DEMONSTRATIO MATHEMATICA
Vol. XXIX No 3 1996

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ON GLOBAL SOLUTIONS OF TRANSLATION EQUATION

Various actions of groups and semigroups on sets are used in mathematics and its applications (see for example [5], [4]). In several papers such actions were generalized to actions of some classes of groupoids on sets. Partial groupoids and partial actions were also considered.

In this note we are going to explain, that for operations defined everywhere only the case of associative groupoids, i.e. semigroups, is essential. Our observations can also be helpful in the case of partial functions, but it will not be the subject in this note.

We will apply here standard notation and terminology, as for example in [1], [6], [9]. Let $(G, \cdot)$, or simply G, represent a groupoid. If the multiplication in G has the neutral element then this element will be denoted by e.

Let X be a nonempty set. By a *generalized action of G on X* we will mean here any function $\circ : X \times G \to X$ satisfying the following equation

(1) $$x \circ (a \cdot b) = (x \circ a) \circ b \text{ for any } x \in X \text{ and any } a, b \in G.$$

If G has the neutral element and in addition we have

(2) $$x \circ e = x \quad \text{for any } x \in X$$

then we will say that $\circ$ is *a standard action* of G on X. For standard actions of groups and semigroups on sets see for example [6], [4].

The equation 1, known as *the right translation equation,* was studied as a particular functional equation for example in [7], [8]. These investigations were continued for example in [2], [3].

Immediately from the definition we obtain the following observation about lifting of generalized actions.

PROPOSITION 1. *Let ϕ be a homomorphism from a groupoid G into a groupoid H and let $\circ'$ be a generalized action of H on a nonempty set X. Then the formula*

(3) $$x \circ g = x \circ' \phi(g) \textit{ for any } x \in X \textit{ and } g \in G$$

defines a generalized action of G on X. ∎

Let G be a groupoid and let $\equiv_s$ be the intersection of all congruences $\equiv$ on G, such that the factor groupoid $G/\equiv$ is a semigroup. Because the class of all semigroups is a variety of groupoids, (defined by the identity $a \cdot (b \cdot c) = (a \cdot b) \cdot c$), then the factor groupoid $G/ \equiv_s$ is a semigroup as a subdirect product of semigroups. By definition it's clear that $\equiv_s$ is the smallest congruence on G such that the factor groupoid is a semigroup.

The semigroup $G/\equiv_s$, mentioned above, is very important for our further considerations. It will be represented by G_s and the natural homomorphism of G onto G_s will be denoted by ϕ_s. Now we are ready to formulate our main result.

THEOREM 2. *Let G be a groupoid and X a nonemtpy set. Then any generalized action of G on X is induced by a generalized action of G_s on X and homomorphism ϕ_s, with help of formula* (3).

Proof. Let $\circ : X \times G \to X$ be a generalized action of G on X. Let $\equiv_\circ$ be the equivalence relation on G given by the rule:

(4) $$a \equiv_\circ b \text{ if and only if } x \circ a = x \circ b \text{ for any } x \in X.$$

Then $\equiv_\circ$ is a congruence on G. Indeed, let $a, b, c \in G$ be such that $a \equiv_\circ b$ and let $x \in X$. Then by the formula 1 we have

$$x \circ (a \cdot c) = (x \circ a) \circ c = (x \circ b) \circ c = x \circ (b \cdot c)$$

and

$$x \circ (c \cdot a) = (x \circ c) \circ a = (x \circ c) \circ b = x \circ (c \cdot b).$$

Hence, by definition, $ac \equiv_\circ bc$ and $ca \equiv_\circ cb$.

Let σ be the natural homomorphism of G onto $G/\equiv_\circ$. If we put

(5) $$x \circ'' \sigma(a) = x \circ a \text{ for any } x \in X \text{ and } a \in G$$

then from the definition of the relation $\equiv_\circ$ we have that $\circ''$ is a well defined function. This immediately implies that $\circ''$ is a generalized action of $G/\equiv_\circ$ on X.

Now let $a, b, c \in G$ be arbitrary, and $x \in X$. Then by formula 1 we have

$$x \circ ((a \cdot b) \cdot c) = (x \circ (a \cdot b)) \circ c = ((x \circ a) \circ b) \circ c.$$

and

$$x \circ (a \cdot (b \cdot c)) = (x \circ a) \circ (b \cdot c) = ((x \circ a) \circ b) \circ c.$$

In this way we see that

$$(a \cdot b) \cdot c \equiv_\circ a \cdot (b \cdot c).$$

This means that the factor algebra $G/\equiv_\circ$ is a semigroup. Hence by definition there exists a homomorphism $\beta : G_s \to G/\equiv_\circ$ such that $\sigma(a) = \beta(\phi_s(a))$ for any $a \in G$.

Let us put

$$x \circ' \phi_s(a) = x \circ'' \beta(\phi_s(a)) = x \circ a \text{ for any } x \in X \text{ and } \phi(a) \in G_s.$$

Immediate calculation shows that $\circ'$ is a generalized action of G_s on X. Moreover this action and $\circ$ are connected by formula 3. This completes the proof. ∎

COROLLARY 3. *Let G be a groupoid and X a nonemtpy set. Then the homomorphism ϕ_s induces a one to one correspondence between generalized actions of G on X and generalized actions of the semigroup G_s on X.* ∎

The result below concerns a connection between generalized and standard actions of monoids on sets.

THEOREM 4. *Let X be a nonempty set and G be a monoid. Then any generalized action of G on X is uniquely determined by an idempotent map $\epsilon = \epsilon^2 : X \to X$ and a standard action of G on the set $\epsilon(X)$.*

Proof. Let $\circ$ be a generalized action of G on X. Let us put $\epsilon(x) = x \circ e$ for all $x \in X$. Clearly $\epsilon = \epsilon^2$. Let us also put $Y = \epsilon(X) = X \circ e$.

If $y \in Y$ then by definition $y = x \circ e$ for some $x \in X$. Then for any $g \in G$ we have

$$y \circ g = (x \circ e) \circ g = x \circ (e \cdot g) = x \circ (g \cdot e) = (x \circ g) \circ e \in Y,$$

hence Y is G-invariant. The equality $e = e^2$ implies that the restriction of $\circ$ to $Y \times G$ satisfies conditions 1 and 2. Hence it is a standard action of G on Y.

Now let $\epsilon : X \to X$ be an idempotent map and let $\circ$ be a standard action of G on the set $\epsilon(X)$. Let us extend the operation $\circ$ to $X \times G$ by the formula

$$x \circ g = (\epsilon(x)) \circ g \text{ for all } x \in X,\ g \in G.$$

It is easy to calculate that this extension of the map $\circ$ is a generalized action of G on X. From such considerations the result follows. ∎

Now we will exhibite that for decomposition theorems standard actions of monoids are essential. For this let us agree that G is a monoid, $\circ$ its generalized action on a nonempty set X, and $Y = X \circ e$. From the above theorem we know that the action of G restricted to $Y \times G$ is a standard action. As usual let us agre that a subset $Z \subseteq X$ is *G-invariant* if $Z \circ G \subseteq Z$ and is *e-closed* if it is G-invariant and $x \circ e \in Z$ implies $x \in Z$ for any $x \in X$. Further let $L(Y)$ denotes the set of all G-invariant subsets of Y and $\tilde{L}(X)$ the set of all e-closed subsets of X. Clearly $L(Y)$ and $\tilde{L}(X)$ are complete lattices under inclusion.

THEOREM 5. *Under the above notation the map $\rho : \tilde{L}(X) \to L(Y)$ given by: $\rho(Z) = Z \cap Y$ is a lattice isomorphism. The map ρ^{-1} is given by: $\rho^{-1}(U) = \{x \in X; x \circ e \in U\}$.*

Proof. Because Y is a G-invariant subset of X then clearly ρ is a homomorphism of the lattice $\tilde{L}(X)$ into the lattice $L(Y)$. Let $U \subseteq V$ be e-closed subsets of X such that $\rho(U) = \rho(V)$. If $v \in V$ then by suitable definitions $v \circ e \in V \cap Y = \rho(V)$ Then by assumption $v \circ e \in \rho(U)$, hence $v \in U$ because U is e-closed. This means that $U = V$ and ρ is an injective map.

Now if $U \subseteq Y$ is G-invariant let us put $\gamma(U) = \{x \in X; x \circ e \in U\}$. Because e is the neutral element of multiplication in G then direct calculation gives that $\gamma(U)$ is an e-closed subset of X and $\rho(\gamma(U)) = U$. This means that ρ is a surjective map, hence an isomorphism, and $\rho^{-1} = \gamma$. ■

Results presented in this note are especially useful in the case, when G is a groupoid such that G_s is a group. In this case, according to [6], one can apply the standard decomposition of the set $Y = X \circ e$ into G_s-orbits and then lift this decomposition to X by Theorem 5. In this way one can obtain a generalization and simplification of many results from [2], [3].

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Received September 25, 1995.