|\frac{\partial}{\partial\tau}\mathfrak{U}(0,\tau+\theta\Delta\tau;y,s)\right|dy,$$

where $0<\theta<1$. Hence, in view of (11), we get

$$|\Delta_\tau q_0(\tau,s)| \le \text{const.} M_g|\Delta\tau| \int\limits_0^1 (\tau+\theta\Delta\tau-s)^{-\frac{4}{3}}\times \times \exp[-c_2 y^{\frac{3}{2}}(\tau+\theta\Delta\tau-s)^{-\frac{1}{2}}]dy.$$

Applying the inequality (39), we obtain

$$|\Delta_\tau q_0(\tau,s)| \le \text{const.} M_g|\Delta\tau| \int\limits_0^1 y^{-\frac{3}{2}\sigma}(\tau+\theta\Delta\tau-s)^{-\frac{4}{3}+\frac{\sigma}{2}}dy,$$

where $0<\sigma<\frac{2}{3}$. Thus, we finally get

(48) $$|\Delta_\tau q_0(\tau,s)| \le \text{const.} M_g|\Delta\tau|^{1-\frac{\sigma}{2}}(\tau-s)^{-\frac{4}{3}+\sigma},$$

where $\frac{1}{3}<\sigma<\frac{2}{3}$.

Let us consider the function

$$Q_0(t,s) = \frac{\partial}{\partial t}\int\limits_s^t (t-\tau)^{-\frac{2}{3}} q_0(\tau,s)d\tau.$$

It follows from (47) and (48) that assumptions of lemma of Baderko (see [1], p. 1785) are satisfied, thus we may apply the said lemma and get

(49) $$|Q_0(t,s)| \le \text{const.}(t-s)^{-1+\frac{\sigma}{2}}.$$

We readily observe that

$$w_0(t) = \int\limits_0^t Q_0(t,s)ds.$$

Consequently, applying the inequality (49), we arrive at the estimation (46). Thus, the proof of Lemma 6 is completed.

Now, we proceed to investigate the function $\mathcal{W}(1,t;g)$. We prove the following lemma.

LEMMA 7. *If the function g satisfies the assumptions of Lemma 1, and $|\frac{\partial}{\partial x}g(x,t)| \le M_g$, where $M_g = const. > 0$, $(x,t) \in \mathfrak{D}$, then*

$$(50)\quad \mathcal{R}_{2/3}[\mathbf{W}(1,t;g)] = \int_0^t \int_0^1 \left[\frac{\partial}{\partial t} \int_s^t (t-\tau)^{-\frac{2}{3}} \mathfrak{U}(1,\tau;y,s)d\tau\right] g(y,s)dyds \equiv w_1(t).$$

Moreover, we have

$$(51)\qquad |w_1(t)| \le const. M_g t^{\frac{1}{12}}.$$

Proof. The function $w_1(t)$ can be represented in the form

$$w_1(t) = \frac{\partial}{\partial t} \int_0^t \int_0^\tau \int_0^1 (t-\tau)^{-\frac{2}{3}} \mathfrak{U}(1,\tau;y,s) g(y,s)dydsd\tau,$$

hence, changing the order of integration, we find

$$w_1(t) = \frac{\partial}{\partial t} \int_0^t \int_s^t \int_0^1 (t-\tau)^{-\frac{2}{3}} \mathfrak{U}(1,\tau;y,s) g(y,s)dyd\tau ds.$$

Let us observe that to derive (50), it is sufficient to prove that $\lim_{s\to t} \mathrm{I}_1(t,s) = 0$, where

$$\mathrm{I}_1(t,s) = \int_s^t \int_0^1 (t-\tau)^{-\frac{2}{3}} \mathfrak{U}(1,\tau;y,s) g(y,s)dyd\tau.$$

In view of (9) and of the properties of the function g, we have

$$\begin{aligned} |\mathrm{I}_1(t,s)| &\le \mathrm{const.} M_g \int_s^t \int_0^1 (t-\tau)^{-\frac{2}{3}} (\tau-s)^{-\frac{1}{4}} (1-y)^{-\frac{1}{4}} dyd\tau \\ &\le \mathrm{const.} M_g \int_s^t (t-\tau)^{-\frac{2}{3}} (\tau-s)^{-\frac{1}{4}} dyd\tau \\ &\le \mathrm{const.} M_g (t-s)^{\frac{1}{12}}, \end{aligned}$$

which proves that the integral $\mathrm{I}_1(t,s)$ tends to zero as $s \to t$.

Let us consider the function

$$q_1(\tau,s) = \int_0^1 \mathfrak{U}(1,\tau;y,s) g(y,s)dy.$$

On the basis of the inequality (9), we get

$$(52)\qquad |q_1(\tau,s)| \le \mathrm{const.} M_g \int_0^1 (\tau-s)^{-\frac{1}{4}} (1-y)^{-\frac{1}{4}} dy$$

$$\leq \text{const.} M_g(\tau - s)^{-\frac{1}{4}}.$$

Now, we proceed to estimate the function $\Delta_\tau q_1(\tau, s)$. We have

$$\Delta_\tau q_1(\tau, s) = \int_0^1 [\mathfrak{U}(1, \tau + \Delta\tau; y, s) - \mathfrak{U}(1, \tau; y, s)] g(y, s) dy.$$

Consider the following expression $\Delta\mathfrak{U} \equiv \mathfrak{U}(1, \tau + \Delta\tau; y, s) - \mathfrak{U}(1, \tau; y, s)$. In virtue of the mean value theorem, we obtain

$$\Delta\mathfrak{U} = \Delta\tau \frac{\partial}{\partial\tau} \mathfrak{U}(1, \tau + \theta\Delta\tau; y, s),$$

where $0 < \theta < 1$. It is easy to see that $\Delta\mathfrak{U}$ may be written in the form

$$\Delta\mathfrak{U} = -\Delta\tau \frac{\partial^3}{\partial y^3} \mathfrak{U}(1, \tau + \theta\Delta\tau; y, s),$$

hence, we have

$$\Delta_\tau q_1(\tau, s) = -\Delta\tau \int_0^1 \frac{\partial^3}{\partial y^3} \mathfrak{U}(1, \tau + \theta\Delta\tau; y, s) g(y, s) dy.$$

Integrating by parts, we get

$$\begin{aligned} \Delta_\tau q_1(\tau, s) = & -\Delta\tau \left[g(y, s) \frac{\partial^2}{\partial y^2} \mathfrak{U}(1, \tau + \theta\Delta\tau; y, s) \right]_{y=0}^{y=1} + \\ & + \Delta\tau \int_0^1 \frac{\partial^2}{\partial y^2} \mathfrak{U}(1, \tau + \theta\Delta\tau; y, s) \frac{\partial}{\partial y} g(y, s) dy = \\ = & -\Delta\tau g(0, s) \frac{\partial^2}{\partial y^2} \mathfrak{U}(1, \tau + \theta\Delta\tau; 0, s) + \\ & + \Delta\tau \int_0^1 \frac{\partial^2}{\partial y^2} \mathfrak{U}(1, \tau + \theta\Delta\tau; y, s) \frac{\partial}{\partial y} g(y, s) dy. \end{aligned}$$

On the basis of the assumptions of Lemma 7 and of the inequality (9), we get

$$\tag{53} \begin{aligned} |\Delta_\tau q_1(\tau, s)| &= \text{const.}\, M_g |\Delta\tau| (\tau + \theta\Delta\tau - s)^{-\frac{5}{4}} \\ &\leq \text{const.}\, M_g |\Delta\tau|^{\frac{3}{4} - \sigma} (\tau - s)^{-1+\sigma}, \end{aligned}$$

where $0 < \sigma < \frac{1}{12}$.

Let us consider the function

$$Q_1(t, s) = \frac{\partial}{\partial t} \int_s^t (t - y)^{-\frac{2}{3}} q_1(y, s) dy.$$

It follows from (52) and (53) that assumptions of lemma of Baderko (see [1], p. 1785) are satisfied, thus we may apply the said lemma and get

$$|Q_1(t,s)| \leq \text{const.}(t-s)^{-1+\frac{1}{12}}. \tag{54}$$

We readily observe that

$$w_1(t) = \int_0^t Q_1(t,s)ds.$$

Consequently, applying the inequality (54), we arrive at the estimation (51). Thus, the proof of Lemma 7 is completed.

LEMMA 8. *If the function g satisfies the assumptions of Lemma 7, then*

$$|w_2(t)| \leq \text{const.}M_g t^{\frac{\sigma}{2}}, \tag{55}$$

where $w_2(t) = D_x^2 W(0,t;g)$, *and* $0 < \sigma < \frac{2}{3}$.

Proof. The function $w_2(t)$ can be represented in the form

$$w_2(t) = \int_0^t \int_0^1 D_x^2 \mathfrak{U}(0,t;y,s)g(y,s)dyds.$$

On the basis of the inequality (11), we get

$$|w_2(t)| \leq \text{const.}M_g \int_0^t \int_0^1 (t-s)^{-1}\exp[-c_2 y^{\frac{3}{2}}(t-s)^{-\frac{1}{2}}]dyds.$$

Using (39), we have the inequality

$$|w_2(t)| \leq \text{const.}M_g \int_0^t \int_0^1 y^{-\frac{3}{2}\sigma}(t-s)^{-1+\frac{\sigma}{2}}dyds$$
$$\leq \text{const.}M_g \int_0^t (t-s)^{-1+\frac{\sigma}{2}}ds,$$

with $0 < \sigma < \frac{2}{3}$, which implies the estimate (55). The proof of Lemma 8 is completed.

6. Formulation of the problem

We pose the following boundary–value problem: **find a function** u **being in the domain** $\mathfrak{D}$ **a solution of equation**

$$\mathfrak{L}[u(x,t)] \equiv D_x^3 u(x,t) - D_t u(x,t) = f(x,t) \tag{56}$$

belonging to the class $\mathfrak{C}^{3,1}_{x,t}(\mathfrak{D}) \cap \mathfrak{C}^{2,0}_{x,t}(\overline{\mathfrak{D}})$ and satisfying the following boundary conditions

(57) $$u(x,0) = \psi(x), \qquad 0 \le x \le 1,$$

(58) $$u(0,t) = \phi_0(t), \qquad 0 \le t \le T,$$

(59) $$u(1,t) = \phi_1(t), \qquad 0 \le t \le T,$$

(60) $$D_x^2 u(0,t) = \phi_2(t), \qquad 0 \le t \le T,$$

where $\phi_j, j = 0,1,2$, and ψ are given functions satisfying the compatibility conditions $\phi_0(0) = \psi(0)$, $\phi_1(0) = \psi(1)$, $\phi_2(0) = \psi''(0)$.

We make the following assumptions:

(**A.1**) The function f is defined and continuous for $(x,t) \in \mathfrak{D}$, possesses continuous derivative $\frac{\partial f}{\partial x}$ and satisties the inequalities

$$| f(x,t) | \le M_f, \qquad \left| \frac{\partial f(x,t)}{\partial x} \right| \le M_f,$$

where M_f is a positive constant.

(**A.2**) The function $\phi_j, j = 0,1,2$, are defined and continuous in the interval $[0,T]$.

(**A.3**) The function ψ is defined and continuous for $x \in [0,1]$, possesses continuous derivatives $\psi^{(k)}, k = 1,2,3,4$, satisfying the conditions $|\psi^{(k)}(x)| \le M_\psi$, $k = 0,1,2,3,4$, where M_ψ is a positive constant.

Let us introduce the function v such that

$$v(x,t) = u(x,t) - \psi(x), \quad (x,t) \in \overline{\mathfrak{D}}.$$

In view of (56) – (60) and assumption (**A.3**), we get for v the following boundary value problem

(61) $$\mathfrak{L}[v(x,t)] = f(x,t) - \psi'''(x) \equiv g(x,t) \qquad \text{in} \quad \mathfrak{D},$$

(62) $$v(x,0) = 0, \qquad 0 \le x \le 1,$$

(63) $$v(0,t) = \phi_0(t) - \psi(0) \equiv \varphi_0(t), \qquad 0 \le t \le 1,$$

(64) $$v(1,t) = \phi_1(t) - \psi(1) \equiv \varphi_1(t), \qquad 0 \le t \le 1,$$

(65) $$D_x^2 v(0,t) = \phi_2(t) - \psi''(0) \equiv \varphi_2(t), \qquad 0 \le t \le 1.$$

7. Solution of the problem

We shall look for a solution v of the problem (61)–(65) such that

(66) $$\pi v(x,t) = \int_0^t \mathfrak{U}(x,t;0,s)\beta_0(s)ds + \int_0^t \mathfrak{V}(x,t;0,s)\beta_1(s)ds + \int_0^t \mathfrak{U}(x,t;1,s)\beta_2(s)ds - \mathfrak{W}(x,t;g),$$

where

$$\mathcal{W}(x,t;g) = \int_0^t \int_0^1 \mathcal{U}(x,t;y,s)g(y,s)dyds,$$

$\beta_0, \beta_1, \beta_2$ are unknown functions, and the functions $\mathcal{U}$ and $\mathcal{V}$ are given by the formulae (3) and (4), respectively. The function v, given by formula (66), satisfies the equation (61) and the initial condition (62). Imposing the boundary conditions (63)–(64), we get

$$\pi\varphi_0(t) = \mathbf{Ai}(0)\int_0^t (t-s)^{-\frac{1}{3}}\beta_0(s)ds + \mathbf{Bi}(0)\int_0^t (t-s)^{-\frac{1}{3}}\beta_1(s)ds + \int_0^t \mathcal{U}(0,t;1,s)\beta_2(s)ds - \mathcal{W}(0,t;g), \tag{67}$$

$$\pi\varphi_1(t) = \int_0^t \mathcal{U}(1,t;0,s)\beta_0(s)ds + \int_0^t \mathcal{V}(1,t;0,s)\beta_1(s)ds + \mathbf{Ai}(0)\int_0^t (t-s)^{-\frac{1}{3}}\beta_2(s)ds - \mathcal{W}(1,t;g), \tag{68}$$

and imposing the boundary condition (65), in view of Theorems 1 and 2, we obtain

$$\pi\varphi_2(t) = -\frac{2\pi}{3}\beta_0(t) + \int_0^t D_x^2\mathcal{U}(0,t;1,s)\beta_2(s)ds - D_x^2\mathcal{W}(0,t;g). \tag{69}$$

The equations (67), (68) are the Volterra integral equations of first kind. To reduce them to the Volterra integral equations of second kind we apply the operator $\mathcal{R}_{2/3}$ to both sides of them. In accordance with Lemmas 1–5, the equations (67)–(69) take the form

$$\frac{2}{\sqrt{3}}\mathbf{Ai}(0)\beta_0(t) + \frac{2}{\sqrt{3}}\mathbf{Bi}(0)\beta_1(t) + \frac{1}{\pi}\int_0^t \mathsf{K}_3(t,s)\beta_2(s)ds = \mathsf{F}_0(t), \tag{70}$$

$$\frac{2}{\sqrt{3}}\mathbf{Ai}(0)\beta_2(t) + \frac{1}{\pi}\int_0^t \mathsf{K}_1(t,s)\beta_0(s)ds + \frac{1}{\pi}\int_0^t \mathsf{K}_2(t,s)\beta_1(s)ds = \mathsf{F}_1(t), \tag{71}$$

$$-\frac{2}{3}\beta_0(t) + \frac{1}{\pi}\int_0^t \mathsf{K}_4(t,s)\beta_2(s)ds = \mathsf{F}_2(t), \tag{72}$$

where

$$\mathsf{K}_3(t,s) = \frac{\partial}{\partial t}\int_0^t (t-\tau)^{-\frac{2}{3}}\mathcal{U}(0,\tau;1,s)d\tau,$$

$$\mathsf{K}_1(t,s) = \frac{\partial}{\partial t} \int_0^t (t-\tau)^{-\frac{2}{3}} \mathfrak{U}(1,\tau;0,s)d\tau,$$

$$\mathsf{K}_2(t,s) = \frac{\partial}{\partial t} \int_0^t (t-\tau)^{-\frac{2}{3}} \mathcal{V}(1,\tau;0,s)d\tau,$$

$$\mathsf{K}_4(t,s) = D_x^2 \mathfrak{U}(0,t;1,s),$$

$$\mathsf{F}_k(t) = \frac{\partial}{\partial t} \int_0^t (t-\tau)^{-\frac{2}{3}} \varphi_k(\tau)d\tau + \frac{1}{\pi}\mathcal{R}_{2/3}[\mathcal{W}(k,t;g)], \quad k = 0,1,$$

$$\mathsf{F}_2(t) = \frac{1}{\pi} D_x^2 \mathcal{W}(0,t;g) + \varphi_2(t).$$

We treat the equation (70–(72) as an algebraic system with respect to functions β_0, β_1, β_2. The determinant of the system is of this form

$$\mathbf{W} = \begin{vmatrix} \frac{2}{\sqrt{3}}\mathbf{Ai}(0) & \frac{2}{\sqrt{3}}\mathbf{Bi}(0) & 0 \\ 0 & 0 & \frac{2}{\sqrt{3}}\mathbf{Ai}(0) \\ -\frac{2}{3} & 0 & 0 \end{vmatrix} = -\frac{8}{9}\mathbf{Ai}(0)\mathbf{Bi}(0) \neq 0,$$

then, using the Cramer formulas, we obtain

$$\beta_0(t) - \frac{3}{2\pi} \int_0^t \mathsf{K}_4(t,s)\beta_2(s)ds = -\frac{3}{2}\mathsf{F}_2(t), \tag{73}$$

$$\beta_1(t) + \frac{3\lambda_2}{2\pi} \int_0^t \mathsf{K}_4(t,s)\beta_2(s)ds + \frac{\sqrt{3}\lambda_1}{2\pi} \int_0^t \mathsf{K}_3(t,s)\beta_2(s)ds = \tag{74}$$

$$= \frac{\sqrt{3}\lambda_1}{2}\mathsf{F}_0(t) + \frac{3\lambda_2}{2}\mathsf{F}_2(t),$$

$$\beta_2(t) + \frac{\sqrt{3}\lambda_0}{2\pi} \int_0^t \mathsf{K}_1(t,s)\beta_0(s)ds + \frac{\sqrt{3}\lambda_0}{2\pi} \int_0^t \mathsf{K}_2(t,s)\beta_1(s)ds = \tag{75}$$

$$= \frac{\sqrt{3}\lambda_0}{2}\mathsf{F}_1(t),$$

where

$$\lambda_0 = \frac{1}{\mathbf{Ai}(0)}, \quad \lambda_1 = \frac{1}{\mathbf{Bi}(0)}, \quad \lambda_2 = \frac{\mathbf{Ai}(0)}{\mathbf{Bi}(0)}.$$

It is easy to see that system (73)–(75) can be written in the form

$$\beta_j(t) = \overline{\mathsf{F}}_j(t) + \sum_{i=0}^{2} \int_0^t \overline{\mathsf{K}}_{ij}(t,s)\beta_i(s)ds, \quad j = 0,1,2. \tag{76}$$

From Lemmas 4–6 it follows that (76) is a system of Volterra equations of second kind with weak singularities. Hence, we can assert that there exists a solution of the said system of the form

$$\beta_j(t) = \bar{\mathsf{F}}_j(t) + \sum_{i=0}^{2} \int_0^t \mathfrak{R}_{ij}(t,s)\bar{\mathsf{F}}_i(s)ds, \quad j = 0,1,2, \tag{77}$$

where $\mathfrak{R}_{ij}$ are resolvent kernels of $\overline{\mathsf{K}}_{ij}, i,j = 0,1,2$.

It is clear that the functions $\beta_j, j = 0,1,2$, obtained in this way satisfy the system of integral equations (67)–(69). In accordance with assumptions (**A.1**), (**A.2**) and Lemmas 6 – 8 the said functions are continuous for $t \in [0,T]$. Substituting these functions into formula (66), we obtain a solution v of the problem (61) – (65), whence and by the relation

$$u(x,t) = v(x,t) + \psi(x), \quad (x,t) \in \overline{\mathfrak{D}},$$

we can easily arrive at a solution u of the problem (56)–(60).

As a result of the foregoing considerations we can formulate the following theorem.

THEOREM 3. *If assumptions* (**A.1**)–(**A.3**) *are satisfied, then there exists a function* $u \in \mathfrak{C}^{3,1}_{x,t}(\mathfrak{D}) \cap \mathfrak{C}^{1,0}_{x,t}(\overline{\mathfrak{D}})$ *which is a solution of the problem* (56)–(60).

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