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DIRECT MULTIPLICATION OF FINITE ALGEBRAS DOES NOT PRESERVE FINITE EQUATIONAL BASES: TWO EXAMPLES AND A GENERAL STATEMENT

It is known [1] that the intersection of two finitely based equational theories needn't be finitely based. In other words, the direct product of two algebras having each a finite equational basis (FEB) not necessarily has a FEB. In this paper we give two examples of such non-preservation phenomenon for *finite* algebras and show, on the basis of these examples, that, within "clone equivalence", every finite algebra with a FEB (and with at least 2 elements) is involved in an example of this kind, and that the same is true—with a single exception—for compositional equivalence as well.

EXAMPLE 1. A finite algebra $\mathfrak{A}_1$ with two binary operations that ceases to have a FEB after direct multiplication by an algebra with two constant operations (necessarily with different values). The elements are a, b, c, α, β, γ, δ, and 0. The operations are denoted $\circ$ and $*$ and defined by the equalities

$$a * \gamma = b,\ b * \delta = c,\ b \circ \alpha = b,\ b \circ \beta = c,\ c \circ \alpha = c,$$

and $x \circ y = x * y = 0$ for the remaining 123 "triads" of the form (element, operation, element).

First we prove that the identities of $\mathfrak{A}_1$ are finitely based. Denote by $\mathbf{0}$ the term, say, $u \circ u$ (or any fixed term expressing the zero function). The symbol $\Box$, possibly with a subscript, means "either $\circ$ or $*$". A bracket-free text of the form

(1) $$\pi_0 \Box_1 \pi_1 \Box_2 \pi_2 \Box_3 \ldots \Box_n \pi_n \ ,$$

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where π_i are variables, is understood as the term

$$((\ldots((\pi_0\Box_1\pi_1)\Box_2\pi_2)\Box_3\ldots)\Box_{n-1}\pi_{n-1})\Box_n\pi_n.$$

The proof itself is quite "Lyndon-style". The laws

$$x\Box_1(y\Box_2 z) = \mathbf{0}\Box x = x\Box\mathbf{0} = \mathbf{0}$$

(their validity for $\mathfrak{A}_1$, as well as the validity of the finitely many other laws introduced in the proof, can be easily verified by the reader) suffice to "annihilate", i.e. reduce to $\mathbf{0}$, any term that has a subterm of the form $T_1\Box_1(T_2\Box_2 T_3)$, thus disposing of all terms but those of the form (1). Now let t be a term of this form. If t contains at least three $*$'s then the law

$$x\circ y\circ z * u = x\circ y * u\circ z \tag{2}$$

enables us to shift all $*$'s in (1) for t, together with the variables immediately following the $*$'s, to the left as far as possible, and then to annihilate t using some law of the form $\tau = \mathbf{0}$ with three $*$'s and a limited number of $\circ$'s in τ (as well as other laws introduced above; that, throughout the proof, goes without saying). Similarly, if there are two $*$'s among the $\Box_i$'s in (1) but $\Box_1$ is not $*$, then $t = \mathbf{0}$ follows from $x\circ y * z * u = \mathbf{0}$. From now on, t is (1) with at most two asterisks, and if any of $\Box_i$ with $i \geq 2$ is an $*$, then the only other possible $*$ in t is $\Box_1$.

Now take the following variables in t (from 1 to 3 in number): the leftmost variable π_0 and, if any, the variable(s) immediately following the $*$(s) in (1) for t. If any one of these coincides with one of the remaining π_i's, or if any two of the three coincide with each other, then, again, using (2) and $x\circ y\circ z = x\circ z\circ y$ and shifting the second occurrence of the "bad" variable to the left, we have $t = \mathbf{0}$ as a consequence of finitely many laws, such as $x\Box_1 y\Box_2 z\Box_3 y = \mathbf{0}$ etc.

If t contains two $*$'s, the first necessarily being $\Box_1$, then the second $*$ is shifted to the extreme right. That may require another law, $x * y * z\circ u = x * y\circ u * z$. A unique occurrence of $*$ (other than $\Box_1$) is similarly shifted to the right.

Thus an arbitrary term t has been reduced to one of the following five forms (in which ξ, ζ, θ are pairwise distinct variables not occurring among variables $\eta_1,\ldots,\eta_n$, which needn't be distinct as yet):

(F0) $\mathbf{0}$;
(F1) $\xi\circ\eta_1\circ\ldots\circ\eta_n,\ n\geq 0$;
(F2) $\xi\circ\eta_1\circ\ldots\circ\eta_n * \zeta,\ n\geq 1$;
(F3) $\xi * \zeta\circ\eta_1\circ\ldots\circ\eta_n,\ n\geq 0$;
(F4) $\xi * \zeta\circ\eta_1\circ\ldots\circ\eta_n * \theta,\ n\geq 0$.

As for repetitions among the η's, the law $x\circ y\circ y\circ y = x\circ y\circ y$ reduces the number of occurrences of each η_i in t to 2 at most. In (F2) and (F4) we further reduce it to 1 using $x \circ y \circ y * z = x \circ y * z$. Now we have what can be called a *semicanonical form* for t; the prefix "semi-" here means that the "$\circ \eta_1 \circ \ldots \circ \eta_n$" segment is defined up to rearrangement of the η's in any order. For (F1/2/3/4) we denote by η' the set of all variables occurring in the semicanonical term as one of the η_i's; for (F1) and (F3), let η'' be the set of all those variables in η' that occur in (F1/F3) twice.

LEMMA. *Let t_1 and t_2 be semicanonical terms. If $t_1 = t_2 \in \mathrm{Eq}(\mathfrak{A}_1)$, then both terms fall under the same of the five items* (F0) *to* (F4) *and have the same ξ (for* F1 *to* F4*), the same ζ (for* F2,F3,F4*), the same θ (for* F4*), the same η' (for* F1 *to* F4*), and the same η'' (for* F1 *and* F3*). (Thus, the two terms differ only in ordering of their η's.)*

Proof. We give a description of the function expressed by a semicanonical term t; from this description the statement of the lemma follows immediately (taking into account that t_1 and t_2 of the lemma express the same function). The description just lists all those tuples of values of variables and corresponding values of t for which the latter are not 0. We use, in an informal manner, common notation, e.g. "$\exists! \pi \in \eta'(\pi = \beta)$, $\forall \eta \in \eta' \setminus \{\pi\}(\eta = \alpha)$" means "among the variables in η', there exist exactly one with value β, and all the other variables in η' have value α".

(F0) No non-zero values.

(F1) For $n > 0 : \xi = b, \forall \eta \in \eta'(\eta = \alpha), t = b;$
$\xi = c, \forall \eta \in \eta'(\eta = \alpha), t = c;$
$\xi = b, \exists! \pi \in \eta'(\pi = \beta), \forall \eta \in \eta' \setminus \{\pi\}(\eta = \alpha), t = c.$
For $n = 0$: ...(obvious).

(F2) $\xi = b, \forall \eta \in \eta'(\eta = \alpha), \zeta = \delta, t = c.$

(F3) $\xi = b, \zeta = \delta, \forall \eta \in \eta'(\eta = \alpha), t = c;$
$\xi = a, \zeta = \gamma, \forall \eta \in \eta'(\eta = \alpha), t = b;$
$\xi = a, \zeta = \gamma, \exists! \pi \in \eta'(\pi = \beta), \forall \eta \in \eta' \setminus \{\pi\}(\eta = \alpha), t = c.$

(F4) $\xi = a, \zeta = \gamma, \forall \eta \in \eta'(\eta = \alpha), \theta = \delta, t = c.$

By the lemma just proved, the laws specified (or mentioned) above form a FEB for $\mathfrak{A}_1$.

To prove that the direct product $\mathfrak{A}_1 \times \mathfrak{B}$, where $\mathfrak{B}$ is, say, $\langle \{0,1\}; x \circ y = 0, x * y = 1 \rangle$, has no FEB, note that $\mathrm{Eq}(\mathfrak{A}_1 \times \mathfrak{B})$, the equational theory of the product, consists exactly of those equations $t = \tau \in \mathrm{Eq}(\mathfrak{A}_1)$ for which neither t nor τ is a variable and their senior (external) operations are both $\circ$ or both $*$ (plus equations of the form $\xi = \xi$). Consider the sequence of equations $x * y_1 \circ y_2 \circ \ldots \circ y_n * x = x * x$, $n = 1, 2, \ldots$ The above description of equivalence of terms in $\mathfrak{A}_1$ shows that these equations are (identically) satisfied for $\mathfrak{A}_1$. (But the only way to derive them using finitely many laws is, for n large enough, to shift to left the right $*$ in the left-hand side of the equation, and that is unlawful in $\mathfrak{B}$.) The same description shows, first, that if τ is a term of the form $\xi * \zeta \circ \eta_1 \circ \ldots \circ \eta_m$ with $m+2$ distinct variables, and $\tau = T \in \mathrm{Eq}(\mathfrak{A}_1)$, then T is $\xi*\zeta\circ\pi_1\circ\ldots\circ\pi_r$, with $\cup\{\pi_i\} = \cup\{\eta_j\}$. Furthermore, it shows that if τ is a term of the form $\xi \circ \eta_1 \circ \ldots \circ \eta_m * \zeta$ with $m+2$ distinct variables, and $\tau = T \in \mathrm{Eq}(\mathfrak{A}_1)$, then T is $\xi \circ \pi_1 \circ \ldots \circ \pi_{k*}\zeta \circ \pi_{k+1} \circ \ldots \circ \pi_r$, with $\cup\{\pi_i\} = \cup\{\eta_j\}$. If, in addition, $\tau = T \in \mathrm{Eq}(\mathfrak{A})$, then T is $\xi \circ \pi_1 \circ \ldots \circ \pi_r * \zeta$. Thus (repeated) applications of identities in less than N variables from $\mathrm{Eq}(\mathfrak{A}_1 \times \mathfrak{A})$ can only transform the left-hand side of the N-th equation of the above sequence into terms of the form $x * y_1 \circ \pi_2 \circ \ldots \circ \pi_m * x$, with $\cup\{\pi_i\} = \{y_2, \ldots, y_N\}$. Hence the identities of $\mathfrak{A}_1 \times \mathfrak{B}$ are not finitely based.

EXAMPLE 2. A finite algebra with a single binary operation that ceases to have a FEB after direct multiplication by an algebra satisfying $x \circ y = y$. Let $\mathfrak{A}_2$ have elements 0, b, c, α, β and non-zero "products" $b \circ \alpha = b$, $c \circ \alpha = b \circ \beta = c$ (i.e. $\mathfrak{A}_2$ is the "$bc\alpha\beta 0$-subalgebra" of the "$\circ$-reduct" of $\mathfrak{A}_1$). The following equations form an equational basis for $\mathfrak{A}_2$: $x \circ x = \mathbf{0} \circ x = x \circ \mathbf{0} = x \circ (y \circ z) = \mathbf{0}$; $x \circ y \circ z = x \circ z \circ y$; $x \circ y \circ y \circ y = x \circ y \circ y$. (Proof is similar to Example 1 but much simpler and is left to the reader.) But $\mathfrak{A}_2 \times \mathfrak{B}$, where $\mathfrak{B}$ satisfies $x \circ y = y$ (and has at least two elements), has no FEB. Indeed, if $\tau = \xi \circ \eta_1 \circ \ldots \circ \eta_m \in \mathrm{Eq}(\mathfrak{A}_2)$ (the variables are assumed to be pairwise distinct) then τ is $\xi \circ \zeta_1 \circ \ldots \circ \zeta_m$ with $(\zeta_1 \ldots \zeta_m)$ a permutation of $(\eta_1 \ldots \eta_m)$; hence if $x \circ y_1 \circ \ldots \circ y_n \circ x = \tau$ is a conseqence of n-variable laws in $\mathrm{Eq}(\mathfrak{A}_2 \times \mathfrak{B}) = \mathrm{Eq}(\mathfrak{A}_2) \cap \mathrm{Eq}(\mathfrak{B})$, then τ must be of the form $x \circ y_p \circ \ldots \circ y_q \circ x$, where $p, \ldots, q$ are $1, \ldots, n$ re-arranged; thus $x \circ y_1 \circ \ldots \circ y_n \circ x = x \circ x \in \mathrm{Eq}(\mathfrak{A}_2 \times \mathfrak{B})$ is not a consequence of n-variable laws in $\mathfrak{A}_2 \times \mathfrak{B}$.

COROLLARY. *For every finite algebra $\mathfrak{A}$ of finite type (and at least two elements) there exist a finite algebra $\mathfrak{A}'$ with the same elements as $\mathfrak{A}$ and a finite algebra $\mathfrak{B}$ of the same finite type as $\mathfrak{A}'$ such that (i) $\mathfrak{A}$ and $\mathfrak{A}'$ are term equivalent, i.e. determine the same clone; (ii) $\mathfrak{B}$ has a* FEB, *but (iii) $\mathfrak{A}' \times \mathfrak{B}$ has no* FEB.

Proof. $\mathfrak{A}'$ is derived from $\mathfrak{A}$ by adjoining a single binary operation $\circ$ defined by $x \circ y = y$. As for $\mathfrak{B}$, its $\circ$-reduct is defined to be the $\mathfrak{A}_2$ of Example 2 while the remaining operation are all constants 0. Both existence of a FEB for $\mathfrak{B}$ and non-existence of one for $\mathfrak{A}' \times \mathfrak{B}$ are proved practically in the same manner as for Example 2.

The corollary shows that, up to "clone equivalence", every finite algebra with a FEB is involved (as a factor) in some example of two finite algebras with a FEB whose direct product has no FEB. The rest of the paper is aimed at proving a somewhat stronger version of the Corollary that has to do with ("strict") compositional equivalence rather than term equivalence.

DEFINITION. A *constant* algebra is an algebra whose operations are all constant operations with the same value.

We denote by $[\mathfrak{A}]$ the compositional closure of the fundamental operations of $\mathfrak{A}$. Thus for a constant algebra $\mathfrak{A}$, $[\mathfrak{A}]$ consists of a single function, which is a constant.

THEOREM. *Let $\mathfrak{A}$ be a non-constant finite algebra of finite type. Then for some $\mathfrak{A}'$ of finite type having the same universe as $\mathfrak{A}$ and for some finite $\mathfrak{B}$ of the same type as $\mathfrak{A}'$ (i) $[\mathfrak{A}'] = [\mathfrak{A}]$; (ii) $\mathfrak{B}$ has a FEB; (iii) $\mathfrak{A}' \times \mathfrak{B}$ has no* FEB.

NOTE. If $\mathfrak{A}$ is (compositionally equivalent to) a constant algebra, then for any $\mathfrak{B}$ the direct product $\mathfrak{A} \times \mathfrak{B}$ has a FEB iff $\mathfrak{B}$ has one. This fact is probably known in literature and, in any case, has a simple syntactical proof.

Proof of Theorem. If $[\mathfrak{A}]$ contains a unary non-constant function, say $f(x)$, then let $\mathfrak{A}'$ be $\mathfrak{A}$ with $x \circ y = f(y)$ adjoined, and let $\mathfrak{B}$ be the algebra $\mathfrak{A}_2$ of Example 2 with the original operations (in the type of $\mathfrak{A}$) equal to 0. The proof is similar to the proof for Example 2 (since $T \circ \xi = \tau \circ \eta \in \mathrm{Eq}(\mathfrak{A}')$ implies $\xi = \eta$, and $\xi_1 \circ \xi_2 \circ \ldots \circ \xi_n = \tau \in \mathrm{Eq}(\mathfrak{B})$ implies that no operation symbol but $\circ$ can occur in τ).

If $[\mathfrak{A}]$ contains at least two different constants, say 0 and 1, let $\mathfrak{A}'$ be $\mathfrak{A}$ with $x \circ y = 0$ and $x * y = 1$ adjoined, and let $\mathfrak{A}$ be the $\mathfrak{A}_1$ of Example 1 with the original operations defined as constants 0. The proof is similar to that of Example 1.

There remains only one case to be considered: the only unary function $f(x) \in [\mathfrak{A}]$ is a constant, say 0. Since $\mathfrak{A}$ is not a constant algebra, some $\phi(x_1, \ldots, x_n) \in [\mathfrak{A}]$ is not a constant function; without loss of generality we assume that x_1 is *essential* for ϕ , i.e. $\phi(a_1, b, \ldots, c) \neq \phi(a_2, b, \ldots, c)$ for some $a_1, a_2, b, \ldots, c$ in $\mathfrak{A}$. Clearly $\phi(0, y, \ldots, z) = 0$ identically. We "enrich" $\mathfrak{A}$ by an $(n+1)$-ary operation ψ defined by $\psi(x_0, x_1, \ldots, x_n) = \phi(x_1, \ldots, x_n)$, yielding $\mathfrak{A}'$. Let $\mathfrak{A}$ have the same elements as $\mathfrak{A}_2$ of Example 2 and operations $\psi(x_0, \ldots, x_n) = x_0 \circ x_1$, $\omega(\ldots) = 0$ for $\omega \neq \psi$.

As before, $\mathfrak{B}$ has a FEB. To prove that $\mathfrak{A}' \times \mathfrak{B}$ has none, we first define, for every finite sequence $\pi = (\eta_1, \ldots, \eta_l)$ of pairwise distinct variables, a set of terms A_π in this way: for $l = 1$, A_π consists of all terms $\psi(x, \eta_1, t_2, \ldots, t_n)$ where t_i are arbitrary terms (of the type of $\mathfrak{A}'$); for $l > 1$, A_π consists of all terms $\psi(t, \eta_l, \tau_2, \ldots, \tau_n)$ with any terms τ_i and with t in $A_{\pi'}$, where π' is $(\eta_1, \ldots, \eta_{l-1})$. (We assume that x, as well as $z_2, \ldots, z_n$ below, are not among the η_i's, but no limitations are imposed upon t_j and τ_j). Similarly, let B_π for $l = 1$ consist of a single term $\psi(x, \eta_1, z_2, \ldots, z_n)$, and for $l > 1$ let it consist of all terms $\psi(t, \eta_l, T_2, \ldots, T_n)$ with t in $A_{\pi'}$ above and $T_2, \ldots, T_n$ satisfying $\phi(\eta_l, T_2, \ldots, T_n) = \phi(\eta_l, z_2, \ldots, z_n)$ (for $\mathfrak{A}$).

If $\tau \in B_\pi$ and $t = \tau \in \mathrm{Eq}(\mathfrak{B})$, then, for some permutation $\pi' = (\eta_{i_1}, \ldots, \eta_{i_l})$ of $(\eta_1, \ldots, \eta_l)$, $t \in A_{\pi'}$. (Proved as above. Informally, $\tau = x \circ \eta_1 \circ \ldots \circ \eta_l$ and the other operations of $\mathfrak{B}$ are zeros, so that t is $\psi(t_1, t_2, \ldots)$ and every $\omega \neq \psi$ can occur only in the "..." part of t or in similar parts of t_1, t_2.) In $\mathfrak{A}'$, $\tau = \phi(\eta_l, z_2, \ldots, z_n)$ and $t = \phi(\eta_{i_l}, t_2, \ldots, t_n)$ for some terms $t_2, \ldots, t_n$. If, in addition, $\tau = t \in \mathrm{Eq}(\mathfrak{A}')$ then $i_l = l$ (otherwise $\eta_{i_l} = \mathbf{0}$ would imply $t = \mathbf{0}$ but not $\tau = \mathbf{0}$). Thus for a τ in $B_{(y_1, \ldots, y_N, x)}$ and a t in $B_{(x)}$ we have $\tau = t \in \mathrm{Eq}(\mathfrak{A}' \times \mathfrak{B})$ but it is not derivable from N-variable laws of $\mathfrak{A}' \times \mathfrak{B}$.

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