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TRANSFORMATIONS OF MIXED HYPERGEOMETRIC SERIES

1. Introduction

In an earlier paper [3] we have introduced a new type of hypergeometric series, called *mixed hypergeometric series* involving parameters of which some were ordinary and others were on the base q. In this paper an attempt is made to find some interesting transformations involving these mixed hypergeometric series.

2. Definitions and notations

Let

$$(a)_n = a(a+1)(a+2)\dots(a+n-1), \quad (a)_0 = 1, \tag{2.1}$$

$$(q^a)_n = (1-q^a)(1-q^{a+1})\dots(1-q^{a+n-1}), \quad (q^a)_0 = 1. \tag{2.2}$$

The generalized *bibasic* hypergeometric series is then defined as

$$A+B^{\phi}C+D\left[\begin{matrix} q^{(a)} : q_1^{(b)}; & x \\ q^{(c)} : q_1^{(d)}; & q^{\lambda}, q_1^{\lambda_1} \end{matrix}\right] \tag{2.3}$$

$$= \sum_{n=0}^{\infty} \frac{[q^{(a)}]_n [q_1^{(b)}]_n}{[q^{(c)}]_n [q_1^{(d)}]_n} x^n q^{\lambda n(n+1)/2} q_1^{\lambda_1 n(n+1)/2},$$

where $\lambda, \lambda_1 > 0, |q| < 1, |q_1| < 1$ and for $\lambda = 0 = \lambda_1$ there is $|x| < 1$.

In the numerator and the denominator the terms before the colon are on the base q and those after it are on the base q_1. As usual (a_N) stands for the sequence of N parameters $a_1, a_2, \dots, a_N$. When $N = A$, we simply write (a) for (a_A).

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Also, the q-fractional derivative $D^{\alpha}_{q,x}$ is defined as (cf. Agarwal [1])

$$D^{\alpha}_{q,x} x^{\mu-1} = (1-q)^{-\alpha} \pi_q[{}^{\mu-\alpha}_{\ \mu}] x^{\mu-\alpha-1}, \quad \alpha \neq \mu, \tag{2.4}$$

where $x_q[{}^{\alpha}_{\beta}]$ stands for $\frac{[\alpha]_\infty}{[\beta]_\infty}$ and $[\alpha]_\infty$ for $\prod_{r=0}^{\infty}[1-\alpha q^r]$.

From (2.4) it follows that

$$D^{\alpha-\beta}_{q,x} x^{n+\alpha-1} = \frac{(q^\beta)_\infty (q^\alpha)_n}{(q^\alpha)_\infty (q^\beta)_n} (1-q)^{\beta-\alpha} x^{n+\beta-1}. \tag{2.5}$$

Lastly, the mixed hypergeometric series is defined as follows

$$A+B^Y C+D \begin{bmatrix} (a) : q^{(b)}; & x \\ (c) : q^{(d)}; & q^\lambda \end{bmatrix} = \sum_{n=0}^{\infty} \frac{[(a)]_n [q^{(b)}]_n}{[(c)]_n [q^{(d)}]_n} x^n q^{\lambda n(n+1)/2}, \tag{2.6}$$

where $\lambda > 0, |q| < 1$ and for $\lambda = 0$ there is $|x| < 1$. Also, $[(a)]_n$ stands for $(a_1)_n (a_2)_n \dots (a_A)_n$ and $[q^{(b)}]_n$ stands for $(q^{b_1})_n (q^{b_2})_n \dots (q^{b_B})_n$.

3. Some simple transformations

In this section we obtain some simple transformations involving mixed hypergeometric series in the form of the following theorems.

THEOREM 1. *If* $|z| < 1$ *and* $|\frac{z}{1-z}| < 1$, *then*

$$2+r^Y 2+r \begin{bmatrix} a, b : q^{\alpha_1}, q^{\alpha_2}, \dots, q^{\alpha_r}; z \\ 1, c : q^1, q^{\beta_1}, \dots, q^{\beta_{r-1}} \end{bmatrix} \tag{3.1}$$

$$= \sum_{n=0}^{\infty} \frac{(a)_n (c-b)_n \prod_{i=1}^{r} (q^{\alpha_i})_n}{n!(c)_n (q^1)_n \prod_{j=1}^{r-1} (q^{\beta_j})_n} \left(1 + r^Y 1 + r \begin{bmatrix} a+n : q^{\alpha_1+n}, \dots, q^{\alpha_r+n}; z \\ 1 : q^{1+n}, q^{\beta_1+n}, q^{\beta_{r-1}+n} \end{bmatrix} \right) (-z)^n.$$

Proof. Consider the well known transformation

$$2^{F_1} \begin{bmatrix} a, b; z \\ c; \end{bmatrix} = (1-z)^{-a} \left(2^{F_1} \begin{bmatrix} a, c-b; \frac{-z}{1-z} \\ c; \end{bmatrix} \right) \tag{3.2}$$

(cf. [4], Theorem 20 p. 60). It can also be written as

$$\sum_{n=0}^{\infty} \frac{(a)_n (b)_n z^n}{n!(c)_n} = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(a)_{n+k} (c-b)_n (-1)^n}{n!k!(c)_n} z^{n+k}. \tag{3.3}$$

Multiplying both sides of (3.3) by z^{α_1-1} and operating by $D^{\alpha_1-\beta_1}_{q,z}$, we get

$$\sum_{n=0}^{\infty} \frac{(a)_n (b)_n (q^{\beta_1})_\infty (q^{\alpha_1})_n}{n!(c)_n (q^{\alpha_1})_\infty (q^{\beta_1})_n} (1-q)^{\beta_1-\alpha_1} z^{n+\beta_1-1} \tag{3.4}$$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{(a)_{n+k}(c-b)_n(-1)^n(q^{\beta_1})_\infty(q^{\alpha_1})_{n+k}}{n!k!(c)_n(q^{\alpha_1})_\infty(q^{\beta_1})_{n+k}}(1-q)^{\beta_1-\alpha_1}z^{n+k+\beta_1-1}$$

which can be written as

$$(3.5)\qquad 2+1^{Y}2+1\left[\begin{matrix}a,b:q^{\alpha_1};z\\ 1,c:q^{\beta_1}\end{matrix}\right]$$

$$= \sum_{n=0}^{\infty} \frac{(a)_n(c-b)_n(q^{\alpha_1})_n}{n!(c)_n(q^{\beta_1})_n}\left(1+1^{Y}1+1\left[\begin{matrix}a+n:q^{\alpha_1+n};z\\ 1:q^{\beta_1+n}\end{matrix}\right]\right)(-z)^n.$$

If we continue this process of performing q-derivative operators r times, we arrive at (3.1).

THEOREM 2. *If* $|z|<1$, *then*

$$(3.6)\qquad 2+r^{Y}2+r\left[\begin{matrix}a,b:q^{\alpha_1},q^{\alpha_2},\ldots,q^{\alpha_r};z\\ 1,c:q^{1},q^{\beta_1},\ldots,q^{\beta_{r-1}}\end{matrix}\right]$$

$$= \sum_{n=0}^{\infty} \frac{(c-a)_n(c-b)_n\prod_{i=1}^{r}(q^{\alpha_i})_n}{n!(c)_n(q^1)_n\prod_{j=1}^{r-1}(q^{\beta_j})_n}\left(1+r^{Y}1+r\left[\begin{matrix}a+b-c:q^{\alpha_1+n}\ldots q^{\alpha_r+n};z\\ 1:q^{1+n},q^{\beta_1+n},\ldots,q^{\beta_{r-1}+n}\end{matrix}\right]\right)z^n.$$

Proof. Consider the transformation

$$(3.7)\qquad 2^{F_1}\left[\begin{matrix}a,b;z\\ c;\end{matrix}\right]=(1-z)^{c-a-b}\left(2^{F_1}\left[\begin{matrix}c-a,c-b;z\\ c;\end{matrix}\right]\right)$$

(cf. [4] Theorem 21, p. 60). It can also be written as

$$(3.8)\qquad \sum_{n=0}^{\infty}\frac{(a)_n(b)_n z^n}{n!(c)_n}=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(c-a)_n(c-b)_n(a+b-c)_k}{n!(c)_n k!}z^{n+k}.$$

Now applying the procedure of the proof of Theorem 1, the relation (3.6) can easily be derived.

THEOREM 3. *If* $|z|<1, |1-z|<1, Re(c)<1, Re(c-a-b)>0$ *and none of* $a,b,c,c-a,c-b,c-a-b$ *is an integer, then*

$$(3.9)\qquad \sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{n!(a+b+1-c)_n}\left(1+r^{Y}1+r\left[\begin{matrix}-n:q^{\alpha_1},q^{\alpha_2},\ldots,a^{\alpha_r};z\\ 1:q^{1},q^{\beta_1},\ldots,q^{\beta_{r-1}}\end{matrix}\right]\right)$$

$$=\frac{\Gamma(a+b+1-c)\Gamma(1-c)}{\Gamma(a+1-c)\Gamma(b+1-c)}\left(2+r^{Y}2+r\left[\begin{matrix}a,b:q^{\alpha_1},q^{\alpha_2},\ldots,q^{\alpha_r};z\\ 1,c:q^{1},q^{\beta_1},\ldots,q^{\beta_{r-1}}\end{matrix}\right]\right)$$

$$+\frac{\Gamma(a+b+1-c)\Gamma(c-1)}{\Gamma(a)\Gamma(b)}\left(2+r^{Y}2+r\left[\begin{matrix}a+1-c,b+1-c:q^{\alpha_1},\ldots,q^{\alpha_r};z\\ 1,2-c:q^{1},q^{\beta_1},\ldots,q^{\beta_{r-1}}\end{matrix}\right]\right)z^{1-c}.$$

The proof of (3.9) follows by using the techniques of the proofs of Theorems 1 and 2 on the following known transformation

$$
{}_2F_1\left[\begin{matrix}a, b; 1-z\\ a+b+1-c;\end{matrix}\right] = \frac{\Gamma(1+a+b-c)\Gamma(1-c)}{\Gamma(a+1-c)\Gamma(b+1-c)}\left({}_2F_1\left[\begin{matrix}a, b; z\\ c;\end{matrix}\right]\right)
+\frac{\Gamma(a+b+1-c)\Gamma(c-1)}{\Gamma(a)\Gamma(b)}z^{1-c}\left({}_2F_1\left[\begin{matrix}a+1-c, b+1-c; z\\ 2-c;\end{matrix}\right]\right)
$$

(cf. [4] Theorem 22, p. 62).

4. Some quadratic transformations

We now establish certain quadratic transformations involving mixed hypergeometric series. For this we consider the following known transformations:

1) If $2b$ is neither zero nor a negative integer and $|x|<1$, $|4x(1+x)^{-2}|<1$, then

$$
(4.1)\qquad (1+x)^{-2a}\left({}_2F_1\left[\begin{matrix}a, b; \frac{4x}{(1+x)^2}\\ 2b;\end{matrix}\right]\right) = {}_2F_1\left[\begin{matrix}a, a-b+\frac{1}{2}; x^2\\ b+\frac{1}{2};\end{matrix}\right]
$$

(cf. [4] Theorem 23, p. 65).

2) If $2b$ is neither zero nor a negative integer and $|y|<1/2$, $|y/(1-y)|<1$, then

$$
(4.2)\qquad (1-y)^{-a}\left({}_2F_1\left[\begin{matrix}\frac{1}{2}a, \frac{1}{2}a+\frac{1}{2}; & \\ & \frac{y^2}{(1-y)^2}\\ b+\frac{1}{2};\end{matrix}\right]\right) = {}_2F_1\left[\begin{matrix}a, b; 2y\\ 2b;\end{matrix}\right]
$$

(cf. [4] Theorem 24, p. 65].

3) If $a+b+\frac{1}{2}$ is neither zero nor a negative integer and $|x|<1$, $|4x(1-x)|<1$, then

$$
(4.3)\qquad {}_2F_1\left[\begin{matrix}a, b; & \\ & 4x(1-x)\\ a+b+\frac{1}{2};\end{matrix}\right] = {}_2F_1\left[\begin{matrix}2a, 2b; x\\ a+b+\frac{1}{2};\end{matrix}\right]
$$

(cf. [4] Theorem 25, p. 67).

4) If c is neither zero nor a negative integer and $|x|<1$, $|4x(1-x)|<1$, then

$$
{}_2F_1\left[\begin{matrix}a, 1-a; x\\ c;\end{matrix}\right] = (1-x)^{c-1}\left({}_2F_1\left[\begin{matrix}\frac{1}{2}c-\frac{1}{2}a, \frac{1}{2}c+\frac{1}{2}a-\frac{1}{2}; & \\ & 4x(1-x)\\ c;\end{matrix}\right]\right)
$$

(cf. [4] Theorem 27, p. 69).

Adopting the procedure obtaining transformations given in Theorems 1, 2 and 3, we obtain the following transformations (given in the form of theorems) corresponding to (4.1), (4.2) and (4.4), respectively:

THEOREM 4. *If* $2b$ *is neither zero nor a negative integer and* $|z| < 1$, *then*

$$\sum_{n=0}^{\infty} \frac{(a)_n(b)_n \prod_{i=1}^{r}(q^{\alpha_i})_n}{n!(2b)_n \prod_{j=1}^{r}(q^{\beta_j})_n} \left(1 + r^Y 1 + r \begin{bmatrix} 2a + 2b : q^{\alpha_1+n}, q^{\alpha_2+n}, \ldots, q^{\alpha_r+n}; -z \\ 1 : q^{\beta_1+n}, q^{\beta_2+n}, \ldots, q^{\beta_r+n} \end{bmatrix}\right) (4z)^n$$

$$= 2{+}2r^Y 2{+}2r \begin{bmatrix} a, a-b+\frac{1}{2} : q_1^{\frac{1}{2}\alpha_1}, q_1^{\frac{1}{2}\alpha_2}, \ldots, q_1^{\frac{1}{2}\alpha_r}, q_1^{\frac{1}{2}\alpha_1+\frac{1}{2}}, q_1^{\frac{1}{2}\alpha_2+\frac{1}{2}}, \ldots, q_1^{\frac{1}{2}\alpha_r+\frac{1}{2}}; z \\ 1, b+\frac{1}{2} : q_1^{\frac{1}{2}\beta_1}, q_1^{\frac{1}{2}\beta_2}, \ldots, q_1^{\frac{1}{2}\beta_r}, q_1^{\frac{1}{2}\beta_1+\frac{1}{2}}, q_1^{\frac{1}{2}\beta_2+\frac{1}{2}}, \ldots, q_1^{\frac{1}{2}\beta_r+\frac{1}{2}} \end{bmatrix},$$

where $q_1 = q^2$ *and* $(q_1^{\frac{1}{2}\alpha})_n = (1 - q_1^{\frac{1}{2}\alpha})(1 - q_1^{\frac{1}{2}\alpha+1}) \ldots (1 - q_1^{\frac{1}{2}\alpha+n-1})$, *i.e.*, $(q_1^{\frac{1}{2}\alpha})_n = (1 - q^{\alpha})(1 - q^{\alpha+2}) \ldots (1 - q^{\alpha+2n-2})$.

THEOREM 5. *If* $2b$ *is neither zero nor a negative integer and* $|z| < \frac{1}{2}$, *then*

(4.6)
$$\sum_{n=0}^{\infty} \frac{(\frac{1}{2}a)_n(\frac{1}{2}a+\frac{1}{2})_n \prod_{i=1}^{r}(q^{\alpha_i})_{2n}}{n!(b+\frac{1}{2})_n \prod_{j=1}^{r}(q^{\beta_j})_{2n}} \left(1 + r^Y 1 + r \begin{bmatrix} 2n + a : q^{\alpha_1+2n}, \ldots, q^{\alpha_r+2n}; z \\ 1 : q^{\beta_1+2n}, \ldots, q^{\beta_r+2n} \end{bmatrix}\right) z^{2n}$$

$$= 2 + r^Y 2 + r \begin{bmatrix} a, b : q^{\alpha_1}, q^{\alpha_2}, \ldots, q^{\alpha_r}; 2z \\ 1, 2b : q^{\beta_1}, q^{\beta_2}, \ldots, q^{\beta_r} \end{bmatrix}.$$

THEOREM 6. *If* $a+b+\frac{1}{2}$ *is neither zero nor a negative integer and* $|z| < 1$, *then*

(4.7)
$$\sum_{n=0}^{\infty} \frac{(a)_n(b)_n \prod_{i=1}^{r}(q^{\alpha_i})_n (4z)^n}{n!(a+b+\frac{1}{2})_n \prod_{j=1}^{r}(q^{\beta_i})_n} \left(1 + r^Y 1 + r \begin{bmatrix} -n : q^{\alpha_1+n}, q^{\alpha_2+n}, \ldots, q^{\alpha_r+n}; z \\ 1 : q^{\beta_1+n}, q^{\beta_2+n}, \ldots, q^{\beta_r+n} \end{bmatrix}\right)$$

$$= 2 + r^Y 2 + r \begin{bmatrix} 2a, 2b : q^{\alpha_1}, q^{\alpha_2}, \ldots, q^{\alpha_r}; z \\ 1, a+b+\frac{1}{2} : q^{\beta_1}, q^{\beta_2}, \ldots, q^{\beta_r} \end{bmatrix}.$$

THEOREM 7. *If* c *is neither zero nor a negative integer and* $|z| < 1$, *then*

(4.8)
$$2 + r^Y 2 + r \begin{bmatrix} a, 1-a : q^{\alpha_1}, q^{\alpha_2}, \ldots, q^{\alpha_r}; z \\ 1, c : q^{\beta_1}, q^{\beta_2}, \ldots, q^{\beta_r} \end{bmatrix}$$

$$= \sum_{n=0}^{\infty} \frac{(\frac{1}{2}c - \frac{1}{2}a)_n(\frac{1}{2}c + \frac{1}{2}a - \frac{1}{2})_n \prod_{i=1}^{r}(q^{\alpha_i})_n}{n!(c)_n \prod_{j=1}^{r}(q^{\beta_j})_n}$$

$$\times \left(1 + r^Y 1 + r \left[\begin{matrix} -n - c + 1 : q^{\alpha_1+n}, q^{\alpha_2+n}, \ldots, q^{\alpha_r+n}; z \\ 1 : q^{\beta_1+n}, q^{\beta_2+n}, \ldots, q^{\beta_r+n} \end{matrix}\right]\right) (4z)^n.$$

References

[1] R. P. Agarwal, *Fractional q-derivatives and q-integrals and certain hypergeometric functions*, Ganita, 27(1976), 25–12.

[2] F. H. Jackson, *On basic double hypergeometric functions*, Quart. J. Math. (1942), 70–80.

[3] M. A. Khan, A. H. Khan, *A note on mixed hypergeometric series*, Acta Math. Vietnam, 14 (1989), 95–98.

[4] E. D. Rainville, *Special Functions.* The Macmillan Co., New York. 1960.

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